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## Complex-pair representation of GA(2,0) multivectors

We found previously that a complex pair representation of a GA(2,0) multivector had a compact geometric product realization. Now that we know the answer, let's work backwards from that representation to verify that everything matches our expectations.

We are representing a multivector of the form

$$M = a + b\mathbf{e}_1\mathbf{e}_2 + x\mathbf{e}_1 + y\mathbf{e}_2, \tag{1.1}$$

as the pair of complex numbers

$$M \sim (a + ib, x + iy). \tag{1.2}$$

Given a pair of multivectors with this complex representation

$$\begin{aligned} M &= (z_1, z_2) \\ N &= (q_1, q_2), \end{aligned} \tag{1.3}$$

we found that our geometric product representation was

$$MN \sim (z_1 q_1 + z_2^* q_2, z_2 q_1 + z_1^* q_2). \tag{1.4}$$

Our task is now to verify that this is correct. Let's set

$$\begin{aligned} z_1 &= a + ib \\ q_1 &= a' + ib' \\ z_2 &= x + iy \\ q_2 &= x' + iy', \end{aligned} \tag{1.5}$$

and proceed with an expansion of the even grade components

$$\begin{aligned} z_1 q_1 + z_2^* q_2 &= (a + ib)\left(a' + ib'\right) + (x - iy)\left(x' + iy'\right) \\ &= aa' - bb' + xx' + yy' + i\left(ba' + ab' + xy' - yx'\right) \\ &= xx' + yy' + i\left(xy' - yx'\right) + \quad aa' - bb' + i\left(ba' + ab'\right). \end{aligned} \tag{1.6}$$

The first terms is clearly the geometric product of two vectors

$$\left(x\mathbf{e}_1 + y\mathbf{e}_2\right)\left(x'\mathbf{e}_1 + y'\mathbf{e}_2\right) = xx' + yy' + i\left(xy' - yx'\right), \tag{1.7}$$

and we are able to verify that the second parts can be factored too

$$\left(a + bi\right)\left(a' + b'i\right) = aa' - bb' + i\left(ba' + ab'\right). \tag{1.8}$$

This leaves us with

$$\left\langle MN\right\rangle_{0,2} = \left\langle M\right\rangle_1 \left\langle N\right\rangle_1 + \left\langle M\right\rangle_{0,2} \left\langle N\right\rangle_{0,2}, \tag{1.9}$$

as expected. This part of our representation checks out.

Now, let's look at the vector component of our representation. First note that to convert from our complex representation of our vector $z = x + iy$ to the standard basis representation of our vector, we need only multiply by $\mathbf{e}_1$ on the left, for example:

$$\mathbf{e}_1\left(x + iy\right) = \mathbf{e}_1 x + \mathbf{e}_1\mathbf{e}_1\mathbf{e}_2 y = \mathbf{e}_1 x + \mathbf{e}_2 y. \tag{1.10}$$

So, for the vector component of our assumed product representation, we have

$$\begin{aligned} \mathbf{e}_1\left(z_2 q_1 + z_1^* q_2\right) &= \mathbf{e}_1\left(x + iy\right)\left(a' + ib'\right) + \mathbf{e}_1\left(a - ib\right)\left(x' + iy'\right) \\ &= \mathbf{e}_1\left(x + iy\right)\left(a' + ib'\right) + \left(a + ib\right)\mathbf{e}_1\left(x' + iy'\right) \\ &= \left\langle M\right\rangle_1 \left\langle N\right\rangle_{0,2} + \left\langle M\right\rangle_{0,2} \left\langle N\right\rangle_1, \end{aligned} \tag{1.11}$$

as expected.

Our complex-pair realization of the geometric product checks out.