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# A complex-pair representation of GA(2,0).

## 1.1 Motivation.

Suppose that we want to represent GA(2,0) (Euclidean) multivectors as a pair of complex numbers, with a structure like

$$M = (m_1, m_2), \tag{1.1}$$

where

$$\begin{aligned} \left\langle M \right\rangle_{0,2} &\sim m_1 \\ \left\langle M \right\rangle_{1} &\sim m_2. \end{aligned} \tag{1.2}$$

Specifically

$$\begin{aligned} \left\langle M \right\rangle_{0} &= \mathrm{Re}(m_1) \\ \left\langle M \right\rangle_{1} \cdot \mathbf{e}_1 &= \mathrm{Re}(m_2) \\ \left\langle M \right\rangle_{1} \cdot \mathbf{e}_2 &= \mathrm{Im}(m_2) \\ \left\langle M \right\rangle_{2} i^{-1} &= \mathrm{Im}(m_1), \end{aligned} \tag{1.3}$$

where $i \sim \mathbf{e}_1 \mathbf{e}_2$.

## 1.2 Multivector product representation.

Let's figure out how we can represent the various GA products, starting with the geometric product. Let

$$\begin{aligned} M &= \left\langle M \right\rangle_{0,2} + \left\langle M \right\rangle_{1} = (m_1, m_2) = (m_{11} + m_{12} i, m_{21} + m_{22} i) \\ N &= \left\langle N \right\rangle_{0,2} + \left\langle N \right\rangle_{1} = (n_1, n_2) = (n_{11} + n_{12} i, n_{21} + n_{22} i), \end{aligned} \tag{1.4}$$

so

$$\begin{aligned} MN &= \left\langle M \right\rangle_{0,2} \left\langle N \right\rangle_{0,2} + \left\langle M \right\rangle_{1} \left\langle N \right\rangle_{1} \\ &\quad + \left\langle M \right\rangle_{1} \left\langle N \right\rangle_{0,2} + \left\langle M \right\rangle_{0,2} \left\langle N \right\rangle_{1} \end{aligned} \tag{1.5}$$

The first two terms have only even grades, and the second two terms are vectors. The complete representation of the even grade components of this multivector product is

$$\left\langle MN \right\rangle_{0,2} \sim \left( m_1 n_1 + \mathrm{Re}(m_2 n_2^*) - i\, \mathrm{Im}(m_2 n_2^*), 0 \right), \tag{1.6}$$

or

$$\begin{aligned}\langle MN\rangle_0 &= \mathrm{Re}\left(m_1 n_1 + m_2 n_2^*\right) \\ \langle MN\rangle_2 i^{-1} &= \mathrm{Im}\left(m_1 n_1 - m_2 n_2^*\right).\end{aligned} \tag{1.7}$$

For the vector components we have

$$\begin{aligned}\langle MN\rangle_1 &= \langle M\rangle_1\langle N\rangle_0 + \langle M\rangle_0\langle N\rangle_1 + \langle M\rangle_1\langle N\rangle_2 + \langle M\rangle_2\langle N\rangle_1 \\ &= \langle M\rangle_1 n_{11} + m_{11}\langle N\rangle_1 + \langle M\rangle_1 i n_{12} + i m_{12}\langle N\rangle_1.\end{aligned} \tag{1.8}$$

For these,

$$\langle M\rangle_1 i = \left(m_{21}\mathbf{e}_1 + m_{22}\mathbf{e}_2\right)\mathbf{e}_{12} \quad = -m_{22}\mathbf{e}_1 + m_{21}\mathbf{e}_2, \tag{1.9}$$

and

$$i\langle N\rangle_1 = \mathbf{e}_{12}\left(n_{21}\mathbf{e}_1 + n_{22}\mathbf{e}_2\right) \quad = n_{22}\mathbf{e}_1 - n_{21}\mathbf{e}_2. \tag{1.10}$$

Comparing to

$$i(a+ib) = -b + ia, \tag{1.11}$$

we see that

$$\langle MN\rangle_1 \sim \left(0, n_{11}m_2 + m_{11}n_2 + n_{12}im_2 - m_{12}in_2\right). \tag{1.12}$$

If we want the vector coordinates, those are

$$\begin{aligned}\langle MN\rangle_1 \cdot \mathbf{e}_1 &= \mathrm{Re}\left(n_{11}m_2 + m_{11}n_2 + n_{12}im_2 - m_{12}in_2\right) \\ \langle MN\rangle_1 \cdot \mathbf{e}_2 &= \mathrm{Im}\left(n_{11}m_2 + m_{11}n_2 + n_{12}im_2 - m_{12}in_2\right).\end{aligned} \tag{1.13}$$

## 1.3 Summary.

$$MN \sim \left(m_1 n_1 + \mathrm{Re}(m_2 n_2^*) - i\,\mathrm{Im}(m_2 n_2^*), n_{11}m_2 + m_{11}n_2 + n_{12}im_2 - m_{12}in_2\right). \tag{1.14}$$

A sample Mathematica implementation is available, as well as an example notebook (that also doubles as a test case.)

## 1.4 Clarification.

I skipped a step above, showing the correspondances to the dot and wedge product.

Let $z = a + bi$, and $w = c + di$. Then:

$$\begin{aligned}zw^* &= \left(a+bi\right)\left(c-di\right) \\ &= ac + bd - i\left(ad - bc\right).\end{aligned} \tag{1.15}$$

Compare that to the geometric product of two vectors $\mathbf{x} = a\mathbf{e}_1 + b\mathbf{e}_2$, and $\mathbf{y} = c\mathbf{e}_1 + d\mathbf{e}_2$, where we have

$$\begin{aligned}\mathbf{x}\mathbf{y} &= \mathbf{x}\cdot\mathbf{y} + \mathbf{x}\wedge\mathbf{y} \\ &= \left(a\mathbf{e}_1 + b\mathbf{e}_2\right)\left(c\mathbf{e}_1 + d\mathbf{e}_2\right) \\ &= ac + bd + \mathbf{e}_1\mathbf{e}_2\left(ad - bc\right).\end{aligned} \tag{1.16}$$

So we have

$$\begin{aligned} ab + cd &= \mathbf{x} \cdot \mathbf{y} = \mathrm{Re}\left(zw^*\right) \\ ad - bc &= \left(\mathbf{x} \wedge \mathbf{y}\right) \mathbf{e}_{12}^{-1} = -\,\mathrm{Im}\left(zw^*\right). \end{aligned} \tag{1.17}$$

We see that $\left(zw^*\right)^* = z^* w$ can be used as a representation of the geometric product (setting $i = \mathbf{e}_1 \mathbf{e}_2$ as usual.)

## 1.5 Another simplification.

We have sums of the form

$$\mathrm{Re}(z)w \pm \mathrm{Im}(z)iw \tag{1.18}$$

above. Let's see if those can be simplified. For the positive case we have

$$\begin{aligned} \mathrm{Re}(z)w + \mathrm{Im}(z)iw &= \frac{1}{2}\left(z + z^*\right) w + \frac{1}{2}\left(z - z^*\right) w \\ &= zw, \end{aligned} \tag{1.19}$$

and for the negative case, we have

$$\begin{aligned} \mathrm{Re}(z)w - \mathrm{Im}(z)iw &= \frac{1}{2}\left(z + z^*\right) w - \frac{1}{2}\left(z - z^*\right) w \\ &= z^* w. \end{aligned} \tag{1.20}$$

This, with the vector-vector product simplification above, means that we can represent the full multivector product in this representation as just

$$MN \sim \left(m_1 n_1 + m_2^* n_2, m_2 n_1 + m_1^* n_2\right). \tag{1.21}$$