PEETER JOOT
GRADUATE QUANTUM MECHANICS

# GRADUATE QUANTUM MECHANICS PEETER JOOT 

Notes and problems from UofT PHY1520H 2015
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submods=" figures / phy356-qmI figures / phy456-qmII
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    phy1520-quantum julia mathematica latex "
for i in $submods ; do
    git submodule update --init $i
    (cd $i && git checkout master)
done
export PATH=`pwd`/latex/bin:$PATH
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make
```

I reserve the right to impose dictatorial control over any editing and content decisions, and may not accept merge requests as-is, or at all. That said, I will probably not refuse reasonable suggestions or merge requests.

Dedicated to:
My awesome siblings, Krista, Erik, Karin and Devin.
I'm so proud of my "little" sisters and brothers, who are each off conquering their part of the world in their own ways.

I love you all!

## PREFACE

This book contains notes for the Winter 2015 session of the University of Toronto Graduate Quantum Mechanics course (PHY1520H), taught by Prof. Arun Paramekanti.

Course Syllabus This course will discuss the following topics in quantum mechanics (time permitting)

1. Basics - Postulates, Wavefunctions, Density matrices, Measurements
2. Time evolution - Schrodinger picture, Heisenberg picture, Interaction picture
3. Harmonic oscillator - Operator method, Wavefunctions, Coherent states
4. Particle in a magnetic field - Local gauge invariance, 2D Landau levels
5. Symmetries - Parity, Translations, Rotations, Time-reversal
6. Angular momentum, Spin, and Angular momentum addition
7. Time-independent perturbation theory
8. Time-dependent perturbation theory
9. Variation approach
10. Scattering theory
11. Dirac equation - one dimension
12. Path integrals

## This document contains:

- Lecture notes.
- Personal notes exploring auxiliary details.
- Worked practice problems.
- Assigned problems.
- Links to Mathematica notebooks associated with problems and course material.

My thanks go to Professor Paramekanti for teaching this course, to Nishant Bhatt for providing me with a copy his notes for lecture 18 , which are incorporated herein, and to Hana and Marcos Basso for finding errors.

Peeter Joot peeterjoot@pm.me

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### 1.1 CLASSICAL MECHANICS.

We'll be talking about one body physics for most of this course. In classical mechanics we can figure out the particle trajectories using both of ( $\mathbf{r}, \mathbf{p}$, where

$$
\begin{align*}
\frac{d \mathbf{r}}{d t} & =\frac{1}{m} \mathbf{p}  \tag{1.1}\\
\frac{d \mathbf{p}}{d t} & =\nabla V
\end{align*}
$$

A two dimensional phase space as sketched in fig. 1.1 shows the trajectory of a point particle subject to some equations of motion


Figure 1.1: One dimensional classical phase space example.

### 1.2 QUANTUM MECHANICS.

For this lecture, we'll work with natural units, setting

$$
\begin{equation*}
\hbar=1 \tag{1.2}
\end{equation*}
$$

In QM we are no longer allowed to think of position and momentum, but have to start asking about state vectors $|\Psi\rangle$.

We'll consider the state vector with respect to some basis, for example, in a position basis, we write

$$
\begin{equation*}
\langle x \mid \Psi\rangle=\Psi(x), \tag{1.3}
\end{equation*}
$$

a complex numbered "wave function", the probability amplitude for a particle in $|\Psi\rangle$ to be in the vicinity of $x$.

We could also consider the state in a momentum basis

$$
\begin{equation*}
\langle p \mid \Psi\rangle=\Psi(p), \tag{1.4}
\end{equation*}
$$

a probability amplitude with respect to momentum $p$. More precisely,

$$
\begin{equation*}
|\Psi(x)|^{2} d x \geq 0, \tag{1.5}
\end{equation*}
$$

is the probability of finding the particle in the range $(x, x+d x)$. To have meaning as a probability, we require

$$
\begin{equation*}
\int_{-\infty}^{\infty}|\Psi(x)|^{2} d x=1 \tag{1.6}
\end{equation*}
$$

The average position can be calculated using this probability density function. For example

$$
\begin{equation*}
\langle x\rangle=\int_{-\infty}^{\infty}|\Psi(x)|^{2} x d x, \tag{1.7}
\end{equation*}
$$

or

$$
\begin{equation*}
\langle f(x)\rangle=\int_{-\infty}^{\infty}|\Psi(x)|^{2} f(x) d x . \tag{1.8}
\end{equation*}
$$

Similarly, calculation of an average of a function of momentum can be expressed as

$$
\begin{equation*}
\langle f(p)\rangle=\int_{-\infty}^{\infty}|\Psi(p)|^{2} f(p) d p \tag{1.9}
\end{equation*}
$$

### 1.3 TRANSFORMATION FROM A POSITION TO MOMENTUM BASIS.

We have a problem, if we which to compute an average in momentum space such as $\langle p\rangle$, when given a wavefunction $\Psi(x)$.

How do we convert

$$
\begin{equation*}
\Psi(p) \stackrel{?}{\leftrightarrow} \Psi(x) \tag{1.10}
\end{equation*}
$$

or equivalently

$$
\begin{equation*}
\langle p \mid \Psi\rangle \stackrel{?}{\leftrightarrow}\langle x \mid \Psi\rangle . \tag{1.11}
\end{equation*}
$$

Such a conversion can be performed by virtue of an the assumption that we have a complete orthonormal basis, for which we can introduce identity operations such as

$$
\begin{equation*}
\int_{-\infty}^{\infty} d p|p\rangle\langle p|=1 \tag{1.12}
\end{equation*}
$$

or

$$
\begin{equation*}
\int_{-\infty}^{\infty} d x|x\rangle\langle x|=1 \tag{1.13}
\end{equation*}
$$

Some interpretations:

1. $\left|x_{0}\right\rangle \leftrightarrow$ sits at $x=x_{0}$
2. $\left\langle x \mid x^{\prime}\right\rangle \leftrightarrow \delta\left(x-x^{\prime}\right)$
3. $\left\langle p \mid p^{\prime}\right\rangle \leftrightarrow \delta\left(p-p^{\prime}\right)$
4. $\left\langle x \mid p^{\prime}\right\rangle=\frac{e^{i p x}}{\sqrt{V}}$, where $V$ is the volume of the box containing the particle. We'll define the appropriate normalization for an infinite box volume later.

The delta function interpretation of the braket $\left\langle p \mid p^{\prime}\right\rangle$ justifies the identity operator, since we recover any state in the basis when operating with it. For example, in momentum space

$$
\begin{align*}
1|p\rangle & =\left(\int_{-\infty}^{\infty} d p^{\prime}\left|p^{\prime}\right\rangle\left\langle p^{\prime}\right|\right)|p\rangle \\
& =\int_{-\infty}^{\infty} d p^{\prime}\left|p^{\prime}\right\rangle\left\langle p^{\prime} \mid p\right\rangle  \tag{1.14}\\
& =\int_{-\infty}^{\infty} d p^{\prime}\left|p^{\prime}\right\rangle \delta\left(p-p^{\prime}\right) \\
& =|p\rangle
\end{align*}
$$

This also the determination of an integral operator representation for the delta function

$$
\begin{align*}
\delta\left(x-x^{\prime}\right) & =\left\langle x \mid x^{\prime}\right\rangle \\
& =\int d p\langle x \mid p\rangle\left\langle p \mid x^{\prime}\right\rangle  \tag{1.15}\\
& =\frac{1}{V} \int d p e^{i p x} e^{-i p x^{\prime}}
\end{align*}
$$

or

$$
\begin{equation*}
\delta\left(x-x^{\prime}\right)=\frac{1}{V} \int d p e^{i p\left(x-x^{\prime}\right)} \tag{1.16}
\end{equation*}
$$

Here we used the fact that $\langle p \mid x\rangle=\langle x \mid p\rangle^{*}$.
FIXME: do we have a justification for that conjugation with what was defined here so far?

The conversion from a position basis to momentum space is now possible

$$
\begin{align*}
\langle p \mid \Psi\rangle & =\Psi(p) \\
& =\int_{-\infty}^{\infty}\langle p \mid x\rangle\langle x \mid \Psi\rangle d x  \tag{1.17}\\
& =\int_{-\infty}^{\infty} \frac{e^{-i p x}}{\sqrt{V}} \Psi(x) d x
\end{align*}
$$

The momentum space to position space conversion can be written as

$$
\begin{equation*}
\Psi(x)=\int_{-\infty}^{\infty} \frac{e^{i p x}}{\sqrt{V}} \Psi(p) d p \tag{1.18}
\end{equation*}
$$

Now we can go back and figure out the an expectation

$$
\begin{align*}
\langle p\rangle & =\int \Psi^{*}(p) \Psi(p) p d p \\
& =\int d p\left(\int_{-\infty}^{\infty} \frac{e^{i p x}}{\sqrt{V}} \Psi^{*}(x) d x\right)\left(\int_{-\infty}^{\infty} \frac{e^{-i p x^{\prime}}}{\sqrt{V}} \Psi\left(x^{\prime}\right) d x^{\prime}\right) p \\
& =\int d p d x d x^{\prime} \Psi^{*}(x) \frac{1}{V} e^{i p\left(x-x^{\prime}\right)} \Psi\left(x^{\prime}\right) p \\
& =\int d p d x d x^{\prime} \Psi^{*}(x) \frac{1}{V}\left(-i \frac{\partial e^{i p\left(x-x^{\prime}\right)}}{\partial x}\right) \Psi\left(x^{\prime}\right)  \tag{1.19}\\
& =\int d p d x \Psi^{*}(x)\left(-i \frac{\partial}{\partial x}\right) \frac{1}{V} \int d x^{\prime} e^{i p\left(x-x^{\prime}\right)} \Psi\left(x^{\prime}\right) \\
& =\int d x \Psi^{*}(x)\left(-i \frac{\partial}{\partial x}\right) \int d x^{\prime}\left(\frac{1}{V} \int d p e^{i p\left(x-x^{\prime}\right)}\right) \Psi\left(x^{\prime}\right) \\
& =\int d x \Psi^{*}(x)\left(-i \frac{\partial}{\partial x}\right) \int d x^{\prime} \delta\left(x-x^{\prime}\right) \Psi\left(x^{\prime}\right) \\
& =\int d x \Psi^{*}(x)\left(-i \frac{\partial}{\partial x}\right) \Psi(x) .
\end{align*}
$$

Here we've essentially calculated the position space representation of the momentum operator, allowing identifications of the following form

$$
\begin{align*}
& p \leftrightarrow-i \frac{\partial}{\partial x}  \tag{1.20}\\
& p^{2} \leftrightarrow-\frac{\partial^{2}}{\partial x^{2}} . \tag{1.21}
\end{align*}
$$

Alternate starting point. Most of the above results followed from the claim that $\langle x \mid p\rangle=e^{i p x}$. Note that this position space representation of the momentum operator can also be taken as the starting point. Given that, the exponential representation of the position-momentum braket follows

$$
\begin{equation*}
\langle x| P|p\rangle=-i \hbar \frac{\partial}{\partial x}\langle x \mid p\rangle, \tag{1.2}
\end{equation*}
$$

but $\langle x| P|p\rangle=p\langle x \mid p\rangle$, providing a differential equation for $\langle x \mid p\rangle$

$$
\begin{equation*}
p\langle x \mid p\rangle=-i \hbar \frac{\partial}{\partial x}\langle x \mid p\rangle, \tag{1.23}
\end{equation*}
$$

with solution

$$
\begin{equation*}
i p x / \hbar=\ln \langle x \mid p\rangle+\text { const }, \tag{1.24}
\end{equation*}
$$

or

$$
\begin{equation*}
\langle x \mid p\rangle \propto e^{i p x / \hbar} \tag{1.25}
\end{equation*}
$$

### 1.4 MATRIX INTERPRETATION.

1. Ket's $|\Psi\rangle \leftrightarrow$ column vector
2. Bra's $\langle\Psi| \leftrightarrow$ (row vector) $^{*}$
3. Operators $\leftrightarrow$ matrices that act on vectors.

$$
\begin{equation*}
\hat{p}|\Psi\rangle \rightarrow\left|\Psi^{\prime}\right\rangle \tag{1.26}
\end{equation*}
$$

### 1.5 TIME EVOLUTION.

For a state subject to the equations of motion given by the Hamiltonian operator $\hat{H}$

$$
\begin{equation*}
i \frac{\partial}{\partial t}|\Psi\rangle=\hat{H}|\Psi\rangle \tag{1.27}
\end{equation*}
$$

the time evolution is given by

$$
\begin{equation*}
|\Psi(t)\rangle=e^{-i \hat{H} t}|\Psi(0)\rangle \tag{1.28}
\end{equation*}
$$

### 1.6 REVIEW: BASIC CONCEPTS.

We've reviewed the basic concepts that we will encounter in Quantum Mechanics.

1. Abstract state vector. $|\psi\rangle$
2. Basis states. $|x\rangle$
3. Observables, special Hermitian operators. We'll only deal with linear observables.
4. Measurement.

We can either express the wave functions $\psi(x)=\langle x \mid \psi\rangle$ in terms of a basis for the observable, or can express the observable in terms of the basis of the wave function (position or momentum for example).

We saw that the position space representation of a momentum operator (also an observable) was

$$
\begin{equation*}
\hat{p} \rightarrow-i \hbar \frac{\partial}{\partial x} \tag{1.29}
\end{equation*}
$$

In general we can find the matrix element representation of any operator by considering its representation in a given basis. For example, in a position basis, that would be

$$
\begin{equation*}
\left\langle x^{\prime}\right| \hat{A}|x\rangle \leftrightarrow A_{x x^{\prime}} \tag{1.30}
\end{equation*}
$$

The Hermitian property of the observable means that $A_{x x^{\prime}}=A_{x^{\prime} x}^{*}$

$$
\begin{align*}
\int d x\left\langle x^{\prime}\right| \hat{A}|x\rangle\langle x \mid \psi\rangle & =\left\langle x^{\prime} \mid \phi\right\rangle \\
& \leftrightarrow A_{x^{\prime} x} \psi_{x}  \tag{1.31}\\
& =\phi_{x^{\prime}}
\end{align*}
$$

## Example 1.1: Measurement example

Consider a polarization apparatus as sketched in fig. 1.2, where the output is of the form $I_{\text {out }}=I_{\text {in }} \cos ^{2} \theta$.


Figure 1.2: Polarizer apparatus.

A general input state can be written in terms of each of the possible polarizations

$$
\begin{equation*}
\alpha|\mathfrak{\downarrow}\rangle+\beta|\leftrightarrow\rangle \sim \cos \theta|\downarrow\rangle+\sin \theta|\leftrightarrow\rangle . \tag{1.32}
\end{equation*}
$$

Here $|\alpha|^{2}$ is the probability that the input state is in the upwards polarization state, and $|\beta|^{2}$ is the probability that the input state is in the downwards polarization state.

The measurement of the polarization results in an output state that has a specific polarization. That measurement is said to collapse the wavefunction.

When attempting a measurement, looking for a specific value, effects the state of the system, and is call a strong or projective measurement. Such a measurement is
(i) Probabilistic.
(ii) Requires many measurements.

This measurement process results a determination of the eigenvalue of the operator. The eigenvalue production of measurement is why we demand that operators be Hermitian.

It is also possible to try to do a weaker (perturbative) measurement, where some information is extracted from the input state without completely altering it.

## Time evolution

1. Schrödinger picture. The time evolution process is governed by a Schrödinger equation of the following form

$$
\begin{equation*}
i \hbar \frac{\partial}{\partial t}|\Psi(t)\rangle=\hat{H}|\Psi(t)\rangle . \tag{1.33}
\end{equation*}
$$

This Hamiltonian could be, for example,

$$
\begin{equation*}
\hat{H}=\frac{\hat{p}^{2}}{2 m}+V(x), \tag{1.34}
\end{equation*}
$$

Such a representation of time evolution is expressed in terms of operators $\hat{x}, \hat{p}, \hat{H}, \cdots$ that are independent of time.
2. Heisenberg picture.

Suppose we have a state $|\Psi(t)\rangle$ and operate on this with an operator

$$
\begin{equation*}
\hat{A}|\Psi(t)\rangle \tag{1.35}
\end{equation*}
$$

This will have time evolution of the form

$$
\begin{equation*}
\hat{A} e^{-i \hat{H} t / \hbar}|\Psi(0)\rangle, \tag{1.36}
\end{equation*}
$$

or in matrix element form

$$
\begin{equation*}
\langle\phi(t)| \hat{A}|\Psi(t)\rangle=\langle\phi(0)| e^{i \hat{H} t / \hbar} \hat{A} e^{-i \hat{H} t / \hbar}|\Psi(0)\rangle \tag{1.37}
\end{equation*}
$$

We work with states that do not evolve in time $|\phi(0)\rangle,|\Psi(0)\rangle, \cdots$, but operators do evolve in time according to

$$
\begin{equation*}
\hat{A}(t)=e^{i \hat{H} t / \hbar} \hat{A} e^{-i \hat{H} t / \hbar} \tag{1.38}
\end{equation*}
$$

Density operator We can have situations where it is impossible to determine a single state that describes the system. For example, given the gas in the room that you are sitting in, there are things that we can measure, but it is impossible to describe the state that describes all the particles and also impossible to construct a Hamiltonian that governs all the interactions of those many particles.

We need a probabilistic description to even describe such a complex system, and to be able to deal with concepts like entanglement.

Suppose we have a complex system that can be partitioned into two subsets, left and right, as sketched in fig. 1.3.


Figure 1.3: System partitioned into separate set of states.

If the states in each partition can be enumerated separately, we can write the state of the system as sums over the probability amplitudes that for the combined states.

$$
\begin{equation*}
|\Psi\rangle=\sum_{m, n} C_{m, n}|m\rangle|n\rangle . \tag{1.39}
\end{equation*}
$$

Here $C_{m, n}$ is the probability amplitude to find the state in the combined state $|m\rangle|n\rangle$.

As an example of such a system, we could investigate a two particle configuration where spin up or spin down can be separately measured for each particle.

$$
\begin{equation*}
|\psi\rangle=\frac{1}{\sqrt{2}}(|\uparrow\rangle|\downarrow\rangle+|\downarrow\rangle|\uparrow\rangle) . \tag{1.40}
\end{equation*}
$$

Considering such a system we could ask questions such as

- What is the probability that the left half is in state $m$ ? This would be

$$
\begin{equation*}
\sum_{n}\left|C_{m, n}\right|^{2} \tag{1.41}
\end{equation*}
$$

- Probability that the left half is in state $m$, and the probability that the right half is in state $n$ ? That is

$$
\begin{equation*}
\left|C_{m, n}\right|^{2} \tag{1.42}
\end{equation*}
$$

We define the density operator

$$
\begin{equation*}
\hat{\rho}=|\Psi\rangle\langle\Psi| \tag{1.43}
\end{equation*}
$$

This is idempotent

$$
\begin{align*}
\hat{\rho}^{2} & =(|\Psi\rangle\langle\Psi|)(|\Psi\rangle\langle\Psi|)  \tag{1.44}\\
& =|\Psi\rangle\langle\Psi|
\end{align*}
$$

An example of a partitioned system with four total states (two spin 1/2 particles) is sketched in fig. 1.4.

An example of a partitioned system with eight total states (three spin $1 / 2$ particles) is sketched in fig. 1.5.


Figure 1.4: Two spins.


Figure 1.5: Three spins.

The density matrix

$$
\begin{equation*}
\hat{\rho}=|\Psi\rangle\langle\Psi| \tag{1.45}
\end{equation*}
$$

is clearly an operator as can be seen by applying it to a state

$$
\begin{equation*}
\hat{\rho}|\phi\rangle=|\Psi\rangle(\langle\Psi \mid \phi\rangle) . \tag{1.46}
\end{equation*}
$$

The quantity in braces is just a complex number.
After expanding the pure state $|\Psi\rangle$ in terms of basis states for each of the two partitions

$$
\begin{equation*}
|\Psi\rangle=\sum_{m, n} C_{m, n}|m\rangle_{\mathrm{L}}|n\rangle_{\mathrm{R}} \tag{1.47}
\end{equation*}
$$

With L and R implied for $|m\rangle,|n\rangle$ indexed states respectively, this can be written

$$
\begin{equation*}
|\Psi\rangle=\sum_{m, n} C_{m, n}|m\rangle|n\rangle \tag{1.48}
\end{equation*}
$$

The density operator is

$$
\begin{equation*}
\hat{\rho}=\sum_{m, n} C_{m, n} C_{m^{\prime}, n^{\prime}}^{*}|m\rangle|n\rangle \sum_{m^{\prime}, n^{\prime}}\left\langle m^{\prime}\right|\left\langle n^{\prime}\right| . \tag{1.49}
\end{equation*}
$$

Suppose we trace over the right partition of the state space, defining such a trace as the reduced density operator $\hat{\rho}_{\text {red }}$

$$
\begin{align*}
\hat{\rho}_{\mathrm{red}} \equiv \operatorname{tr}_{\mathrm{R}}(\hat{\rho}) & =\sum_{\tilde{n}}\langle\tilde{n}| \hat{\rho}|\tilde{n}\rangle \\
& =\sum_{\tilde{n}}\langle\tilde{n}|\left(\sum_{m, n} C_{m, n}|m\rangle|n\rangle\right)\left(\sum_{m^{\prime}, n^{\prime}} C_{m^{\prime}, n^{\prime}}^{*}\left\langle m^{\prime}\right|\left\langle n^{\prime}\right|\right)|\tilde{n}\rangle  \tag{1.50}\\
& =\sum_{\tilde{n}} \sum_{m, n} \sum_{m^{\prime}, n^{\prime}} C_{m, n} C_{m^{\prime}, n^{\prime}}^{*}|m\rangle \delta_{\tilde{n} n}\left\langle m^{\prime}\right| \delta_{\tilde{n} n^{\prime}} \\
& =\sum_{\tilde{n}, m, m^{\prime}} C_{m, \tilde{n}} C_{m^{\prime}, \tilde{n}}^{*}|m\rangle\left\langle m^{\prime}\right|
\end{align*}
$$

Computing the matrix element of $\hat{\rho}_{\text {red }}$, we have

$$
\begin{align*}
\langle\tilde{m}| \hat{\rho}_{\mathrm{red}}|\tilde{m}\rangle & =\sum_{m, m^{\prime}, \tilde{n}} C_{m, \tilde{n}} C_{m^{\prime}, \tilde{n}}^{*}\langle\tilde{m} \mid m\rangle\left\langle m^{\prime} \mid \tilde{m}\right\rangle  \tag{1.51}\\
& =\sum_{\tilde{n}}\left|C_{\tilde{m}, \tilde{n}}\right|^{2}
\end{align*}
$$

This is the probability that the left partition is in state $\tilde{m}$.

### 1.7 AVERAGE OF AN OBSERVABLE.

Suppose we have two spin half particles. For such a system the total magnetization is

$$
\begin{equation*}
S_{\text {Total }}=S_{1}^{z}+S_{1}^{z} \tag{1.52}
\end{equation*}
$$

as sketched in fig. 1.6.


Figure 1.6: Magnetic moments from two spins.

The average of some observable is

$$
\begin{equation*}
\langle\hat{A}\rangle=\sum_{m, n, m^{\prime}, n^{\prime}} C_{m, n}^{*} C_{m^{\prime}, n^{\prime}}\langle m|\langle n| \hat{A}\left|n^{\prime}\right\rangle\left|m^{\prime}\right\rangle \tag{1.53}
\end{equation*}
$$

Consider the trace of the density operator observable product

$$
\begin{equation*}
\operatorname{tr}(\hat{\rho} \hat{A})=\sum_{m, n}\langle m n \mid \Psi\rangle\langle\Psi| \hat{A}|m, n\rangle \tag{1.54}
\end{equation*}
$$

Let

$$
\begin{equation*}
|\Psi\rangle=\sum_{m, n} C_{m n}|m, n\rangle \tag{1.55}
\end{equation*}
$$

so that

$$
\begin{align*}
\operatorname{tr}(\hat{\rho} \hat{A}) & =\sum_{m, n, m^{\prime}, n^{\prime}, m^{\prime \prime}, n^{\prime \prime}} C_{m^{\prime}, n^{\prime}} C_{m^{\prime \prime}, n^{\prime \prime}}^{*}\left\langle m n \mid m^{\prime}, n^{\prime}\right\rangle\left\langle m^{\prime \prime}, n^{\prime \prime}\right| \hat{A}|m, n\rangle  \tag{1.56}\\
& =\sum_{m, n, m^{\prime \prime}, n^{\prime \prime}} C_{m, n} C_{m^{\prime \prime}, n^{\prime \prime}}^{*}\left\langle m^{\prime \prime}, n^{\prime \prime}\right| \hat{A}|m, n\rangle
\end{align*}
$$

This is just

$$
\begin{equation*}
\langle\Psi| \hat{A}|\Psi\rangle=\operatorname{tr}(\hat{\rho} \hat{A}) \tag{1.57}
\end{equation*}
$$

### 1.8 LEFT OBSERVABLES.

Consider

$$
\begin{align*}
\langle\Psi| \hat{A}_{\mathrm{L}}|\Psi\rangle & =\operatorname{tr}\left(\hat{\rho} \hat{A}_{\mathrm{L}}\right) \\
& =\operatorname{tr}_{\mathrm{L}} \operatorname{tr}_{\mathrm{R}}\left(\hat{\rho} \hat{A}_{\mathrm{L}}\right) \\
& \left.=\operatorname{tr}_{\mathrm{L}}\left(\left(\operatorname{tr}_{\mathrm{R}} \hat{\rho}\right) \hat{A}_{\mathrm{L}}\right)\right)  \tag{1.58}\\
& \left.=\operatorname{tr}_{\mathrm{L}}\left(\hat{\rho}_{\mathrm{red}} \hat{A}_{\mathrm{L}}\right)\right) .
\end{align*}
$$

We see

$$
\begin{equation*}
\langle\Psi| \hat{A}_{\mathrm{L}}|\Psi\rangle=\operatorname{tr}_{\mathrm{L}}\left(\hat{\rho}_{\mathrm{red}, \mathrm{~L}} \hat{A}_{\mathrm{L}}\right) \tag{1.59}
\end{equation*}
$$

We find that we don't need to know the state of the complete system to answer questions about portions of the system, but instead just need $\hat{\rho}$, a "probability operator" that provides all the required information about the partitioning of the system.

### 1.9 PURE States vs. MiXed states.

For pure states we can assign a state vector and talk about reduced scenarios. For mixed states we must work with reduced density matrices.

## Example 1.2: Two particle spin half pure states

Consider

$$
\begin{align*}
& \left|\psi_{1}\right\rangle=\frac{1}{\sqrt{2}}(|\uparrow \downarrow\rangle-|\downarrow \uparrow\rangle)  \tag{1.60}\\
& \left|\psi_{2}\right\rangle=\frac{1}{\sqrt{2}}(|\uparrow \downarrow\rangle+|\uparrow \uparrow\rangle) . \tag{1.61}
\end{align*}
$$

For the first pure state the density operator is

$$
\begin{equation*}
\hat{\rho}=\frac{1}{2}(|\uparrow \downarrow\rangle-|\downarrow \uparrow\rangle)(\langle\uparrow \downarrow|-\langle\downarrow \uparrow|) \tag{1.62}
\end{equation*}
$$

What are the reduced density matrices?

$$
\begin{align*}
\hat{\rho}_{\mathrm{L}} & =\operatorname{tr}_{\mathrm{R}}(\hat{\rho}) \\
& =\frac{1}{2}(-1)(-1)|\downarrow\rangle\langle\downarrow|+\frac{1}{2}(+1)(+1)|\uparrow\rangle\langle\uparrow|, \tag{1.63}
\end{align*}
$$

so the matrix representation of this reduced density operator is

$$
\hat{\rho}_{\mathrm{L}}=\frac{1}{2}\left[\begin{array}{ll}
1 & 0  \tag{1.64}\\
0 & 1
\end{array}\right] .
$$

For the second pure state the density operator is

$$
\begin{equation*}
\hat{\rho}=\frac{1}{2}(|\uparrow \downarrow\rangle+|\uparrow \uparrow\rangle)(\langle\uparrow \downarrow|+\langle\uparrow \uparrow|) . \tag{1.65}
\end{equation*}
$$

This has a reduced density matrix

$$
\begin{align*}
\hat{\rho}_{\mathrm{L}} & =\operatorname{tr}_{\mathrm{R}}(\hat{\rho}) \\
& =\frac{1}{2}|\uparrow\rangle\langle\uparrow|+\frac{1}{2}|\uparrow\rangle\langle\uparrow|  \tag{1.66}\\
& =|\uparrow\rangle\langle\uparrow| .
\end{align*}
$$

This has a matrix representation

$$
\hat{\rho}_{\mathrm{L}}=\left[\begin{array}{ll}
1 & 0  \tag{1.67}\\
0 & 0
\end{array}\right] .
$$

In this second example, we have more information about the left partition. That will be seen as a zero entanglement entropy in the problem set. In contrast we have less information about the first state, and will find a non-zero positive entanglement entropy in that case.

### 1.10 ENTROPY WHEN DENSITY OPERATOR HAS ZERO EIGENVALUES.

In the class notes and the text [15] the Von Neumann entropy is defined as

$$
\begin{equation*}
S=-\operatorname{tr}(\rho \ln \rho) . \tag{1.68}
\end{equation*}
$$

In one of our problems I had trouble evaluating this, having calculated a density operator matrix representation

$$
\begin{equation*}
\rho=E \wedge E^{-1} \tag{1.69}
\end{equation*}
$$

where

$$
E=\frac{1}{\sqrt{2}}\left[\begin{array}{cc}
1 & 1  \tag{1.70}\\
1 & -1
\end{array}\right]
$$

and

$$
\wedge=\left[\begin{array}{ll}
1 & 0  \tag{1.71}\\
0 & 0
\end{array}\right]
$$

The usual method of evaluating a function of a matrix is to assume the function has a power series representation, and that a similarity transformation of the form $A=E \wedge E^{-1}$ is possible, so that

$$
\begin{equation*}
f(A)=E f(\wedge) E^{-1} \tag{1.72}
\end{equation*}
$$

however, when attempting to do this with the matrix of eq. (1.69) leads to an undesirable result

$$
\ln \rho=\frac{1}{2}\left[\begin{array}{cc}
1 & 1  \tag{1.73}\\
1 & -1
\end{array}\right]\left[\begin{array}{cc}
\ln 1 & 0 \\
0 & \ln 0
\end{array}\right]\left[\begin{array}{cc}
1 & 1 \\
1 & -1
\end{array}\right] .
$$

The $\ln 0$ makes the evaluation of this matrix logarithm rather unpleasant. To give meaning to the entropy expression, we have to do two things, the first is treating the trace operation as a higher precedence than the logarithms that it contains. That is

$$
\begin{align*}
-\operatorname{tr}(\rho \ln \rho) & =-\operatorname{tr}\left(E \wedge E^{-1} E \ln \wedge E^{-1}\right) \\
& =-\operatorname{tr}\left(E \wedge \ln \wedge E^{-1}\right) \\
& =-\operatorname{tr}\left(E^{-1} E \wedge \ln \wedge\right)  \tag{1.74}\\
& =-\operatorname{tr}(\wedge \ln \wedge) \\
& =-\sum_{k} \wedge_{k k} \ln \wedge_{k k}
\end{align*}
$$

Now the matrix of the logarithm need not be evaluated, but we still need to give meaning to $\wedge_{k k} \ln \wedge_{k k}$ for zero diagonal entries. This can be done by considering a limiting scenario

$$
\begin{align*}
-\lim _{a \rightarrow 0} a \ln a & =-\lim _{x \rightarrow \infty} e^{-x} \ln e^{-x} \\
& =\lim _{x \rightarrow \infty} x e^{-x}  \tag{1.75}\\
& =0 .
\end{align*}
$$

The entropy can now be expressed in the unambiguous form, summing over all the non-zero eigenvalues of the density operator

$$
\begin{equation*}
S=-\sum_{\wedge_{k k} \neq 0} \wedge_{k k} \ln \wedge_{k k} . \tag{1.76}
\end{equation*}
$$

### 1.11 Problems.

## Exercise 1.1 Pauli and $2 \times 2$ matrixes. ([15] pr. 1.2)

Given an arbitrary $2 \times 2$ matrix $X=a_{0}+\boldsymbol{\sigma} \cdot \mathbf{a}$, show the relationships between $a_{\mu}$ and $\operatorname{tr}(X), \operatorname{tr}\left(\sigma_{k} X\right)$, and $X_{i j}$.
Answer for Exercise 1.1
Observe that each of the Pauli matrices $\sigma_{k}$ are traceless

$$
\begin{align*}
& \sigma_{x}=\left[\begin{array}{ll}
0 & 1 \\
1 & 0
\end{array}\right] \\
& \sigma_{y}=\left[\begin{array}{cc}
0 & -i \\
i & 0
\end{array}\right],  \tag{1.77}\\
& \sigma_{z}=\left[\begin{array}{cc}
1 & 0 \\
0 & -1
\end{array}\right]
\end{align*}
$$

so $\operatorname{tr}(X)=2 a_{0}$. Note that $\operatorname{tr}\left(\sigma_{k} \sigma_{m}\right)=2 \delta_{k m}$, so $\operatorname{tr}\left(\sigma_{k} X\right)=2 a_{k}$.
Notationally, it would seem to make sense to define $\sigma_{0} \equiv I$, so that $\operatorname{tr}\left(\sigma_{\mu} X\right)=a_{\mu}$. I don't know if that is common practice.

For the opposite relations, given

$$
\begin{aligned}
X & =a_{0}+\boldsymbol{\sigma} \cdot \mathbf{a} \\
& =\left[\begin{array}{ll}
1 & 0 \\
0 & 1
\end{array}\right] a_{0}+\left[\begin{array}{ll}
0 & 1 \\
1 & 0
\end{array}\right] a_{1}+\left[\begin{array}{cc}
0 & -i \\
i & 0
\end{array}\right] a_{2}+\left[\begin{array}{cc}
1 & 0 \\
0 & -1
\end{array}\right] a_{3} \\
& =\left[\begin{array}{ll}
a_{0}+a_{3} & a_{1}-i a_{2} \\
a_{1}+i a_{2} & a_{0}-a_{3}
\end{array}\right] \\
& =\left[\begin{array}{ll}
X_{11} & X_{12} \\
X_{21} & X_{22}
\end{array}\right],
\end{aligned}
$$

so

$$
\begin{align*}
& a_{0}=\frac{1}{2}\left(X_{11}+X_{22}\right) \\
& a_{1}=\frac{1}{2}\left(X_{12}+X_{21}\right) \\
& a_{2}=\frac{1}{2 i}\left(X_{21}-X_{12}\right)  \tag{1.79}\\
& a_{3}=\frac{1}{2}\left(X_{11}-X_{22}\right) .
\end{align*}
$$

## Exercise 1.2 Rotation transformation. ([15] pr. 1.3)

Determine the structure and determinant of the transformation

$$
\begin{equation*}
\boldsymbol{\sigma} \cdot \mathbf{a} \rightarrow \boldsymbol{\sigma} \cdot \mathbf{a}^{\prime}=\exp (i \boldsymbol{\sigma} \cdot \hat{\mathbf{n}} \phi / 2) \boldsymbol{\sigma} \cdot \mathbf{a} \exp (-i \boldsymbol{\sigma} \cdot \hat{\mathbf{n}} \phi / 2) \tag{1.80}
\end{equation*}
$$

Answer for Exercise 1.2
Knowing Geometric Algebra, this is recognized as a rotation transformation. In GA, $i$ is treated as a pseudoscalar (which commutes with all grades in $\mathbb{R}^{3}$ ), and the expression can be reduced to one involving dot and wedge products. Let's see how can this be reduced using only the Pauli matrix toolbox.

First, consider the determinant of one of the exponentials. Showing that one such exponential has unit determinant is sufficient. The matrix representation of the unit normal is

$$
\begin{align*}
\boldsymbol{\sigma} \cdot \hat{\mathbf{n}} & =n_{x}\left[\begin{array}{ll}
0 & 1 \\
1 & 0
\end{array}\right]+n_{y}\left[\begin{array}{cc}
0 & -i \\
i & 0
\end{array}\right]+n_{z}\left[\begin{array}{cc}
1 & 0 \\
0 & -1
\end{array}\right] \\
& =\left[\begin{array}{cc}
n_{z} & n_{x}-i n_{y} \\
n_{x}+i n_{y} & -n_{z}
\end{array}\right] . \tag{1.81}
\end{align*}
$$

This is expected to have a unit square, and does

$$
\begin{align*}
(\boldsymbol{\sigma} \cdot \hat{\mathbf{n}})^{2} & =\left[\begin{array}{cc}
n_{z} & n_{x}-i n_{y} \\
n_{x}+i n_{y} & -n_{z}
\end{array}\right]\left[\begin{array}{cc}
n_{z} & n_{x}-i n_{y} \\
n_{x}+i n_{y} & -n_{z}
\end{array}\right] \\
& =\left(n_{x}^{2}+n_{y}^{2}+n_{z}^{2}\right)\left[\begin{array}{ll}
1 & 0 \\
0 & 1
\end{array}\right]  \tag{1.82}\\
& =1
\end{align*}
$$

This allows for a cosine and sine expansion of the exponential, as in

$$
\begin{align*}
\exp (i \boldsymbol{i} \cdot \hat{\mathbf{n}} \theta) & =\cos \theta+i \boldsymbol{\sigma} \cdot \hat{\mathbf{n}} \sin \theta \\
& =\cos \theta\left[\begin{array}{ll}
1 & 0 \\
0 & 1
\end{array}\right]+i \sin \theta\left[\begin{array}{cc}
n_{z} & n_{x}-i n_{y} \\
n_{x}+i n_{y} & -n_{z}
\end{array}\right]  \tag{1.83}\\
& =\left[\begin{array}{cc}
\cos \theta+i n_{z} \sin \theta & \left(n_{x}-i n_{y}\right) i \sin \theta \\
\left(n_{x}+i n_{y}\right) i \sin \theta & \cos \theta-i n_{z} \sin \theta
\end{array}\right]
\end{align*}
$$

This has determinant

$$
\begin{align*}
|\exp (i \sigma \cdot \hat{\mathbf{n}} \theta)| & =\cos ^{2} \theta+n_{z}^{2} \sin ^{2} \theta-\left(-n_{x}^{2}+-n_{y}^{2}\right) \sin ^{2} \theta \\
& =\cos ^{2} \theta+\left(n_{x}^{2}+n_{y}^{2}+n_{z}^{2}\right) \sin ^{2} \theta  \tag{1.84}\\
& =1
\end{align*}
$$

as expected.
Next step is to show that this transformation is a rotation, and determine the sense of the rotation. Let $C=\cos \phi / 2, S=\sin \phi / 2$, so that

$$
\begin{align*}
\sigma \cdot \mathbf{a}^{\prime} & =\exp (i \sigma \cdot \hat{\mathbf{n}} \phi / 2) \sigma \cdot \mathbf{a} \exp (-i \sigma \cdot \hat{\mathbf{n}} \phi / 2) \\
& =(C+i \sigma \cdot \hat{\mathbf{n}} S) \sigma \cdot \mathbf{a}(C-i \sigma \cdot \hat{\mathbf{n}} S) \\
& =(C+i \sigma \cdot \hat{\mathbf{n}} S)(C \sigma \cdot \mathbf{a}-i \sigma \cdot \mathbf{a} \sigma \cdot \hat{\mathbf{n}} S) \\
& =C^{2} \boldsymbol{\sigma} \cdot \mathbf{a}+\boldsymbol{\sigma} \cdot \hat{\mathbf{n}} \boldsymbol{\sigma} \cdot \mathbf{a} \boldsymbol{\sigma} \cdot \hat{\mathbf{n}} S^{2}+i(-\sigma \cdot \mathbf{a} \sigma \cdot \hat{\mathbf{n}}+\sigma \cdot \hat{\mathbf{n}} \sigma \cdot \mathbf{a}) S C \\
& =\frac{1}{2}(1+\cos \phi) \sigma \cdot \mathbf{a}+\sigma \cdot \hat{\mathbf{n}} \sigma \cdot \mathbf{a} \sigma \cdot \hat{\mathbf{n}} \frac{1}{2}(1-\cos \phi)+i[\sigma \cdot \hat{\mathbf{n}}, \sigma \cdot \mathbf{a}] \frac{1}{2} \sin \phi \\
& =\frac{1}{2} \boldsymbol{\sigma} \cdot \hat{\mathbf{n}}\{\sigma \cdot \hat{\mathbf{n}}, \sigma \cdot \mathbf{a}\}+\frac{1}{2} \sigma \cdot \hat{\mathbf{n}}[\sigma \cdot \hat{\mathbf{n}}, \sigma \cdot \mathbf{a}] \cos \phi+\frac{1}{2} i[\sigma \cdot \hat{\mathbf{n}}, \sigma \cdot \mathbf{a}] \sin \phi \tag{1.85}
\end{align*}
$$

Observe that the angle dependent portion can be written in a compact exponential form

$$
\begin{align*}
\sigma \cdot \mathbf{a}^{\prime} & =\frac{1}{2} \sigma \cdot \hat{\mathbf{n}}\{\sigma \cdot \hat{\mathbf{n}}, \sigma \cdot \mathbf{a}\}+(\cos \phi+i \sigma \cdot \hat{\mathbf{n}} \sin \phi) \frac{1}{2} \sigma \cdot \hat{\mathbf{n}}[\sigma \cdot \hat{\mathbf{n}}, \sigma \cdot \mathbf{a}] \\
& =\frac{1}{2} \sigma \cdot \hat{\mathbf{n}}\{\sigma \cdot \hat{\mathbf{n}}, \sigma \cdot \mathbf{a}\}+\exp (i \sigma \cdot \hat{\mathbf{n}} \phi) \frac{1}{2} \sigma \cdot \hat{\mathbf{n}}[\sigma \cdot \hat{\mathbf{n}}, \sigma \cdot \mathbf{a}] \tag{1.86}
\end{align*}
$$

The anticommutator and commutator products with the unit normal can be identified as projections and rejections respectively. Consider the symmetric product first

$$
\begin{aligned}
\frac{1}{2}\{\sigma \cdot \hat{\mathbf{n}}, \boldsymbol{\sigma} \cdot \mathbf{a}\} & =\frac{1}{2} \sum n_{r} a_{s}\left(\sigma_{r} \sigma_{s}+\sigma_{s} \sigma_{r}\right) \\
& =\frac{1}{2} \sum_{r \neq s} n_{r} a_{s}\left(\sigma_{r} \sigma_{s}+\sigma_{s} \sigma_{r}\right)+\frac{1}{2} \sum_{r} n_{r} a_{r} 2 \\
& =2 \hat{\mathbf{n}} \cdot \mathbf{a} .
\end{aligned}
$$

This shows that

$$
\begin{equation*}
\frac{1}{2} \sigma \cdot \hat{\mathbf{n}}\{\sigma \cdot \hat{\mathbf{n}}, \sigma \cdot \mathbf{a}\}=(\hat{\mathbf{n}} \cdot \mathbf{a}) \sigma \cdot \hat{\mathbf{n}}, \tag{1.88}
\end{equation*}
$$

which is the projection of $\mathbf{a}$ in the direction of the normal $\hat{\mathbf{n}}$. To show that the commutator term is the rejection, consider the sum of the two

$$
\begin{align*}
\frac{1}{2} \sigma \cdot \hat{\mathbf{n}}\{\sigma \cdot \hat{\mathbf{n}}, \sigma \cdot \mathbf{a}\}+\frac{1}{2} \sigma \cdot \hat{\mathbf{n}}[\sigma \cdot \hat{\mathbf{n}}, \sigma \cdot \mathbf{a}] & =\sigma \cdot \hat{\mathbf{n}} \sigma \cdot \hat{\mathbf{n}} \sigma \cdot \mathbf{a}  \tag{1.89}\\
& =\sigma \cdot \mathbf{a}
\end{align*}
$$

so we must have

$$
\begin{equation*}
\sigma \cdot \mathbf{a}-(\hat{\mathbf{n}} \cdot \mathbf{a}) \sigma \cdot \hat{\mathbf{n}}=\frac{1}{2} \sigma \cdot \hat{\mathbf{n}}[\sigma \cdot \hat{\mathbf{n}}, \sigma \cdot \mathbf{a}] . \tag{1.90}
\end{equation*}
$$

This is the component of a that has the projection in the $\hat{\mathbf{n}}$ direction removed. Looking back to eq. (1.86), the transformation leaves components of the vector that are colinear with the unit normal unchanged, and applies an exponential operation to the component that lies in what is presumed to be the rotation plane. To verify that this latter portion of the transformation is a rotation, and to determine the sense of the rotation, let's expand the factor of the sine of eq. (1.85).

That is

$$
\begin{align*}
\frac{i}{2}[\boldsymbol{\sigma} \cdot \hat{\mathbf{n}}, \boldsymbol{\sigma} \cdot \mathbf{a}] & =\frac{i}{2} \sum n_{r} a_{s}\left[\sigma_{r}, \sigma_{s}\right] \\
& =\frac{i}{2} \sum n_{r} a_{s} 2 i \epsilon_{r s t} \sigma_{t} \\
& =-\sum \sigma_{t} n_{r} a_{s} \epsilon_{r s t}  \tag{1.91}\\
& =-\sigma \cdot(\hat{\mathbf{n}} \times \mathbf{a}) \\
& =\sigma \cdot(\mathbf{a} \times \hat{\mathbf{n}})
\end{align*}
$$

Since $\mathbf{a} \times \hat{\mathbf{n}}=(\mathbf{a}-\hat{\mathbf{n}}(\hat{\mathbf{n}} \cdot \mathbf{a})) \times \hat{\mathbf{n}}$, this vector is seen to lie in the plane normal to $\hat{\mathbf{n}}$, but perpendicular to the rejection of $\hat{\mathbf{n}}$ from $\mathbf{a}$. That completes the demonstration that this is a rotation transformation.

To understand the sense of this rotation, consider $\hat{\mathbf{n}}=\hat{\mathbf{z}}, \mathbf{a}=\hat{\mathbf{x}}$, so

$$
\begin{equation*}
\sigma \cdot(\mathbf{a} \times \hat{\mathbf{n}})=\sigma \cdot(\hat{\mathbf{x}} \times \hat{\mathbf{z}})=-\sigma \cdot \hat{\mathbf{y}} \tag{1.92}
\end{equation*}
$$

and

$$
\begin{equation*}
\sigma \cdot \mathbf{a}^{\prime}=\hat{\mathbf{x}} \cos \phi-\hat{\mathbf{y}} \sin \phi \tag{1.93}
\end{equation*}
$$

showing that this rotation transformation has a clockwise sense.
Exercise 1.3 Some bra-ket manipulation problems. ([15] pr. 1.4)
Using braket logic expand
a. $\operatorname{tr} X Y$,
b. $(X Y)^{\dagger}$,
c. $e^{i f(A)}$, where $A$ is Hermitian with a complete set of eigenvalues.
d. $\sum_{a^{\prime}} \Psi_{a^{\prime}}\left(\mathbf{x}^{\prime}\right)^{*} \Psi_{a^{\prime}}\left(\mathbf{x}^{\prime \prime}\right)$, where $\Psi_{a^{\prime}}\left(\mathbf{x}^{\prime \prime}\right)=\left\langle\mathbf{x}^{\prime} \mid a^{\prime}\right\rangle$.

Answer for Exercise 1.3

Part a.

$$
\begin{align*}
\operatorname{tr} X Y & =\sum_{a}\langle a| X Y|a\rangle \\
& =\sum_{a, b}\langle a| X|b\rangle\langle b| Y|a\rangle \\
& =\sum_{a, b}\langle b| Y|a\rangle\langle a| X|b\rangle  \tag{1.94}\\
& =\sum_{a, b}\langle b| Y X|b\rangle \\
& =\operatorname{tr} Y X
\end{align*}
$$

Part b.

$$
\begin{align*}
\langle a|(X Y)^{\dagger}|b\rangle & =(\langle b| X Y|a\rangle)^{*} \\
& =\sum_{c}(\langle b| X|c\rangle\langle c| Y|a\rangle)^{*} \\
& =\sum_{c}(\langle b| X|c\rangle)^{*}(\langle c| Y|a\rangle)^{*}  \tag{1.95}\\
& =\sum_{c}(\langle c| Y|a\rangle)^{*}(\langle b| X|c\rangle)^{*} \\
& =\sum_{c}\langle a| Y^{\dagger}|c\rangle\langle c| X^{\dagger}|b\rangle \\
& =\langle a| Y^{\dagger} X^{\dagger}|b\rangle,
\end{align*}
$$

so $(X Y)^{\dagger}=Y^{\dagger} X^{\dagger}$.
Part c. Let's presume that the function $f$ has a Taylor series representation

$$
\begin{equation*}
f(A)=\sum_{r} b_{r} A^{r} . \tag{1.96}
\end{equation*}
$$

If the eigenvalues of $A$ are given by

$$
\begin{equation*}
A\left|a_{s}\right\rangle=a_{s}\left|a_{s}\right\rangle \tag{1.97}
\end{equation*}
$$

this operator can be expanded like

$$
\begin{align*}
A & =\sum_{a_{s}} A\left|a_{s}\right\rangle\left\langle a_{s}\right| \\
& =\sum_{a_{s}} a_{s}\left|a_{s}\right\rangle\left\langle a_{s}\right|, \tag{1.98}
\end{align*}
$$

To compute powers of this operator, consider first the square

$$
\begin{align*}
A^{2} & =\sum_{a_{s}} a_{s}\left|a_{s}\right\rangle\left\langle a_{s}\right| \sum_{a_{r}} a_{r}\left|a_{r}\right\rangle\left\langle a_{r}\right| \\
& =\sum_{a_{s}, a_{r}} a_{s} a_{r}\left|a_{s}\right\rangle\left\langle a_{s}\right|\left|a_{r}\right\rangle\left\langle a_{r}\right| \\
& =\sum_{a_{s}, a_{r}} a_{s} a_{r}\left|a_{s}\right\rangle \delta_{s r}\left\langle a_{r}\right|  \tag{1.99}\\
& =\sum_{a_{s}} a_{s}^{2}\left|a_{s}\right\rangle\left\langle a_{s}\right| .
\end{align*}
$$

The pattern for higher powers will clearly just be

$$
\begin{equation*}
A^{k}=\sum_{a_{s}} a_{s}^{k}\left|a_{s}\right\rangle\left\langle a_{s}\right| \tag{1.100}
\end{equation*}
$$

so the expansion of $f(A)$ will be

$$
\begin{align*}
f(A) & =\sum_{r} b_{r} A^{r} \\
& =\sum_{r} b_{r} \sum_{a_{s}} a_{s}^{r}\left|a_{s}\right\rangle\left\langle a_{s}\right| \\
& =\sum_{a_{s}}\left(\sum_{r} b_{r} a_{s}^{r}\right)\left|a_{s}\right\rangle\left\langle a_{s}\right|  \tag{1.101}\\
& =\sum_{a_{s}} f\left(a_{s}\right)\left|a_{s}\right\rangle\left\langle a_{s}\right|
\end{align*}
$$

The exponential expansion is

$$
\begin{align*}
e^{i f(A)} & =\sum_{t} \frac{i^{t}}{t!} f^{t}(A) \\
& =\sum_{t} \frac{i^{t}}{t!}\left(\sum_{a_{s}} f\left(a_{s}\right)\left|a_{s}\right\rangle\left\langle a_{s}\right|\right)^{t}  \tag{1.102}\\
& =\sum_{t} \frac{i^{t}}{t!} \sum_{a_{s}} f^{t}\left(a_{s}\right)\left|a_{s}\right\rangle\left\langle a_{s}\right| \\
& =\sum_{a_{s}} e^{i f\left(a_{s}\right)}\left|a_{s}\right\rangle\left\langle a_{s}\right|
\end{align*}
$$

Part d.

$$
\begin{align*}
\sum_{a^{\prime}} \Psi_{a^{\prime}}\left(\mathbf{x}^{\prime}\right)^{*} \Psi_{a^{\prime}}\left(\mathbf{x}^{\prime \prime}\right) & =\sum_{a^{\prime}}\left\langle\mathbf{x}^{\prime} \mid a^{\prime}\right\rangle^{*}\left\langle\mathbf{x}^{\prime \prime} \mid a^{\prime}\right\rangle \\
& =\sum_{a^{\prime}}\left\langle a^{\prime} \mid \mathbf{x}^{\prime}\right\rangle\left\langle\mathbf{x}^{\prime \prime} \mid a^{\prime}\right\rangle \\
& =\sum_{a^{\prime}}\left\langle\mathbf{x}^{\prime \prime} \mid a^{\prime}\right\rangle\left\langle a^{\prime} \mid \mathbf{x}^{\prime}\right\rangle  \tag{1.103}\\
& =\left\langle\mathbf{x}^{\prime \prime} \mid \mathbf{x}^{\prime}\right\rangle \\
& =\delta\left(\mathbf{x}^{\prime \prime}-\mathbf{x}^{\prime}\right)
\end{align*}
$$

a. Determine the matrix representation of $|\alpha\rangle\langle\beta|$ given a complete set of eigenvectors $\left|a^{r}\right\rangle$.
b. Verify with $|\alpha\rangle=\left|s_{z}=\hbar / 2\right\rangle,\left|s_{x}=\hbar / 2\right\rangle$.

Answer for Exercise 1.4

Part $a$. Forming the matrix element

$$
\begin{align*}
\left\langle a^{r}\right|(|\alpha\rangle\langle\beta|)\left|a^{s}\right\rangle & =\left\langle a^{r} \mid \alpha\right\rangle\left\langle\beta \mid a^{s}\right\rangle  \tag{1.104}\\
& =\left\langle a^{r} \mid \alpha\right\rangle\left\langle a^{s} \mid \beta\right\rangle^{*},
\end{align*}
$$

the matrix representation is seen to be

$$
\begin{align*}
|\alpha\rangle\langle\beta| \sim & {\left[\begin{array}{ccc}
\left\langle a^{1}\right|(|\alpha\rangle\langle\beta|)\left|a^{1}\right\rangle & \left\langle a^{1}\right|(|\alpha\rangle\langle\beta|)\left|a^{2}\right\rangle & \cdots \\
\left\langle a^{2}\right|(|\alpha\rangle\langle\beta|)\left|a^{1}\right\rangle & \left\langle a^{2}\right|(|\alpha\rangle\langle\beta|)\left|a^{2}\right\rangle & \cdots \\
\vdots & \vdots & \\
& \cdots
\end{array}\right] }  \tag{1.105}\\
& =\left[\begin{array}{ccc}
\left\langle a^{1} \mid \alpha\right\rangle\left\langle a^{1} \mid \beta\right\rangle^{*} & \left\langle a^{1} \mid \alpha\right\rangle\left\langle a^{2} \mid \beta\right\rangle^{*} & \cdots \\
\left\langle a^{2} \mid \alpha\right\rangle\left\langle a^{1} \mid \beta\right\rangle^{*} & \left\langle a^{2} \mid \alpha\right\rangle\left\langle a^{2} \mid \beta\right\rangle^{*} & \cdots \\
\vdots & \vdots & \ddots
\end{array}\right]
\end{align*}
$$

Part b. First compute the spin-z representation of $\left|s_{x}=\hbar / 2\right\rangle$.

$$
\begin{align*}
\left(S_{x}-\hbar / 2 I\right)\left[\begin{array}{l}
a \\
b
\end{array}\right] & =\left(\left[\begin{array}{cc}
0 & \hbar / 2 \\
\hbar / 2 & 0
\end{array}\right]-\left[\begin{array}{cc}
\hbar / 2 & 0 \\
0 & \hbar / 2
\end{array}\right]\right) \\
& =\left[\begin{array}{l}
a \\
b
\end{array}\right]  \tag{1.106}\\
& =\frac{\hbar}{2}\left[\begin{array}{cc}
-1 & 1 \\
1 & -1
\end{array}\right]\left[\begin{array}{l}
a \\
b
\end{array}\right]
\end{align*}
$$

so $\left|s_{x}=\hbar / 2\right\rangle \propto(1,1)$. Normalized we have

$$
\begin{align*}
& |\alpha\rangle=\left|s_{z}=\hbar / 2\right\rangle=\left[\begin{array}{l}
1 \\
0
\end{array}\right]  \tag{1.107}\\
& |\beta\rangle=\left|s_{z}=\hbar / 2\right\rangle \frac{1}{\sqrt{2}}\left[\begin{array}{l}
1 \\
1
\end{array}\right] .
\end{align*}
$$

Using eq. (1.105) the matrix representation is

$$
|\alpha\rangle\langle\beta| \sim\left[\begin{array}{ll}
(1)(1 / \sqrt{2})^{*} & (1)(1 / \sqrt{2})^{*}  \tag{1.108}\\
(0)(1 / \sqrt{2})^{*} & (0)(1 / \sqrt{2})^{*}
\end{array}\right]=\frac{1}{\sqrt{2}}\left[\begin{array}{ll}
1 & 1 \\
0 & 0
\end{array}\right] .
$$

This can be confirmed with direct computation

$$
\begin{align*}
|\alpha\rangle\langle\beta| & =\left[\begin{array}{l}
1 \\
0
\end{array}\right] \frac{1}{\sqrt{2}}\left[\begin{array}{ll}
1 & 1
\end{array}\right] \\
& =\frac{1}{\sqrt{2}}\left[\begin{array}{ll}
1 & 1 \\
0 & 0
\end{array}\right] . \tag{1.109}
\end{align*}
$$

Exercise 1.5 Eigenvalue of sum of kets. ([15] pr. 1.6)
Given eigenkets $|i\rangle,|j\rangle$ of an operator $A$, what are the conditions that $|i\rangle+|j\rangle$ is also an eigenvector?
Answer for Exercise 1.5
Let $A|i\rangle=i|i\rangle, A|j\rangle=j|j\rangle$, and suppose that the sum is an eigenket. Then there must be a value $a$ such that

$$
\begin{equation*}
A(|i\rangle+|j\rangle)=a(|i\rangle+|j\rangle), \tag{1.110}
\end{equation*}
$$

so

$$
\begin{equation*}
i|i\rangle+j|j\rangle=a(|i\rangle+|j\rangle) \tag{1.111}
\end{equation*}
$$

Operating with $\langle i|,\langle j|$ respectively, gives

$$
\begin{align*}
& i=a \\
& j=a, \tag{1.112}
\end{align*}
$$

so for the sum to be an eigenket, both of the corresponding energy eigenvalues must be identical (i.e. linear combinations of degenerate eigenkets are also eigenkets).

Exercise 1.6 Null operator. ([15] pr. 1.7)
Given eigenkets $\left|a^{\prime}\right\rangle$ of operator $A$
a. show that

$$
\begin{equation*}
\prod_{a^{\prime}}\left(A-a^{\prime}\right) \tag{1.113}
\end{equation*}
$$

is the null operator.
b.

$$
\begin{equation*}
\prod_{a^{\prime \prime} \neq a^{\prime}} \frac{\left(A-a^{\prime \prime}\right)}{a^{\prime}-a^{\prime \prime}} \tag{1.114}
\end{equation*}
$$

c. Illustrate using $S_{z}$ for a spin $1 / 2$ system.

Answer for Exercise 1.6

Part $a$. Application of $|a\rangle$, the eigenket of $A$ with eigenvalue $a$ to any term $A-a^{\prime}$ scales $|a\rangle$ by $a-a^{\prime}$, so the product operating on $|a\rangle$ is

$$
\begin{equation*}
\prod_{a^{\prime}}\left(A-a^{\prime}\right)|a\rangle=\prod_{a^{\prime}}\left(a-a^{\prime}\right)|a\rangle . \tag{1.115}
\end{equation*}
$$

Since $|a\rangle$ is one of the $\left\{\left|a^{\prime}\right\rangle\right\}$ eigenkets of $A$, one of these terms must be zero.

Part $b$. Again, consider the action of the operator on $|a\rangle$,

$$
\begin{equation*}
\prod_{a^{\prime \prime} \neq a^{\prime}} \frac{\left(A-a^{\prime \prime}\right)}{a^{\prime}-a^{\prime \prime}}|a\rangle=\prod_{a^{\prime \prime} \neq a^{\prime}} \frac{\left(a-a^{\prime \prime}\right)}{a^{\prime}-a^{\prime \prime}}|a\rangle . \tag{1.116}
\end{equation*}
$$

If $|a\rangle=\left|a^{\prime}\right\rangle$, then $\prod_{a^{\prime \prime} \neq a^{\prime}} \frac{\left(A-a^{\prime \prime}\right)}{a^{\prime}-a^{\prime \prime}}|a\rangle=|a\rangle$, whereas if it does not, then it equals one of the $a^{\prime \prime}$ energy eigenvalues. This is a representation of the Kronecker delta function

$$
\begin{equation*}
\prod_{a^{\prime \prime} \neq a^{\prime}} \frac{\left(A-a^{\prime \prime}\right)}{a^{\prime}-a^{\prime \prime}}|a\rangle \equiv \delta_{a^{\prime}, a}|a\rangle \tag{1.117}
\end{equation*}
$$

Part c. For operator $S_{z}$ the eigenvalues are $\{\hbar / 2,-\hbar / 2\}$, so the null operator must be

$$
\begin{align*}
\prod_{a^{\prime}}\left(A-a^{\prime}\right) & =\left(\frac{\hbar}{2}\right)^{2}\left(\left[\begin{array}{cc}
1 & 0 \\
0 & -1
\end{array}\right]-\left[\begin{array}{ll}
1 & 0 \\
0 & 1
\end{array}\right]\right)\left(\left[\begin{array}{cc}
1 & 0 \\
0 & -1
\end{array}\right]+\left[\begin{array}{ll}
1 & 0 \\
0 & 1
\end{array}\right]\right) \\
& =\left[\begin{array}{ll}
0 & 0 \\
0 & -2
\end{array}\right]\left[\begin{array}{ll}
2 & 0 \\
0 & 0
\end{array}\right]  \tag{1.118}\\
& =\left[\begin{array}{ll}
0 & 0 \\
0 & 0
\end{array}\right]
\end{align*}
$$

For the delta representation, consider the $| \pm\rangle$ states and their eigenvalue. The delta operators are

$$
\begin{align*}
\prod_{a^{\prime \prime} \neq \hbar / 2} \frac{\left(A-a^{\prime \prime}\right)}{\hbar / 2-a^{\prime \prime}} & =\frac{S_{z}-(-\hbar / 2) I}{\hbar / 2-(-\hbar / 2)} \\
& =\frac{1}{2}\left(\sigma_{z}+I\right) \\
& =\frac{1}{2}\left(\left[\begin{array}{cc}
1 & 0 \\
0 & -1
\end{array}\right]+\left[\begin{array}{ll}
1 & 0 \\
0 & 1
\end{array}\right]\right)  \tag{1.119}\\
& =\frac{1}{2}\left[\begin{array}{ll}
2 & 0 \\
0 & 0
\end{array}\right] \\
& =\left[\begin{array}{ll}
1 & 0 \\
0 & 0
\end{array}\right] .
\end{align*}
$$

$$
\begin{align*}
\prod_{a^{\prime \prime} \neq-\hbar / 2} \frac{\left(A-a^{\prime \prime}\right)}{-\hbar / 2-a^{\prime \prime}} & =\frac{S_{z}-(\hbar / 2) I}{-\hbar / 2-\hbar / 2} \\
& =\frac{1}{2}\left(\sigma_{z}-I\right) \\
& =\frac{1}{2}\left(\left[\begin{array}{cc}
1 & 0 \\
0 & -1
\end{array}\right]-\left[\begin{array}{ll}
1 & 0 \\
0 & 1
\end{array}\right]\right)  \tag{1.120}\\
& =\frac{1}{2}\left[\begin{array}{cc}
0 & 0 \\
0 & -2
\end{array}\right] \\
& =\left[\begin{array}{ll}
0 & 0 \\
0 & 1
\end{array}\right] .
\end{align*}
$$

These clearly have the expected delta function property acting on kets


## Exercise 1.7 Spin half general normal. ([15] pr. 1.9)

Construct $|\mathbf{S} \cdot \hat{\mathbf{n}} ;+\rangle$, where $\hat{\mathbf{n}}=(\cos \alpha \sin \beta, \sin \alpha \sin \beta, \cos \beta)^{\mathrm{T}}$ such that

$$
\begin{equation*}
\mathbf{S} \cdot \hat{\mathbf{n}}|\mathbf{S} \cdot \hat{\mathbf{n}} ;+\rangle=\frac{\hbar}{2}|\mathbf{S} \cdot \hat{\mathbf{n}} ;+\rangle, \tag{1.121}
\end{equation*}
$$

Solve this as an eigenvalue problem.

## Answer for Exercise 1.7

The spin operator for this direction is

$$
\begin{align*}
\mathbf{S} \cdot \hat{\mathbf{n}} & =\frac{\hbar}{2} \boldsymbol{\sigma} \cdot \hat{\mathbf{n}} \\
& =\frac{\hbar}{2}\left(\cos \alpha \sin \beta\left[\begin{array}{ll}
0 & 1 \\
1 & 0
\end{array}\right]+\sin \alpha \sin \beta\left[\begin{array}{cc}
0 & -i \\
i & 0
\end{array}\right]+\cos \beta\left[\begin{array}{cc}
1 & 0 \\
0 & -1
\end{array}\right]\right) \\
& =\frac{\hbar}{2}\left[\begin{array}{cc}
\cos \beta & e^{-i \alpha} \sin \beta \\
e^{i \alpha} \sin \beta & -\cos \beta
\end{array}\right] . \tag{1.122}
\end{align*}
$$

Observed that this is traceless and has a $-\hbar / 2$ determinant like any of the $x, y, z$ spin operators.

Assuming that this has an $\hbar / 2$ eigenvalue (to be verified later), the eigenvalue problem is

$$
\begin{align*}
0 & =\mathbf{S} \cdot \hat{\mathbf{n}}-\hbar / 2 I \\
& =\frac{\hbar}{2}\left[\begin{array}{cc}
\cos \beta-1 & e^{-i \alpha} \sin \beta \\
e^{i \alpha} \sin \beta & -\cos \beta-1
\end{array}\right]  \tag{1.123}\\
& =\hbar\left[\begin{array}{cc}
-\sin ^{2} \frac{\beta}{2} & e^{-i \alpha} \sin \frac{\beta}{2} \cos \frac{\beta}{2} \\
e^{i \alpha} \sin \frac{\beta}{2} \cos \frac{\beta}{2} & -\cos ^{2} \frac{\beta}{2}
\end{array}\right] .
\end{align*}
$$

This has a zero determinant as expected, and the eigenvector $(a, b)$ will satisfy

$$
\begin{align*}
& 0=-\sin ^{2} \frac{\beta}{2} a+e^{-i \alpha} \sin \frac{\beta}{2} \cos \frac{\beta}{2} b \\
&=\sin \frac{\beta}{2}\left(-\sin \frac{\beta}{2} a+e^{-i \alpha} b \cos \frac{\beta}{2}\right)  \tag{1.124}\\
& {\left[\begin{array}{l}
a \\
b
\end{array}\right] \propto\left[\begin{array}{c}
\cos \frac{\beta}{2} \\
e^{i \alpha} \sin \frac{\beta}{2}
\end{array}\right] . } \tag{1.125}
\end{align*}
$$

This is appropriately normalized, so the ket for $\mathbf{S} \cdot \hat{\mathbf{n}}$ is

$$
\begin{equation*}
|\mathbf{S} \cdot \hat{\mathbf{n}} ;+\rangle=\cos \frac{\beta}{2}|+\rangle+e^{i \alpha} \sin \frac{\beta}{2}|-\rangle . \tag{1.126}
\end{equation*}
$$

Note that the other eigenvalue is

$$
\begin{equation*}
|\mathbf{S} \cdot \hat{\mathbf{n}} ;-\rangle=-\sin \frac{\beta}{2}|+\rangle+e^{i \alpha} \cos \frac{\beta}{2}|-\rangle . \tag{1.127}
\end{equation*}
$$

It is straightforward to show that these are orthogonal and that this has the $-\hbar / 2$ eigenvalue.

## Exercise 1.8 Two state Hamiltonian. ([15] pr. 1.10)

Solve the eigenproblem for

$$
\begin{equation*}
H=a(|1\rangle\langle 1|-|2\rangle\langle 2|+|1\rangle\langle 2|+|2\rangle\langle 1|) \tag{1.128}
\end{equation*}
$$

Answer for Exercise 1.8
In matrix form the Hamiltonian is

$$
H=a\left[\begin{array}{cc}
1 & 1  \tag{1.129}\\
1 & -1
\end{array}\right]
$$

The eigenvalue problem is

$$
\begin{align*}
0 & =|H-\lambda I| \\
& =(a-\lambda)(-a-\lambda)-a^{2} \\
& =(-a+\lambda)(a+\lambda)-a^{2}  \tag{1.130}\\
& =\lambda^{2}-a^{2}-a^{2},
\end{align*}
$$

or

$$
\begin{equation*}
\lambda= \pm \sqrt{2} a \tag{1.131}
\end{equation*}
$$

An eigenket proportional to $(\alpha, \beta)$ must satisfy

$$
\begin{equation*}
0=(1 \mp \sqrt{2}) \alpha+\beta \tag{1.132}
\end{equation*}
$$

so
or

That is

Exercise 1.9 Spin half and dispersion. (2015 ps1.3; [15] pr. 1.12)
A spin $1 / 2 \operatorname{system} \mathbf{S} \cdot \hat{\mathbf{n}}$, with $\hat{\mathbf{n}}=\sin \theta \hat{\mathbf{x}}+\cos \theta \hat{\mathbf{z}}$, is in state with eigenvalue $\hbar / 2$.
a. If $S_{x}$ is measured. What is the probability of getting $+\hbar / 2$ ?
b. Evaluate the dispersion in $S_{x}$, that is,

$$
\begin{equation*}
\left\langle\left(S_{x}-\left\langle S_{x}\right\rangle\right)^{2}\right\rangle \tag{1.136}
\end{equation*}
$$

Answer for Exercise 1.9

Part a. In matrix form the spin operator for the system is

$$
\begin{align*}
\mathbf{S} \cdot \hat{\mathbf{n}} & =\frac{\hbar}{2}\left(\cos \theta\left[\begin{array}{cc}
1 & 0 \\
0 & -1
\end{array}\right]+\sin \theta\left[\begin{array}{ll}
0 & 1 \\
1 & 0
\end{array}\right]\right)  \tag{1.137}\\
& =\frac{\hbar}{2}\left[\begin{array}{cc}
\cos \theta & \sin \theta \\
\sin \theta & -\cos \theta
\end{array}\right] .
\end{align*}
$$

An eigenket $|\mathbf{S} \cdot \hat{\mathbf{n}} ;+\rangle=(a, b)^{\mathrm{T}}$ must satisfy

$$
\begin{align*}
0 & =(\cos \theta-1) a+\sin \theta b \\
& =\left(-2 \sin ^{2} \frac{\theta}{2}\right) a+2 \sin \frac{\theta}{2} \cos \frac{\theta}{2} b  \tag{1.138}\\
& =-\sin \frac{\theta}{2} a+\cos \frac{\theta}{2} b,
\end{align*}
$$

so the eigenstate is

$$
|\mathbf{S} \cdot \hat{\mathbf{n}} ;+\rangle=\left[\begin{array}{c}
\cos \frac{\theta}{2}  \tag{1.139}\\
\sin \frac{\theta}{2}
\end{array}\right] .
$$

Pick $\left|S_{x} ; \pm\right\rangle=\frac{1}{\sqrt{2}}\left[\begin{array}{c}1 \\ \pm 1\end{array}\right]$ as the basis for the $S_{x}$ operator. Then, for the probability that the system will end up in the $+\hbar / 2$ state of $S_{x}$, we have

$$
\begin{align*}
P & =\left|\left\langle S_{x} ;+\mid \mathbf{S} \cdot \hat{\mathbf{n}} ;+\right\rangle\right|^{2} \\
& =\left\lvert\, \frac{1}{\sqrt{2}}\left[\begin{array}{l}
1 \\
1
\end{array}\right]\left[\left.\begin{array}{c}
\cos \frac{\theta}{2} \\
\sin \frac{\theta}{2}
\end{array}\right|^{2}\right.\right. \\
& =\frac{1}{2}\left|\left[\begin{array}{ll}
1 & 1
\end{array}\right]\left[\begin{array}{c}
\cos \frac{\theta}{2} \\
\sin \frac{\theta}{2}
\end{array}\right]\right|^{2}  \tag{1.140}\\
& =\frac{1}{2}\left(\cos \frac{\theta}{2}+\sin \frac{\theta}{2}\right)^{2} \\
& =\frac{1}{2}\left(1+2 \cos \frac{\theta}{2} \sin \frac{\theta}{2}\right) \\
& =\frac{1}{2}(1+\sin \theta) .
\end{align*}
$$

This is a reasonable seeming result, with $P \in[0,1]$. Some special values also further validate this

$$
\begin{align*}
& \theta=0,|\mathbf{S} \cdot \hat{\mathbf{n}} ;+\rangle=\left[\begin{array}{l}
1 \\
0
\end{array}\right]=\left|S_{z} ;+\right\rangle=\frac{1}{\sqrt{2}}\left|S_{x} ;+\right\rangle+\frac{1}{\sqrt{2}}\left|S_{x} ;-\right\rangle \\
& \theta=\pi / 2,|\mathbf{S} \cdot \hat{\mathbf{n}} ;+\rangle=\frac{1}{\sqrt{2}}\left[\begin{array}{l}
1 \\
1
\end{array}\right]=\left|S_{x} ;+\right\rangle  \tag{1.141}\\
& \theta=\pi,|\mathbf{S} \cdot \hat{\mathbf{n}} ;+\rangle=\left[\begin{array}{l}
0 \\
1
\end{array}\right]=\left|S_{z} ;-\right\rangle=\frac{1}{\sqrt{2}}\left|S_{x} ;+\right\rangle-\frac{1}{\sqrt{2}}\left|S_{x} ;-\right\rangle,
\end{align*}
$$

where we see that the probabilities are in proportion to the projection of the initial state onto the measured state $\left|S_{x} ;+\right\rangle$.

Part b. The $S_{x}$ expectation is

$$
\begin{align*}
\left\langle S_{x}\right\rangle & =\frac{\hbar}{2}\left[\begin{array}{ll}
\cos \frac{\theta}{2} & \sin \frac{\theta}{2}
\end{array}\right]\left[\begin{array}{ll}
0 & 1 \\
1 & 0
\end{array}\right]\left[\begin{array}{cc}
\cos \frac{\theta}{2} \\
\sin \frac{\theta}{2}
\end{array}\right] \\
& =\frac{\hbar}{2}\left[\begin{array}{ll}
\cos \frac{\theta}{2} & \sin \frac{\theta}{2}
\end{array}\right]\left[\begin{array}{l}
\sin \frac{\theta}{2} \\
\cos \frac{\theta}{2}
\end{array}\right]  \tag{1.142}\\
& =\frac{\hbar}{2} 2 \sin \frac{\theta}{2} \cos \frac{\theta}{2} \\
& =\frac{\hbar}{2} \sin \theta .
\end{align*}
$$

Note that $S_{x}^{2}=(\hbar / 2)^{2} I$, so

$$
\begin{align*}
\left\langle S_{x}^{2}\right\rangle & =\left(\frac{\hbar}{2}\right)^{2}\left[\begin{array}{ll}
\cos \frac{\theta}{2} & \sin \frac{\theta}{2}
\end{array}\right]\left[\begin{array}{c}
\cos \frac{\theta}{2} \\
\sin \frac{\theta}{2}
\end{array}\right] \\
& =\left(\frac{\hbar}{2}\right)^{2} \cos ^{2} \frac{\theta}{2}+\sin ^{2} \frac{\theta}{2}  \tag{1.143}\\
& =\left(\frac{\hbar}{2}\right)^{2}
\end{align*}
$$

The dispersion is

$$
\begin{align*}
\left\langle\left(S_{x}-\left\langle S_{x}\right\rangle\right)^{2}\right\rangle & =\left\langle S_{x}^{2}\right\rangle-\left\langle S_{x}\right\rangle^{2} \\
& =\left(\frac{\hbar}{2}\right)^{2}\left(1-\sin ^{2} \theta\right)  \tag{1.144}\\
& =\left(\frac{\hbar}{2}\right)^{2} \cos ^{2} \theta
\end{align*}
$$

At $\theta=\pi / 2$ the dispersion is 0 , which is expected since $|\mathbf{S} \cdot \hat{\mathbf{n}} ;+\rangle=\left|S_{x} ;+\right\rangle$ at that point. Similarly, the dispersion is maximized at $\theta=0, \pi$ where the $|\mathbf{S} \cdot \hat{\mathbf{n}} ;+\rangle$ component in the $\left|S_{x} ;+\right\rangle$ direction is minimized.

## Exercise 1.10 Cascading Stern-Gerlach. ([15] pr. 1.13)

Three Stern-Gerlach type measurements are performed, the first that prepares the state in a $\left|S_{z} ;+\right\rangle$ state, the next in a $|\mathbf{S} \cdot \hat{\mathbf{n}} ;+\rangle$ state where $\hat{\mathbf{n}}=\cos \beta \hat{\mathbf{z}}+\sin \beta \hat{\mathbf{x}}$, and the last performing a $S_{z} \hbar / 2$ state measurement, as illustrated in fig. 1.7.


Figure 1.7: Cascaded Stern-Gerlach type measurements.
What is the intensity of the final $s_{z}=-\hbar / 2$ beam? What is the orientation for the second measuring apparatus to maximize the intensity of this beam?

## Answer for Exercise 1.10

The spin operator for the second apparatus is

$$
\begin{align*}
\mathbf{S} \cdot \hat{\mathbf{n}} & =\frac{\hbar}{2}\left(\sin \beta\left[\begin{array}{ll}
0 & 1 \\
1 & 0
\end{array}\right]+\cos \beta\left[\begin{array}{cc}
1 & 0 \\
0 & -1
\end{array}\right]\right)  \tag{1.145}\\
& =\frac{\hbar}{2}\left[\begin{array}{cc}
\cos \beta & \sin \beta \\
\sin \beta & -\cos \beta
\end{array}\right] .
\end{align*}
$$

The intensity of the final $\left|S_{z} ;-\right\rangle$ beam is

$$
\begin{equation*}
P=|\langle-\mid \mathbf{S} \cdot \hat{\mathbf{n}} ;+\rangle\langle\mathbf{S} \cdot \hat{\mathbf{n}} ;+\mid+\rangle|^{2}, \tag{1.146}
\end{equation*}
$$

(i.e. the second apparatus applies a projection operator $|\mathbf{S} \cdot \hat{\mathbf{n}} ;+\rangle\langle\mathbf{S} \cdot \hat{\mathbf{n}} ;+|$ to the initial $|+\rangle$ state, and then the $|-\rangle$ states are selected out of that.

The $\mathbf{S} \cdot \hat{\mathbf{n}}$ eigenket is found to be

$$
|\mathbf{S} \cdot \hat{\mathbf{n}} ;+\rangle=\left[\begin{array}{c}
\cos \frac{\beta}{2}  \tag{1.147}\\
\sin \frac{\beta}{2}
\end{array}\right],
$$

so

$$
\begin{align*}
P & =\left|\left[\begin{array}{ll}
0 & 1
\end{array}\right]\left[\begin{array}{ll}
\cos \frac{\beta}{2} \\
\sin \frac{\beta}{2}
\end{array}\right]\left[\begin{array}{ll}
\cos \frac{\beta}{2} & \sin \frac{\beta}{2}
\end{array}\right]\left[\begin{array}{l}
1 \\
0
\end{array}\right]\right|^{2} \\
& =\left|\cos \frac{\beta}{2} \sin \frac{\beta}{2}\right|^{2}  \tag{1.148}\\
& =\left|\frac{1}{2} \sin \beta\right|^{2} \\
& =\frac{1}{4} \sin ^{2} \beta .
\end{align*}
$$

This is maximized when $\beta=\pi / 2$, or $\hat{\mathbf{n}}=\hat{\mathbf{x}}$. At this angle the state leaving the second apparatus is

$$
\begin{align*}
{\left[\begin{array}{c}
\cos \frac{\beta}{2} \\
\sin \frac{\beta}{2}
\end{array}\right]\left[\begin{array}{ll}
\cos \frac{\beta}{2} & \sin \frac{\beta}{2}
\end{array}\right]\left[\begin{array}{l}
1 \\
0
\end{array}\right] } & =\frac{1}{2}\left[\begin{array}{l}
1 \\
1
\end{array}\right]\left[\begin{array}{ll}
1 & 1
\end{array}\right]\left[\begin{array}{l}
1 \\
0
\end{array}\right] \\
& =\frac{1}{2}\left[\begin{array}{l}
1 \\
1
\end{array}\right]  \tag{1.149}\\
& =\frac{1}{2}|+\rangle+\frac{1}{2}|-\rangle
\end{align*}
$$

so the state after filtering the $|-\rangle$ states is $\frac{1}{2}|-\rangle$ with intensity (probability density) of $1 / 4$ relative to a unit normalize input $|+\rangle$ state to the $\mathbf{S} \cdot \hat{\mathbf{n}}$ apparatus.

## Exercise 1.11 Anticommuting operators and eigenkets. ([15] pr.1.16)

Given two Hermitian operators that anticommute

$$
\begin{equation*}
\{A, B\}=A B+B A=0 \tag{1.150}
\end{equation*}
$$

is it possible to have a simultaneous eigenket of $A$ and $B$ ? Prove or illustrate your assertion.
Answer for Exercise 1.11
Suppose that such a simultaneous non-zero eigenket $|\alpha\rangle$ exists, then

$$
\begin{equation*}
A|\alpha\rangle=a|\alpha\rangle \tag{1.151}
\end{equation*}
$$

and

$$
\begin{equation*}
B|\alpha\rangle=b|\alpha\rangle \tag{1.152}
\end{equation*}
$$

This gives

$$
\begin{equation*}
(A B+B A)|\alpha\rangle=(A b+B a)|\alpha\rangle=2 a b|\alpha\rangle \tag{1.153}
\end{equation*}
$$

If this is zero, one of the operators must have a zero eigenvalue. Knowing that we can construct an example of such operators. In matrix form, let

$$
A=\left[\begin{array}{ccc}
1 & 0 & 0  \tag{1.154a}\\
0 & -1 & 0 \\
0 & 0 & a
\end{array}\right]
$$

$$
B=\left[\begin{array}{lll}
0 & 1 & 0  \tag{1.154b}\\
1 & 0 & 0 \\
0 & 0 & b
\end{array}\right] .
$$

These are both Hermitian, and anticommute provided at least one of $a, b$ is zero. These have a common eigenket

$$
|\alpha\rangle=\left[\begin{array}{l}
0  \tag{1.155}\\
0 \\
1
\end{array}\right] .
$$

A zero eigenvalue of one of the commuting operators may not be a sufficient condition for such anticommutation.

## Exercise 1.12 Non-commuting observables, degeneracy. ([15] pr. 1.17)

Show that non-commuting operators that both commute with the Hamiltonian, have, in general, degenerate energy eigenvalues. That is

$$
\begin{equation*}
[A, H]=[B, H]=0, \tag{1.156}
\end{equation*}
$$

but

$$
\begin{equation*}
[A, B] \neq 0 . \tag{1.157}
\end{equation*}
$$

a. Consider $L_{x}, L_{z}$ and a central force Hamiltonian $H=\mathbf{p}^{2} / 2 m+V(r)$ as examples.
b. Construct some simple matrix examples that illustrate the degeneracy conditions.
c. Prove the general case.

Answer for Exercise 1.12

Part a. Let's start with demonstrate these commutators act as expected in these cases. With $\mathbf{L}=\mathbf{x} \times \mathbf{p}$, we have

$$
\begin{align*}
& L_{x}=y p_{z}-z p_{y} \\
& L_{y}=z p_{x}-x p_{z}  \tag{1.158}\\
& L_{z}=x p_{y}-y p_{x} .
\end{align*}
$$

The $L_{x}, L_{z}$ commutator is

$$
\begin{align*}
{\left[L_{x}, L_{z}\right] } & =\left[y p_{z}-z p_{y}, x p_{y}-y p_{x}\right] \\
& =\left[y p_{z}, x p_{y}\right]-\left[y p_{z}, y p_{x}\right]-\left[z p_{y}, x p_{y}\right]+\left[z p_{y}, y p_{x}\right] \\
& =x p_{z}\left[y, p_{y}\right]+z p_{x}\left[p_{y}, y\right]  \tag{1.159}\\
& =i \hbar\left(x p_{z}-z p_{x}\right) \\
& =-i \hbar L_{y}
\end{align*}
$$

Cyclically permuting the indexes shows that no pairs of different $\mathbf{L}$ components commute. For $L_{y}, L_{x}$ that is

$$
\begin{align*}
{\left[L_{y}, L_{x}\right] } & =\left[z p_{x}-x p_{z}, y p_{z}-z p_{y}\right] \\
& =\left[z p_{x}, y p_{z}\right]-\left[z p_{x}, z p_{y}\right]-\left[x p_{z}, y p_{z}\right]+\left[x p_{z}, z p_{y}\right] \\
& =y p_{x}\left[z, p_{z}\right]+x p_{y}\left[p_{z}, z\right]  \tag{1.160}\\
& =i \hbar\left(y p_{x}-x p_{y}\right) \\
& =-i \hbar L_{z},
\end{align*}
$$

and for $L_{z}, L_{y}$

$$
\begin{align*}
{\left[L_{z}, L_{y}\right] } & =\left[x p_{y}-y p_{x}, z p_{x}-x p_{z}\right] \\
& =\left[x p_{y}, z p_{x}\right]-\left[x p_{y}, x p_{z}\right]-\left[y p_{x}, z p_{x}\right]+\left[y p_{x}, x p_{z}\right] \\
& =z p_{y}\left[x, p_{x}\right]+y p_{z}\left[p_{x}, x\right]  \tag{1.161}\\
& =i \hbar\left(z p_{y}-y p_{z}\right) \\
& =-i \hbar L_{x} .
\end{align*}
$$

If these angular momentum components are also shown to commute with themselves (which they do), the commutator relations above can be summarized as

$$
\begin{equation*}
\left[L_{a}, L_{b}\right]=i \hbar \epsilon_{a b c} L_{c} \tag{1.162}
\end{equation*}
$$

In the example to consider, we'll have to consider the commutators with $\mathbf{p}^{2}$ and $V(r)$. Picking any one component of $\mathbf{L}$ is sufficient due to the symmetries of the problem. For example

$$
\begin{align*}
{\left[L_{x}, \mathbf{p}^{2}\right] } & =\left[y p_{z}-z p_{y}, p_{x}^{2}+p_{y}^{2}+p_{z}^{2}\right] \\
& =\left[y p_{z}, p_{x}^{\not 2}+p_{y}^{2}+p_{z}^{2}\right]-\left[z p_{y}, p_{x}^{\not 2}+p_{y}^{22}+p_{z}^{2}\right] \\
& =p_{z}\left[y, p_{y}^{2}\right]-p_{y}\left[z, p_{z}^{2}\right]  \tag{1.163}\\
& =p_{z} 2 i \hbar p_{y}-p_{y} 2 i \hbar p_{z} \\
& =0 .
\end{align*}
$$

How about the commutator of $\mathbf{L}$ with the potential? It is sufficient to consider one component again, for example

$$
\begin{align*}
{\left[L_{x}, V\right] } & =\left[y p_{z}-z p_{y}, V\right] \\
& =y\left[p_{z}, V\right]-z\left[p_{y}, V\right] \\
& =-i \hbar y \frac{\partial V(r)}{\partial z}+i \hbar z \frac{\partial V(r)}{\partial y} \\
& =-i \hbar y \frac{\partial V}{\partial r} \frac{\partial r}{\partial z}+i \hbar z \frac{\partial V}{\partial r} \frac{\partial r}{\partial y}  \tag{1.164}\\
& =-i \hbar y \frac{\partial V}{\partial r} \frac{z}{r}+i \hbar z \frac{\partial V}{\partial r} \frac{y}{r} \\
& =0 .
\end{align*}
$$

This has shown that all the components of $\mathbf{L}$ commute with a central force Hamiltonian, and each different component of $\mathbf{L}$ do not commute. It does not demonstrate the degeneracy, but I do recall that exists for this system.

Part b. I thought perhaps the problem at hand would be easier if I were to construct some example matrices representing operators that did not commute, but did commuted with a Hamiltonian. I came up with

$$
\begin{aligned}
& A=\left[\begin{array}{cc}
\sigma_{z} & 0 \\
0 & 1
\end{array}\right]=\left[\begin{array}{ccc}
1 & 0 & 0 \\
0 & -1 & 0 \\
0 & 0 & 1
\end{array}\right] \\
& B=\left[\begin{array}{cc}
\sigma_{x} & 0 \\
0 & 1
\end{array}\right]=\left[\begin{array}{lll}
0 & 1 & 0 \\
1 & 0 & 0 \\
0 & 0 & 1
\end{array}\right] . \\
& H=\left[\begin{array}{lll}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 1
\end{array}\right]
\end{aligned}
$$

This system has $[A, H]=[B, H]=0$, and

$$
[A, B]=\left[\begin{array}{ccc}
0 & 2 & 0  \tag{1.166}\\
-2 & 0 & 0 \\
0 & 0 & 0
\end{array}\right]
$$

There is one shared eigenvector between all of $A, B, H$

$$
|3\rangle=\left[\begin{array}{l}
0  \tag{1.167}\\
0 \\
1
\end{array}\right] .
$$

The other eigenvectors for $A$ are

$$
\left|a_{1}\right\rangle=\left[\begin{array}{l}
1  \tag{1.168}\\
0 \\
0
\end{array}\right], \quad\left|a_{2}\right\rangle=\left[\begin{array}{l}
0 \\
1 \\
0
\end{array}\right]
$$

and for $B$

$$
\left|b_{1}\right\rangle=\frac{1}{\sqrt{2}}\left[\begin{array}{l}
1  \tag{1.169}\\
1 \\
0
\end{array}\right], \quad\left|b_{2}\right\rangle=\frac{1}{\sqrt{2}}\left[\begin{array}{c}
1 \\
-1 \\
0
\end{array}\right] .
$$

This clearly has the degeneracy sought.
Looking to [1], it appears that it is possible to construct an even simpler example. Let

$$
\begin{align*}
A & =\left[\begin{array}{ll}
0 & 1 \\
0 & 0
\end{array}\right] \\
B & =\left[\begin{array}{ll}
1 & 0 \\
0 & 0
\end{array}\right]  \tag{1.170}\\
H & =\left[\begin{array}{ll}
0 & 0 \\
0 & 0
\end{array}\right] .
\end{align*}
$$

Here $[A, B]=-A$, and $[A, H]=[B, H]=0$, but the Hamiltonian isn't interesting at all physically.

A less boring example builds on this. Let

$$
\begin{align*}
& A=\left[\begin{array}{lll}
0 & 1 & 0 \\
0 & 0 & 0 \\
0 & 0 & 1
\end{array}\right] \\
& B=\left[\begin{array}{lll}
1 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 1
\end{array}\right]  \tag{1.171}\\
& H=\left[\begin{array}{lll}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 1
\end{array}\right] .
\end{align*}
$$

Here $[A, B] \neq 0$, and $[A, H]=[B, H]=0$. I don't see a way for any exception to be constructed.

Part c. The concrete examples above give some intuition for solving the more abstract problem. Suppose that we are working in a basis that simultaneously diagonalizes operator $A$ and the Hamiltonian $H$. To make life easy consider the simplest case where this basis is also an eigenbasis for the second operator $B$ for all but two of that operators eigenvectors. For such a system let's write

$$
\begin{align*}
H|1\rangle & =\epsilon_{1}|1\rangle \\
H|2\rangle & =\epsilon_{2}|2\rangle  \tag{1.172}\\
A|1\rangle & =a_{1}|1\rangle \\
A|2\rangle & =a_{2}|2\rangle
\end{align*}
$$

where $|1\rangle$, and $|2\rangle$ are not eigenkets of $B$. Because $B$ also commutes with $H$, we must have

$$
\begin{align*}
H B|1\rangle & =H \sum_{n}|n\rangle\langle n| B|1\rangle \\
& =\sum_{n} \epsilon_{n}|n\rangle B_{n 1} \tag{1.173}
\end{align*}
$$

and

$$
\begin{align*}
B H|1\rangle & =B \epsilon_{1}|1\rangle \\
& =\epsilon_{1} \sum_{n}|n\rangle\langle n| B|1\rangle  \tag{1.174}\\
& =\epsilon_{1} \sum_{n}|n\rangle B_{n 1} .
\end{align*}
$$

We can now compute the action of the commutators on $|1\rangle,|2\rangle$,

$$
\begin{equation*}
[B, H]|1\rangle=\sum_{n}\left(\epsilon_{1}-\epsilon_{n}\right)|n\rangle B_{n 1} \tag{1.175}
\end{equation*}
$$

Similarly

$$
\begin{equation*}
[B, H]|2\rangle=\sum_{n}\left(\epsilon_{2}-\epsilon_{n}\right)|n\rangle B_{n 2} \tag{1.176}
\end{equation*}
$$

However, for those kets $|m\rangle \in\{|3\rangle,|4\rangle, \cdots\}$ that are eigenkets of $B$, with $B|m\rangle=b_{m}|m\rangle$, we have

$$
\begin{align*}
{[B, H]|m\rangle } & =B \epsilon_{m}|m\rangle-H b_{m}|m\rangle \\
& =b_{m} \epsilon_{m}|m\rangle-\epsilon_{m} b_{m}|m\rangle  \tag{1.177}\\
& =0
\end{align*}
$$

The sums in eq. (1.175) and eq. (1.176) reduce to

$$
\begin{align*}
{[B, H]|1\rangle } & =\sum_{n=1}^{2}\left(\epsilon_{1}-\epsilon_{n}\right)|n\rangle B_{n 1}  \tag{1.178}\\
& =\left(\epsilon_{1}-\epsilon_{2}\right)|2\rangle B_{21}
\end{align*}
$$

and

$$
\begin{align*}
{[B, H]|2\rangle } & =\sum_{n=1}^{2}\left(\epsilon_{2}-\epsilon_{n}\right)|n\rangle B_{n 2}  \tag{1.179}\\
& =\left(\epsilon_{2}-\epsilon_{1}\right)|1\rangle B_{12}
\end{align*}
$$

Since the commutator is zero, the matrix elements of the commutator must all be zero, in particular

$$
\begin{align*}
\langle 1|[B, H]|1\rangle & =\left(\epsilon_{1}-\epsilon_{2}\right) B_{21}\langle 1 \mid 2\rangle=0 \\
\langle 2|[B, H]|1\rangle & =\left(\epsilon_{1}-\epsilon_{2}\right) B_{21}\langle 1 \mid 1\rangle \\
\langle 1|[B, H]|2\rangle & =\left(\epsilon_{2}-\epsilon_{1}\right) B_{12}\langle 1 \mid 2\rangle=0  \tag{1.180}\\
\langle 2|[B, H]|2\rangle & =\left(\epsilon_{2}-\epsilon_{1}\right) B_{12}\langle 2 \mid 2\rangle .
\end{align*}
$$

We must either have

- $B_{21}=B_{12}=0$, or
- $\epsilon_{1}=\epsilon_{2}$.

If the first condition were true we would have

$$
\begin{align*}
B|1\rangle & =|n\rangle\langle n| B|1\rangle \\
& =|n\rangle B_{n 1}  \tag{1.181}\\
& =|1\rangle B_{11},
\end{align*}
$$

and $B|2\rangle=B_{22}|2\rangle$. This contradicts the requirement that $|1\rangle,|2\rangle$ not be eigenkets of $B$, leaving only the second option. That second option means there must be a degeneracy in the system.

## Exercise 1.13 Uncertainty relation. ([15] pr. 1.20)

Find the ket that maximizes the uncertainty product

$$
\begin{equation*}
\left\langle\left(\Delta S_{x}\right)^{2}\right\rangle\left\langle\left(\Delta S_{y}\right)^{2}\right\rangle \tag{1.182}
\end{equation*}
$$

and compare to the uncertainty bound $\frac{1}{4}\left|\left\langle\left[S_{x}, S_{y}\right]\right\rangle\right|^{2}$.
Answer for Exercise 1.13
To parameterize the ket space, consider first the kets that where both components are both not zero, where a single complex number can parameterize the ket

$$
|s\rangle=\left[\begin{array}{c}
\beta^{\prime} e^{i \phi^{\prime}}  \tag{1.183}\\
\alpha^{\prime} e^{i \theta^{\prime}}
\end{array}\right] \propto\left[\begin{array}{c}
1 \\
\alpha e^{i \theta}
\end{array}\right]
$$

The expectation values with respect to this ket are

$$
\begin{align*}
\left\langle S_{x}\right\rangle & =\frac{\hbar}{2}\left[\begin{array}{ll}
1 & \alpha e^{-i \theta}
\end{array}\right]\left[\begin{array}{cc}
0 & 1 \\
1 & 0
\end{array}\right]\left[\begin{array}{c}
1 \\
\alpha e^{i \theta}
\end{array}\right] \\
& =\frac{\hbar}{2}\left[\begin{array}{ll}
1 & \alpha e^{-i \theta}
\end{array}\right]\left[\begin{array}{c}
\alpha e^{i \theta} \\
1
\end{array}\right]  \tag{1.184}\\
& =\frac{\hbar}{2} \alpha e^{i \theta}+\alpha e^{-i \theta} \\
& =\frac{\hbar}{2} 2 \alpha \cos \theta \\
& =\hbar \alpha \cos \theta .
\end{align*}
$$

$$
\begin{align*}
\left\langle S_{y}\right\rangle & =\frac{\hbar}{2}\left[\begin{array}{ll}
1 & \alpha e^{-i \theta}
\end{array}\right]\left[\begin{array}{cc}
0 & -i \\
i & 0
\end{array}\right]\left[\begin{array}{c}
1 \\
\alpha e^{i \theta}
\end{array}\right] \\
& =\frac{i \hbar}{2}\left[\begin{array}{ll}
1 & \alpha e^{-i \theta}
\end{array}\right]\left[\begin{array}{c}
-\alpha e^{i \theta} \\
1
\end{array}\right]  \tag{1.185}\\
& =\frac{-i \alpha \hbar}{2} 2 i \sin \theta \\
& =\alpha \hbar \sin \theta .
\end{align*}
$$

With $C_{\theta}=\cos \theta, S_{\theta}=\sin \theta$, the variances are

$$
\begin{aligned}
\left(\Delta S_{x}\right)^{2} & =\left(\frac{\hbar}{2}\left[\begin{array}{cc}
-2 \alpha C_{\theta} & 1 \\
1 & -2 \alpha C_{\theta}
\end{array}\right]\right)^{2} \\
& =\frac{\hbar^{2}}{4}\left[\begin{array}{cc}
-2 \alpha C_{\theta} & 1 \\
1 & -2 \alpha C_{\theta}
\end{array}\right]\left[\begin{array}{cc}
-2 \alpha C_{\theta} & 1 \\
1 & -2 \alpha C_{\theta}
\end{array}\right] \\
& =\frac{\hbar^{2}}{4}\left[\begin{array}{cc}
4 \alpha^{2} C_{\theta}^{2}+1 & -4 \alpha C_{\theta} \\
-4 \alpha C_{\theta} & 4 \alpha^{2} C_{\theta}^{2}+1
\end{array}\right]
\end{aligned}
$$

and

$$
\begin{align*}
\left(\Delta S_{y}\right)^{2} & =\left(\frac{\hbar}{2}\left[\begin{array}{cc}
-2 \alpha S_{\theta} & -i \\
i & -2 \alpha S_{\theta}
\end{array}\right]\right)^{2} \\
& =\frac{\hbar^{2}}{4}\left[\begin{array}{cc}
-2 \alpha S_{\theta} & -i \\
i & -2 \alpha S_{\theta}
\end{array}\right]\left[\begin{array}{cc}
-2 \alpha S_{\theta} & -i \\
i & -2 \alpha S_{\theta}
\end{array}\right]  \tag{1.187}\\
& =\frac{\hbar^{2}}{4}\left[\begin{array}{cc}
4 \alpha^{2} S_{\theta}^{2}+1 & 4 \alpha i S_{\theta} \\
-4 \alpha i S_{\theta} & 4 \alpha^{2} S_{\theta}^{2}+1
\end{array}\right]
\end{align*}
$$

The uncertainty factors are

$$
\begin{align*}
\left\langle\left(\Delta S_{x}\right)^{2}\right\rangle & =\frac{\hbar^{2}}{4}\left[\begin{array}{ll}
1 & \alpha e^{-i \theta}
\end{array}\right]\left[\begin{array}{cc}
4 \alpha^{2} C_{\theta}^{2}+1 & -4 \alpha C_{\theta} \\
-4 \alpha C_{\theta} & 4 \alpha^{2} C_{\theta}^{2}+1
\end{array}\right]\left[\begin{array}{c}
1 \\
\alpha e^{i \theta}
\end{array}\right] \\
& =\frac{\hbar^{2}}{4}\left[\begin{array}{ll}
1 & \alpha e^{-i \theta}
\end{array}\right]\left[\begin{array}{c}
4 \alpha^{2} C_{\theta}^{2}+1-4 \alpha^{2} C_{\theta} e^{i \theta} \\
-4 \alpha C_{\theta}+4 \alpha^{3} C_{\theta}^{2} e^{i \theta}+\alpha e^{i \theta}
\end{array}\right] \\
& =\frac{\hbar^{2}}{4}\left(4 \alpha^{2} C_{\theta}^{2}+1-4 \alpha^{2} C_{\theta} e^{i \theta}-4 \alpha^{2} C_{\theta} e^{-i \theta}+4 \alpha^{4} C_{\theta}^{2}+\alpha^{2}\right) \\
& =\frac{\hbar^{2}}{4}\left(4 \alpha^{2} C_{\theta}^{2}+1-8 \alpha^{2} C_{\theta}^{2}+4 \alpha^{4} C_{\theta}^{2}+\alpha^{2}\right) \\
& =\frac{\hbar^{2}}{4}\left(-4 \alpha^{2} C_{\theta}^{2}+1+4 \alpha^{4} C_{\theta}^{2}+\alpha^{2}\right) \\
& =\frac{\hbar^{2}}{4}\left(4 \alpha^{2} C_{\theta}^{2}\left(\alpha^{2}-1\right)+\alpha^{2}+1\right) \tag{1.188}
\end{align*}
$$

and

$$
\begin{align*}
\left\langle\left(\Delta S_{y}\right)^{2}\right\rangle & =\frac{\hbar^{2}}{4}\left[\begin{array}{ll}
1 & \alpha e^{-i \theta}
\end{array}\right]\left[\begin{array}{cc}
4 \alpha^{2} S_{\theta}^{2}+1 & 4 \alpha i S_{\theta} \\
-4 \alpha i S_{\theta} & 4 \alpha^{2} S_{\theta}^{2}+1
\end{array}\right]\left[\begin{array}{c}
1 \\
\alpha e^{i \theta}
\end{array}\right] \\
& =\frac{\hbar^{2}}{4}\left[\begin{array}{ll}
1 & \alpha e^{-i \theta}
\end{array}\right]\left[\begin{array}{c}
4 \alpha^{2} S_{\theta}^{2}+1+4 \alpha^{2} i S_{\theta} e^{i \theta} \\
-4 \alpha i S_{\theta}+4 \alpha^{3} S_{\theta}^{2} e^{i \theta}+\alpha e^{i \theta}
\end{array}\right] \\
& =\frac{\hbar^{2}}{4}\left(4 \alpha^{2} S_{\theta}^{2}+1+4 \alpha^{2} i S_{\theta} e^{i \theta}-4 \alpha^{2} i S_{\theta} e^{-i \theta}+4 \alpha^{4} S_{\theta}^{2}+\alpha^{2}\right) \\
& =\frac{\hbar^{2}}{4}\left(-4 \alpha^{2} S_{\theta}^{2}+1+4 \alpha^{4} S_{\theta}^{2}+\alpha^{2}\right) \\
& =\frac{\hbar^{2}}{4}\left(4 \alpha^{2} S_{\theta}^{2}\left(\alpha^{2}-1\right)+\alpha^{2}+1\right) \tag{1.189}
\end{align*}
$$

The uncertainty product can finally be calculated

$$
\begin{aligned}
\left\langle\left(\Delta S_{x}\right)^{2}\right\rangle & \left\langle\left(\Delta S_{y}\right)^{2}\right\rangle \\
& =\left(\frac{\hbar}{2}\right)^{4}\left(4 \alpha^{2} C_{\theta}^{2}\left(\alpha^{2}-1\right)+\alpha^{2}+1\right)\left(4 \alpha^{2} S_{\theta}^{2}\left(\alpha^{2}-1\right)+\alpha^{2}+1\right) \\
& =\left(\frac{\hbar}{2}\right)^{4}\left(4 \alpha^{4} \sin ^{2}(2 \theta)\left(\alpha^{2}-1\right)+4 \alpha^{2}\left(\alpha^{4}-1\right)+\left(\alpha^{2}+1\right)^{2}\right)
\end{aligned}
$$

The maximum occurs when $f=\sin ^{2} 2 \theta$ is extremized. Those points are

$$
\begin{align*}
0 & =\frac{\partial f}{\partial \theta} \\
& =2 \sin 2 \theta \cos 2 \theta  \tag{1.191}\\
& =4 \sin 4 \theta .
\end{align*}
$$

Those points are at $4 \theta=\pi n$, for integer $n$, or

$$
\begin{equation*}
\theta=\frac{\pi}{4} n, n \in[0,7], \tag{1.192}
\end{equation*}
$$

Minimums will occur when

$$
\begin{equation*}
0<\frac{\partial^{2} f}{\partial \theta^{2}}=8 \cos 4 \theta, \tag{1.193}
\end{equation*}
$$

or

$$
\begin{equation*}
n=0,2,4,6 . \tag{1.194}
\end{equation*}
$$

At these points $\sin ^{2} 2 \theta$ takes the values

$$
\begin{equation*}
\sin ^{2}\left(2 \frac{\pi}{4}\{0,2,4,6\}\right)=\sin ^{2}(\pi\{0,1,2,3\}) \in\{0\}, \tag{1.195}
\end{equation*}
$$

so the maximization of the uncertainty product can be reduced to that of

$$
\begin{equation*}
\left\langle\left(\Delta S_{x}\right)^{2}\right\rangle\left\langle\left(\Delta S_{y}\right)^{2}\right\rangle=\left(\frac{\hbar}{2}\right)^{4}\left(4 \alpha^{2}\left(\alpha^{4}-1\right)+\left(\alpha^{2}+1\right)^{2}\right) \tag{1.196}
\end{equation*}
$$

We seek

$$
\begin{align*}
0 & =\frac{\partial}{\partial \alpha}\left(4 \alpha^{2}\left(\alpha^{4}-1\right)+\left(\alpha^{2}+1\right)^{2}\right) \\
& =\left(8 \alpha\left(\alpha^{4}-1\right)+16 \alpha^{5}+4\left(\alpha^{2}+1\right) \alpha\right)  \tag{1.197}\\
& =4 \alpha\left(2 \alpha^{4}-2+4 \alpha^{4}+4 \alpha^{2}+4\right) \\
& =8 \alpha\left(3 \alpha^{4}+2 \alpha^{2}+1\right) .
\end{align*}
$$

The only real root of this polynomial is $\alpha=0$, so the ket where both $|+\rangle$ and $|-\rangle$ are not zero that maximizes the uncertainty product is

$$
|s\rangle=\left[\begin{array}{l}
1  \tag{1.198}\\
0
\end{array}\right]=|+\rangle .
$$

The search for this maximizing value excluded those kets proportional to $\left[\begin{array}{l}0 \\ 1\end{array}\right]^{\mathrm{T}}=|-\rangle$. Let's see the values of this uncertainty product at both $| \pm\rangle$, and compare to the uncertainty commutator. First $|s\rangle=|+\rangle$

$$
\begin{align*}
& \left\langle S_{x}\right\rangle=\left[\begin{array}{ll}
1 & 0
\end{array}\right]\left[\begin{array}{ll}
0 & 1 \\
1 & 0
\end{array}\right]\left[\begin{array}{l}
1 \\
0
\end{array}\right]=0  \tag{1.199}\\
& \left\langle S_{y}\right\rangle=\left[\begin{array}{ll}
1 & 0
\end{array}\right]\left[\begin{array}{cc}
0 & -i \\
i & 0
\end{array}\right]\left[\begin{array}{l}
1 \\
0
\end{array}\right]=0 \tag{1.200}
\end{align*}
$$

so

$$
\begin{align*}
& \left\langle\left(\Delta S_{x}\right)^{2}\right\rangle=\left(\frac{\hbar}{2}\right)^{2}\left[\begin{array}{ll}
1 & 0
\end{array}\right]\left[\begin{array}{l}
1 \\
0
\end{array}\right]=\left(\frac{\hbar}{2}\right)^{2}  \tag{1.201}\\
& \left\langle\left(\Delta S_{y}\right)^{2}\right\rangle=\left(\frac{\hbar}{2}\right)^{2}\left[\begin{array}{ll}
1 & 0
\end{array}\right]\left[\begin{array}{l}
1 \\
0
\end{array}\right]=\left(\frac{\hbar}{2}\right)^{2} . \tag{1.202}
\end{align*}
$$

For the commutator side of the uncertainty relation we have

$$
\begin{align*}
\frac{1}{4}\left|\left\langle\left[S_{x}, S_{y}\right]\right\rangle\right|^{2} & =\frac{1}{4}\left|\left\langle i \hbar S_{z}\right\rangle\right|^{2} \\
& =\left(\frac{\hbar}{2}\right)^{4}\left|\left[\begin{array}{ll}
1 & 0
\end{array}\right]\left[\begin{array}{cc}
1 & 0 \\
0 & -1
\end{array}\right]\left[\begin{array}{l}
1 \\
0
\end{array}\right]\right|^{2} \tag{1.203}
\end{align*}
$$

so for the $|+\rangle$ state we have an equality condition for the uncertainty relation

$$
\begin{equation*}
\left\langle\left(\Delta S_{x}\right)^{2}\right\rangle\left\langle\left(\Delta S_{y}\right)^{2}\right\rangle=\frac{1}{4}\left|\left\langle\left[S_{x}, S_{y}\right]\right\rangle\right|^{2}=\left(\frac{\hbar}{2}\right)^{4} \tag{1.204}
\end{equation*}
$$

It's reasonable to guess that the $|-\rangle$ state also matches the equality condition. Let's check

$$
\begin{align*}
& \left\langle S_{x}\right\rangle=\left[\begin{array}{ll}
0 & 1
\end{array}\right]\left[\begin{array}{ll}
0 & 1 \\
1 & 0
\end{array}\right]\left[\begin{array}{l}
0 \\
1
\end{array}\right]=0  \tag{1.205}\\
& \left\langle S_{y}\right\rangle=\left[\begin{array}{ll}
0 & 1
\end{array}\right]\left[\begin{array}{cc}
0 & -i \\
i & 0
\end{array}\right]\left[\begin{array}{l}
0 \\
1
\end{array}\right]=0, \tag{1.206}
\end{align*}
$$

so $\left\langle\left(\Delta S_{x}\right)^{2}\right\rangle=\left\langle\left(\Delta S_{y}\right)^{2}\right\rangle=(\hbar / 2)^{2}$.
For the commutator side of the uncertainty relation will be identical, so the equality of eq. (1.204) is satisfied for both $| \pm\rangle$. Note that it wasn't explicitly verified that $|-\rangle$ maximized the uncertainty product, but I don't feel like working through that second set of algebraic mess.

We can see by example that equality does not mean that the equality condition means that the product is maximized. For example, it is straightforward to show that $\left|S_{x} ; \pm\right\rangle$ also satisfy the equality condition of the uncertainty relation. However, in that case the product is not maximized, but is zero.

Exercise 1.14 Degenerate ket space example. ([15] pr. 1.23)
Consider operators with representation

$$
A=\left[\begin{array}{ccc}
a & 0 & 0  \tag{1.207}\\
0 & -a & 0 \\
0 & 0 & -a
\end{array}\right], \quad B=\left[\begin{array}{ccc}
b & 0 & 0 \\
0 & 0 & -i b \\
0 & i b & 0
\end{array}\right]
$$

Show that these both have degeneracies, commute, and compute a simultaneous ket space for both operators.
Answer for Exercise 1.14
The eigenvalues and eigenvectors for $A$ can be read off by inspection, with values of $a,-a,-a$, and kets

$$
\left|a_{1}\right\rangle=\left[\begin{array}{l}
1  \tag{1.208}\\
0 \\
0
\end{array}\right], \quad\left|a_{2}\right\rangle=\left[\begin{array}{l}
0 \\
1 \\
0
\end{array}\right], \quad\left|a_{3}\right\rangle=\left[\begin{array}{l}
0 \\
0 \\
1
\end{array}\right]
$$

Notice that the lower-right $2 \times 2$ submatrix of $B$ is proportional to $\sigma_{y}$, so it's eigenvalues can be formed by inspection

$$
\left|b_{1}\right\rangle=\left[\begin{array}{l}
1  \tag{1.209}\\
0 \\
0
\end{array}\right], \quad\left|b_{2}\right\rangle=\frac{1}{\sqrt{2}}\left[\begin{array}{l}
0 \\
1 \\
i
\end{array}\right], \quad\left|b_{3}\right\rangle=\frac{1}{\sqrt{2}}\left[\begin{array}{c}
0 \\
1 \\
-i
\end{array}\right] .
$$

Computing $B\left|b_{i}\right\rangle$ shows that the eigenvalues are $b, b,-b$ respectively.
Because of the two-fold degeneracy in the $-a$ eigenvalues of $A$, any linear combination of $\left|a_{2}\right\rangle,\left|a_{3}\right\rangle$ will also be an eigenket. In particular,

$$
\begin{align*}
\left|a_{2}\right\rangle+i\left|a_{3}\right\rangle & =\left|b_{2}\right\rangle \\
\left|a_{2}\right\rangle-i\left|a_{3}\right\rangle & =\left|b_{3}\right\rangle \tag{1.210}
\end{align*}
$$

so the basis $\left\{\left|b_{i}\right\rangle\right\}$ is a simultaneous eigenspace for both $A$ and $B$. Because there is a simultaneous eigenspace, the matrices must commute. This can be confirmed with direct computation

$$
\begin{align*}
A B & =a b\left[\begin{array}{ccc}
1 & 0 & 0 \\
0 & -1 & 0 \\
0 & 0 & -1
\end{array}\right]\left[\begin{array}{ccc}
1 & 0 & 0 \\
0 & 0 & -i \\
0 & i & 0
\end{array}\right] \\
& =a b\left[\begin{array}{ccc}
1 & 0 & 0 \\
0 & 0 & i \\
0 & -i & 0
\end{array}\right], \tag{1.211}
\end{align*}
$$

and

$$
\begin{align*}
B A & =a b\left[\begin{array}{ccc}
1 & 0 & 0 \\
0 & 0 & -i \\
0 & i & 0
\end{array}\right]\left[\begin{array}{ccc}
1 & 0 & 0 \\
0 & -1 & 0 \\
0 & 0 & -1
\end{array}\right] \\
& =a b\left[\begin{array}{ccc}
1 & 0 & 0 \\
0 & 0 & i \\
0 & -i & 0
\end{array}\right] . \tag{1.212}
\end{align*}
$$

## Exercise 1.15 Unitary transformation. ([15] pr. 1.26)

Construct the transformation matrix that maps between the $S_{z}$ diagonal basis, to the $S_{x}$ diagonal basis.
Answer for Exercise 1.15
Based on the definition

$$
\begin{equation*}
U\left|a^{(r)}\right\rangle=\left|b^{(r)}\right\rangle \tag{1.213}
\end{equation*}
$$

the matrix elements can be computed

$$
\begin{equation*}
\left\langle a^{(s)}\right| U\left|a^{(r)}\right\rangle=\left\langle a^{(s)} \mid b^{(r)}\right\rangle, \tag{1.214}
\end{equation*}
$$

that is

$$
\begin{align*}
& U=\left[\begin{array}{ll}
\left\langle a^{(1)}\right| U\left|a^{(1)}\right\rangle & \left\langle a^{(1)}\right| U\left|a^{(2)}\right\rangle \\
\left\langle a^{(2)}\right| U\left|a^{(1)}\right\rangle & \left\langle a^{(2)}\right| U\left|a^{(2)}\right\rangle
\end{array}\right] \\
& =\left[\begin{array}{ll}
\left\langle a^{(1)} \mid b^{(1)}\right\rangle & \left\langle a^{(1)} \mid b^{(2)}\right\rangle \\
\left\langle a^{(2)} \mid b^{(1)}\right\rangle & \left\langle a^{(2)} \mid b^{(2)}\right\rangle
\end{array}\right] \\
& =\frac{1}{\sqrt{2}}\left[\begin{array}{ll}
{\left[\begin{array}{ll}
1 & 0
\end{array}\right]\left[\begin{array}{l}
1 \\
1
\end{array}\right]} & {\left[\begin{array}{ll}
1 & 0
\end{array}\right]\left[\begin{array}{c}
1 \\
-1
\end{array}\right]} \\
{\left[\begin{array}{ll}
0 & 1
\end{array}\right]\left[\begin{array}{l}
1 \\
1
\end{array}\right]} & {\left[\begin{array}{ll}
0 & 1
\end{array}\right]\left[\begin{array}{c}
1 \\
-1
\end{array}\right]}
\end{array}\right]  \tag{1.215}\\
& =\frac{1}{\sqrt{2}}\left[\begin{array}{cc}
1 & 1 \\
1 & -1
\end{array}\right] .
\end{align*}
$$

As a similarity transformation, we have

$$
\begin{align*}
\left\langle b^{(r)}\right| S_{z}\left|b^{(s)}\right\rangle & =\left\langle b^{(r)} \mid a^{(t)}\right\rangle\left\langle a^{(t)}\right| S_{z}\left|a^{(u)}\right\rangle\left\langle a^{(u)} \mid b^{(s)}\right\rangle  \tag{1.216}\\
& =\left\langle a^{(r)} \mid U\right\rangle^{\dagger} a^{(t)}\left\langle a^{(t)}\right| S_{z}\left|a^{(u)}\right\rangle\left\langle a^{(u)}\right| U\left|a^{(s)}\right\rangle
\end{align*}
$$

or

$$
\begin{equation*}
S_{z}^{\prime}=U^{\dagger} S_{z} U \tag{1.217}
\end{equation*}
$$

Let's check that the computed similarity transformation does it's job.

$$
\begin{align*}
\sigma_{z}^{\prime} & =U^{\dagger} \sigma_{z} U \\
& =\frac{1}{2}\left[\begin{array}{cc}
1 & 1 \\
1 & -1
\end{array}\right]\left[\begin{array}{cc}
1 & 0 \\
0 & -1
\end{array}\right]\left[\begin{array}{cc}
1 & 1 \\
1 & -1
\end{array}\right] \\
& =\frac{1}{2}\left[\begin{array}{cc}
1 & -1 \\
1 & 1
\end{array}\right]\left[\begin{array}{cc}
1 & 1 \\
1 & -1
\end{array}\right]  \tag{1.218}\\
& =\frac{1}{2}\left[\begin{array}{ll}
0 & 2 \\
2 & 0
\end{array}\right] \\
& =\sigma_{x} .
\end{align*}
$$

The transformation matrix can also be computed more directly

$$
\begin{align*}
U & =U\left|a^{(r)}\right\rangle\left\langle a^{(r)}\right| \\
& =\left|b^{(r)}\right\rangle\left\langle a^{(r)}\right| \\
& =\frac{1}{\sqrt{2}}\left[\begin{array}{l}
1 \\
1
\end{array}\right]\left[\begin{array}{ll}
1 & 0
\end{array}\right]+\frac{1}{\sqrt{2}}\left[\begin{array}{c}
1 \\
-1
\end{array}\right]\left[\begin{array}{ll}
0 & 1
\end{array}\right]  \tag{1.219}\\
& =\frac{1}{\sqrt{2}}\left[\begin{array}{ll}
1 & 0 \\
1 & 0
\end{array}\right]+\frac{1}{\sqrt{2}}\left[\begin{array}{cc}
0 & 1 \\
0 & -1
\end{array}\right] \\
& =\frac{1}{\sqrt{2}}\left[\begin{array}{cc}
1 & 1 \\
1 & -1
\end{array}\right]
\end{align*}
$$

## Exercise 1.16 One dimensional translation operator. ([15] pr. 1.28)

a. Evaluate the classical Poisson bracket

$$
\begin{equation*}
[x, F(p)]_{\text {classical }} \tag{1.220}
\end{equation*}
$$

b. Evaluate the commutator

$$
\begin{equation*}
\left[x, e^{i p a / \hbar}\right] \tag{1.221}
\end{equation*}
$$

c. Using the result in b , prove that

$$
\begin{equation*}
e^{i p a / \hbar}\left|x^{\prime}\right\rangle \tag{1.222}
\end{equation*}
$$

is an eigenstate of the coordinate operator $x$.
Answer for Exercise 1.16

Part a.

$$
\begin{align*}
{[x, F(p)]_{\text {classical }} } & =\frac{\partial x}{\partial x} \frac{\partial F(p)}{\partial p}-\frac{\partial x}{\partial p} \frac{\partial F(p)}{\partial x}  \tag{1.223}\\
& =\frac{\partial F(p)}{\partial p}
\end{align*}
$$

Part b. Having worked backwards through these problems, the answer for this one dimensional problem can be obtained from eq. (1.244) and is

$$
\begin{equation*}
\left[x, e^{i p a / \hbar}\right]=a e^{i p a / \hbar} \tag{1.224}
\end{equation*}
$$

Part c.

$$
\begin{align*}
x e^{i p a / \hbar}\left|x^{\prime}\right\rangle & =\left(\left[x, e^{i p a / \hbar}\right] e^{i p a / \hbar} x+\right)\left|x^{\prime}\right\rangle \\
& =\left(a e^{i p a / \hbar}+e^{i p a / \hbar} x^{\prime}\right)\left|x^{\prime}\right\rangle  \tag{1.225}\\
& =\left(a+x^{\prime}\right)\left|x^{\prime}\right\rangle .
\end{align*}
$$

This demonstrates that $e^{i p a / \hbar}\left|x^{\prime}\right\rangle$ is an eigenstate of $x$ with eigenvalue $a+x^{\prime}$.

## Exercise $1.17 \quad$ Polynomial commutators. ([15] pr. 1.29)

a. For power series $F, G$, verify

$$
\begin{equation*}
\left[x_{k}, G(\mathbf{p})\right]=i \hbar \frac{\partial G}{\partial p_{k}}, \quad\left[p_{k}, F(\mathbf{x})\right]=-i \hbar \frac{\partial F}{\partial x_{k}} . \tag{1.226}
\end{equation*}
$$

b. Evaluate $\left[x^{2}, p^{2}\right]$, and compare to the classical Poisson bracket $\left[x^{2}, p^{2}\right]_{\text {classical }}$.
Answer for Exercise 1.17

Part a. Let

$$
\begin{align*}
& G(\mathbf{p})=\sum_{k l m} a_{k l m} p_{1}^{k} p_{2}^{l} p_{3}^{m} \\
& F(\mathbf{x})=\sum_{k l m} b_{k l m} x_{1}^{k} x_{2}^{l} x_{3}^{m} . \tag{1.227}
\end{align*}
$$

It is simpler to work with a specific $x_{k}$, say $x_{k}=y$. The validity of the general result will still be clear doing so. Expanding the commutator gives

$$
\begin{align*}
{[y, G(\mathbf{p})] } & =\sum_{k l m} a_{k l m}\left[y, p_{1}^{k} p_{2}^{l} p_{3}^{m}\right] \\
& =\sum_{k l m} a_{k l m}\left(y p_{1}^{k} p_{2}^{l} p_{3}^{m}-p_{1}^{k} p_{2}^{l} p_{3}^{m} y\right) \\
& =\sum_{k l m} a_{k l m}\left(p_{1}^{k} y p_{2}^{l} p_{3}^{m}-p_{1}^{k} y p_{2}^{l} p_{3}^{m}\right)  \tag{1.228}\\
& =\sum_{k l m} a_{k l m} p_{1}^{k}\left[y, p_{2}^{l}\right] p_{3}^{m}
\end{align*}
$$

From eq. (1.242), we have $\left[y, p_{2}^{l}\right]=l i \hbar p_{2}^{l-1}$, so

$$
\begin{align*}
{[y, G(\mathbf{p})] } & =\sum_{k l m} a_{k l m} p_{1}^{k}\left[y, p_{2}^{l}\right]\left(l i \hbar p_{2}^{l-1}\right) p_{3}^{m} \\
& =i \hbar \frac{\partial G(\mathbf{p})}{\partial y} \tag{1.229}
\end{align*}
$$

It is straightforward to show that $\left[p, x^{l}\right]=-l i \hbar x^{l-1}$, allowing for a similar computation of the momentum commutator

$$
\begin{align*}
{\left[p_{y}, F(\mathbf{x})\right] } & =\sum_{k l m} b_{k l m}\left[p_{y}, x_{1}^{k} x_{2}^{l} x_{3}^{m}\right] \\
& =\sum_{k l m} b_{k l m}\left(p_{y} x_{1}^{k} x_{2}^{l} x_{3}^{m}-x_{1}^{k} x_{2}^{l} x_{3}^{m} p_{y}\right) \\
& =\sum_{k l m} b_{k l m}\left(x_{1}^{k} p_{y} x_{2}^{l} x_{3}^{m}-x_{1}^{k} p_{y} x_{2}^{l} x_{3}^{m}\right) \\
& =\sum_{k l m} b_{k l m} x_{1}^{k}\left[p_{y}, x_{2}^{l}\right] x_{3}^{m}  \tag{1.230}\\
& =\sum_{k l m} b_{k l m} x_{1}^{k}\left(-l i \hbar x_{2}^{l-1}\right) x_{3}^{m} \\
& =-i \hbar \frac{\partial F(\mathbf{x})}{\partial p_{y}}
\end{align*}
$$

Part $b$. It isn't clear to me how the results above can be used directly to compute $\left[x^{2}, p^{2}\right]$. However, when the first term of such a commutator is a mononomial, it can be expanded in terms of an $x$ commutator

$$
\begin{align*}
{\left[x^{2}, G(\mathbf{p})\right] } & =x^{2} G-G x^{2} \\
& =x(x G)-G x^{2} \\
& =x([x, G]+G x)-G x^{2} \\
& =x[x, G]+(x G) x-G x^{2}  \tag{1.231}\\
& =x[x, G]+([x, G]+G x) x-G x^{2} \\
& =x[x, G]+[x, G] x .
\end{align*}
$$

Similarly,

$$
\begin{equation*}
\left[x^{3}, G(\mathbf{p})\right]=x^{2}[x, G]+x[x, G] x+[x, G] x^{2} \tag{1.232}
\end{equation*}
$$

An induction hypothesis can be formed

$$
\begin{equation*}
\left[x^{k}, G(\mathbf{p})\right]=\sum_{j=0}^{k-1} x^{k-1-j}[x, G] x^{j} \tag{1.233}
\end{equation*}
$$

and demonstrated

$$
\begin{align*}
{\left[x^{k+1}, G(\mathbf{p})\right] } & =x^{k+1} G-G x^{k+1} \\
& =x\left(x^{k} G\right)-G x^{k+1} \\
& =x\left(\left[x^{k}, G\right]+G x^{k}\right)-G x^{k+1} \\
& =x\left[x^{k}, G\right]+(x G) x^{k}-G x^{k+1} \\
& =x\left[x^{k}, G\right]+([x, G]+G x) x^{k}-G x^{k+1} \\
& =x\left[x^{k}, G\right]+[x, G] x^{k}  \tag{1.234}\\
& =x \sum_{j=0}^{k-1} x^{k-1-j}[x, G] x^{j}+[x, G] x^{k} \\
& =\sum_{j=0}^{k-1} x^{(k+1)-1-j}[x, G] x^{j}+[x, G] x^{k} \\
& =\sum_{j=0}^{k} x^{(k+1)-1-j}[x, G] x^{j} .
\end{align*}
$$

That was a bit overkill for this problem, but may be useful later. Application of this to the problem gives

$$
\begin{align*}
{\left[x^{2}, p^{2}\right] } & =x\left[x, p^{2}\right]+\left[x, p^{2}\right] x \\
& =x i \hbar \frac{\partial p^{2}}{\partial x}+i \hbar \frac{\partial p^{2}}{\partial x} x  \tag{1.235}\\
& =x 2 i \hbar p+2 i \hbar p x \\
& =i \hbar(2 x p+2 p x)
\end{align*}
$$

The classical commutator is

$$
\begin{align*}
{\left[x^{2}, p^{2}\right]_{\text {classical }} } & =\frac{\partial x^{2}}{\partial x} \frac{\partial p^{2}}{\partial p}-\frac{\partial x^{2}}{\partial p} \frac{\partial p^{2}}{\partial x} \\
& =2 x 2 p  \tag{1.236}\\
& =2 x p+2 p x
\end{align*}
$$

This demonstrates the expected relation between the classical and quantum commutators

$$
\begin{equation*}
\left[x^{2}, p^{2}\right]=i \hbar\left[x^{2}, p^{2}\right]_{\text {classical }} \tag{1.237}
\end{equation*}
$$

Exercise 1.18 Translation op., position expectation. ([15] pr. 1.30)
The translation operator for a finite spatial displacement is given by

$$
\begin{equation*}
\mathscr{T}(\mathbf{l})=\exp (-i \mathbf{p} \cdot \mathbf{l} / \hbar), \tag{1.238}
\end{equation*}
$$

where $\mathbf{p}$ is the momentum operator.
a. Evaluate

$$
\begin{equation*}
\left[x_{i}, \mathscr{T}(\mathbf{l})\right] . \tag{1.239}
\end{equation*}
$$

b. Demonstrate how the expectation value $\langle\mathbf{x}\rangle$ changes under translation.
Answer for Exercise 1.18

Part a. For clarity, let's set $x_{i}=y$. The general result will be clear despite doing so.

$$
\begin{equation*}
[y, \mathcal{T}(\mathbf{l})]=\sum_{k=0} \frac{1}{k!}\left(\frac{-i}{\hbar}\right)\left[y,(\mathbf{p} \cdot \mathbf{l})^{k}\right] . \tag{1.240}
\end{equation*}
$$

The commutator expands as

$$
\begin{align*}
{\left[y,(\mathbf{p} \cdot \mathbf{l})^{k}\right]+(\mathbf{p} \cdot \mathbf{l})^{k} y } & =y(\mathbf{p} \cdot \mathbf{l})^{k} \\
& =y\left(p_{x} l_{x}+p_{y} l_{y}+p_{z} l_{z}\right)(\mathbf{p} \cdot \mathbf{l})^{k-1} \\
& =\left(p_{x} l_{x} y+y p_{y} l_{y}+p_{z} l_{z} y\right)(\mathbf{p} \cdot \mathbf{l})^{k-1} \\
& =\left(p_{x} l_{x} y+l_{y}\left(p_{y} y+i \hbar\right)+p_{z} l_{z} y\right)(\mathbf{p} \cdot \mathbf{l})^{k-1} \\
& =(\mathbf{p} \cdot \mathbf{l}) y(\mathbf{p} \cdot \mathbf{l})^{k-1}+i \hbar l_{y}(\mathbf{p} \cdot \mathbf{l})^{k-1} \\
& =\cdots \\
& =(\mathbf{p} \cdot \mathbf{l})^{k-1} y(\mathbf{p} \cdot \mathbf{l})^{k-(k-1)}+(k-1) i \hbar l_{y}(\mathbf{p} \cdot \mathbf{l})^{k-1} \\
& =(\mathbf{p} \cdot \mathbf{l})^{k} y+k i \hbar l_{y}(\mathbf{p} \cdot \mathbf{l})^{k-1} . \tag{1.241}
\end{align*}
$$

In the above expansion, the commutation of $y$ with $p_{x}, p_{z}$ has been used. This gives, for $k \neq 0$,

$$
\begin{equation*}
\left[y,(\mathbf{p} \cdot \mathbf{l})^{k}\right]=k i \hbar l_{y}(\mathbf{p} \cdot \mathbf{l})^{k-1} \tag{1.242}
\end{equation*}
$$

Note that this also holds for the $k=0$ case, since $y$ commutes with the identity operator. Plugging back into the $\mathcal{T}$ commutator, we have

$$
\begin{align*}
{[y, \mathcal{T}(\mathbf{l})] } & =\sum_{k=1} \frac{1}{k!}\left(\frac{-i}{\hbar}\right) k i \hbar l_{y}(\mathbf{p} \cdot \mathbf{l})^{k-1} \\
& =l_{y} \sum_{k=1} \frac{1}{(k-1)!}\left(\frac{-i}{\hbar}\right)(\mathbf{p} \cdot \mathbf{l})^{k-1}  \tag{1.243}\\
& =l_{y} \mathcal{T}(\mathbf{l}) .
\end{align*}
$$

The same pattern clearly applies with the other $x_{i}$ values, providing the desired relation.

$$
\begin{equation*}
[\mathbf{x}, \mathcal{T}(\mathbf{l})]=\sum_{m=1}^{3} \mathbf{e}_{m} l_{m} \mathcal{T}(\mathbf{l})=\mathbf{l} \mathcal{T}(\mathbf{l}) \tag{1.244}
\end{equation*}
$$

Part b. Suppose that the translated state is defined as $\left|\alpha_{1}\right\rangle=\mathcal{T}(\mathbf{l})|\alpha\rangle$. The expectation value with respect to this state is

$$
\begin{aligned}
\left\langle\mathbf{x}^{\prime}\right\rangle & =\left\langle\alpha_{\mathbf{l}}\right| \mathbf{x}\left|\alpha_{\mathbf{1}}\right\rangle \\
& =\langle\alpha| \mathscr{T}^{\dagger}(\mathbf{l}) \mathbf{x}(\mathbf{I})|\alpha\rangle \\
& =\langle\alpha| \mathcal{T}^{\dagger} \mathbf{( l )}(\mathbf{x} \mathcal{T}(\mathbf{l}))|\alpha\rangle \\
& \left.=\langle\alpha| \mathcal{T}^{\dagger} \mathbf{(}\right)(\mathcal{T}(\mathbf{l}) \mathbf{x}+\mathbf{I}(\mathbf{I}))|\alpha\rangle \\
& =\langle\alpha| \mathcal{T}^{\dagger} \mathscr{T} \mathbf{x}+\mathbf{l} \mathcal{T}^{\dagger} \mathscr{T}|\alpha\rangle \\
& =\langle\alpha| \mathbf{x}|\alpha\rangle+\mathbf{l}\langle\alpha \mid \alpha\rangle \\
& =\langle\mathbf{x}\rangle+\mathbf{l} .
\end{aligned}
$$

## Exercise 1.19 Density matrix. (2015 psl.1)

Consider a spin- $1 / 2$ particle. The Hilbert space is two-dimensional, let us label the two states as $|\uparrow\rangle$ and $|\downarrow\rangle$. Write down the $2 \times 2$ density matrix which corresponds to the following pure states.
(i) $|\uparrow\rangle$
(ii) $\frac{1}{\sqrt{2}}(|\uparrow\rangle+|\downarrow\rangle)$
(iii) $\frac{1}{\sqrt{2}}(|\uparrow\rangle+i|\downarrow\rangle)$
(iv) At time $t=0$, let us start with the state $\frac{1}{\sqrt{2}}(|\uparrow\rangle+i|\downarrow\rangle)$, and consider time-evolution under the Hamiltonian $\hat{H}=-B S_{z}$, where $S_{z}$ is the zcomponent of the spin operator. This leads to eigenstates $|\uparrow\rangle$ with energy $-B \hbar / 2$, and $|\downarrow\rangle$ with energy $+B \hbar / 2$. The state $\frac{1}{\sqrt{2}}(|\uparrow\rangle+i|\downarrow\rangle)$ is not an eigenstate of this Hamiltonian and it will evolve in time. Find the state and the corresponding $2 \times 2$ density matrix of this system at a later time $t$.

Answer for Exercise 1.19
(i) This state in matrix form is

$$
|\psi\rangle=\left[\begin{array}{l}
1  \tag{1.246}\\
0
\end{array}\right]
$$

for which the density operator representation is

$$
\hat{\rho}=\left[\begin{array}{l}
1  \tag{1.247}\\
0
\end{array}\right]\left[\begin{array}{ll}
1 & 0
\end{array}\right]=\left[\begin{array}{ll}
1 & 0 \\
0 & 0
\end{array}\right] .
$$

(ii) This state in matrix form is

$$
|\psi\rangle=\frac{1}{\sqrt{2}}\left(\left[\begin{array}{l}
1  \tag{1.248}\\
0
\end{array}\right]+\left[\begin{array}{l}
0 \\
1
\end{array}\right]\right)=\frac{1}{\sqrt{2}}\left[\begin{array}{l}
1 \\
1
\end{array}\right]
$$

for which the density operator representation is

$$
\hat{\rho}=\frac{1}{2}\left[\begin{array}{l}
1  \tag{1.249}\\
1
\end{array}\right]\left[\begin{array}{ll}
1 & 1
\end{array}\right]=\frac{1}{2}\left[\begin{array}{ll}
1 & 1 \\
1 & 1
\end{array}\right] .
$$

(iii) This state in matrix form is

$$
|\psi\rangle=\frac{1}{\sqrt{2}}\left(\left[\begin{array}{l}
1  \tag{1.250}\\
0
\end{array}\right]+\left[\begin{array}{l}
0 \\
i
\end{array}\right]\right)=\frac{1}{\sqrt{2}}\left[\begin{array}{l}
1 \\
i
\end{array}\right]
$$

for which the density operator representation is

$$
\hat{\rho}=\frac{1}{2}\left[\begin{array}{l}
1  \tag{1.251}\\
i
\end{array}\right]\left[\begin{array}{ll}
1 & -i
\end{array}\right]=\frac{1}{2}\left[\begin{array}{cc}
1 & -i \\
i & 1
\end{array}\right] .
$$

(iv) The time evolution operator is

$$
\begin{align*}
U(t) & =e^{-i \hat{H} t / \hbar} \\
& =e^{i B S_{z} t / \hbar} \\
& =e^{i B \sigma_{z} t / 2} \\
& =\cos (B t / 2)+i \sigma_{z} \sin (B t / 2) \\
& =\left[\begin{array}{cc}
\cos (B t / 2) & 0 \\
0 & \cos (B t / 2)
\end{array}\right]+\left[\begin{array}{cc}
i \sin (B t / 2) & 0 \\
0 & -i \sin (B t / 2)
\end{array}\right] \\
& =\left[\begin{array}{cc}
e^{i B t / 2} & 0 \\
0 & e^{-i B t / 2}
\end{array}\right] \tag{1.252}
\end{align*}
$$

so the time evolved state is

$$
\begin{align*}
|\psi(t)\rangle & =\frac{1}{\sqrt{2}}\left[\begin{array}{cc}
e^{i B t / 2} & 0 \\
0 & e^{-i B t / 2}
\end{array}\right]\left[\begin{array}{l}
1 \\
i
\end{array}\right] \\
& =\frac{1}{\sqrt{2}}\left[\begin{array}{c}
e^{i B t / 2} \\
i e^{-i B t / 2}
\end{array}\right] \tag{1.253}
\end{align*}
$$

The density operator matrix representation is

$$
\begin{align*}
\hat{\rho}(t) & =\frac{1}{2}\left[\begin{array}{c}
e^{i B t / 2} \\
i e^{-i B t / 2}
\end{array}\right]\left[\begin{array}{ll}
e^{-i B t / 2} & -i e^{i B t / 2}
\end{array}\right] \\
& =\frac{1}{2}\left[\begin{array}{cc}
1 & -i e^{i B t} \\
i e^{-i B t} & 1
\end{array}\right] \tag{1.254}
\end{align*}
$$

As a check observe that this has the right value for $t=0$. This also checks against the slightly messier computation $\hat{p}(t)=U \hat{p}(0) U^{\dagger}$.

## Exercise $1.20 \quad$ Reduced density matrix. (2015 psl.2)

Consider two spin-1/2 particles, the Hilbert space is now 4-dimensional, with states $|\uparrow \uparrow\rangle,|\uparrow \downarrow\rangle,|\downarrow \uparrow\rangle,|\downarrow \downarrow\rangle$. Let us consider the following pure states:
(i) $\frac{1}{2}(|\uparrow \uparrow\rangle-|\uparrow \downarrow\rangle-|\downarrow \downarrow\rangle+|\downarrow \uparrow\rangle)$
(ii) $\frac{1}{\sqrt{2}}(|\uparrow \uparrow\rangle+|\downarrow \downarrow\rangle)$
(iii) $\frac{1}{\sqrt{5}}(|\uparrow \uparrow\rangle+2|\downarrow \downarrow\rangle)$

In each case, obtain the reduced $2 \times 2$ density matrix which describes the first spin, when we trace over the second spin. The von Neumann entanglement entropy is defined via $S_{\mathrm{vN}}=-\operatorname{tr}\left(\rho_{\mathrm{R}} \ln \rho_{\mathrm{R}}\right)$ where $\rho_{\mathrm{R}}$ is the reduced density matrix you have obtained above and the tr now refers to tracing over the first spin. Using the reduced density matrices you have obtained above, compute the corresponding $S_{\mathrm{vN}}$, and simply explain your result in words. Consider the Renyi entropy $S_{n}=\frac{1}{1-n} \ln \left(\operatorname{tr}\left(\rho_{\mathrm{R}}^{n}\right)\right)$. Prove that $S_{n \rightarrow 1}=S_{\mathrm{vN}}$, and compute $S_{n=2}$ for the above $\rho_{\mathrm{R}}$.
Answer for Exercise 1.20
reduced density operator matrix representation Some notation is useful to start. Let $|a\rangle_{1}$ represent a spin state for the first particle and $|a\rangle_{2}$ represent a spin state for the second particle. If the state under consideration is $|\psi\rangle$, the reduced density operator matrix representation, after summing over all the second particle states, is

$$
\begin{equation*}
\rho_{\mathrm{R}}=\sum_{a}{ }_{2}\langle a|(|\psi\rangle\langle\psi|)|a\rangle_{2} . \tag{1.255}
\end{equation*}
$$

Computing these ${ }_{2}\langle a \mid \psi\rangle$ brackets for each state will allow the each of reduced density matrix to be expressed easily.
(i) For this state we have

$$
\begin{align*}
{ }_{2}\langle\uparrow \mid \psi\rangle & =\frac{1}{2}{ }_{2}\langle\uparrow|(|\uparrow \uparrow\rangle-|\uparrow \downarrow\rangle-|\downarrow \downarrow\rangle+|\downarrow \uparrow\rangle) \\
& =\frac{1}{2}\left(|\uparrow\rangle_{1}+|\downarrow\rangle_{1}\right)  \tag{1.256}\\
& =\frac{1}{2}\left[\begin{array}{l}
1 \\
1
\end{array}\right]
\end{align*}
$$

and

$$
\begin{align*}
{ }_{2}\langle\downarrow \mid \psi\rangle & =\frac{1}{2}{ }_{2}\langle\downarrow|(|\uparrow \uparrow\rangle-|\uparrow \downarrow\rangle-|\downarrow \downarrow\rangle+|\downarrow \uparrow\rangle) \\
& =\frac{1}{2}\left(-|\uparrow\rangle_{1}-|\downarrow\rangle_{1}\right)  \tag{1.257}\\
& =\frac{1}{2}\left[\begin{array}{l}
-1 \\
-1
\end{array}\right]
\end{align*}
$$

so the reduced density operator matrix representation is

$$
\begin{align*}
\rho_{\mathrm{R}} & =\frac{1}{4}\left[\begin{array}{l}
1 \\
1
\end{array}\right]\left[\begin{array}{ll}
1 & 1
\end{array}\right]+\frac{1}{4}\left[\begin{array}{l}
-1 \\
-1
\end{array}\right]\left[\begin{array}{ll}
-1 & -1
\end{array}\right] \\
& =\frac{1}{2}\left[\begin{array}{ll}
1 & 1 \\
1 & 1
\end{array}\right] . \tag{1.258}
\end{align*}
$$

(ii) For this state we have

$$
{ }_{2}\langle\uparrow \mid \psi\rangle=\frac{1}{\sqrt{2}}{ }_{2}\langle\uparrow|(|\uparrow \uparrow\rangle+|\downarrow \downarrow\rangle)=\frac{1}{\sqrt{2}}|\uparrow\rangle_{1}=\frac{1}{\sqrt{2}}\left[\begin{array}{l}
1  \tag{1.259}\\
0
\end{array}\right],
$$

and

$$
{ }_{2}\langle\downarrow \mid \psi\rangle=\frac{1}{\sqrt{2}}{ }_{2}\langle\downarrow|(|\uparrow \uparrow\rangle+|\downarrow \downarrow\rangle)=\frac{1}{\sqrt{2}}|\downarrow\rangle_{1}=\frac{1}{\sqrt{2}}\left[\begin{array}{l}
0  \tag{1.260}\\
1
\end{array}\right],
$$

so the reduced density operator matrix representation is

$$
\begin{align*}
\rho_{\mathrm{R}} & =\frac{1}{2}\left[\begin{array}{l}
1 \\
0
\end{array}\right]\left[\begin{array}{ll}
1 & 0
\end{array}\right]+\frac{1}{2}\left[\begin{array}{l}
0 \\
1
\end{array}\right]\left[\begin{array}{ll}
0 & 1
\end{array}\right]  \tag{1.261}\\
& =\frac{1}{2}\left[\begin{array}{ll}
1 & 0 \\
0 & 1
\end{array}\right] .
\end{align*}
$$

(iii) For this state we have

$$
{ }_{2}\langle\uparrow \mid \psi\rangle=\frac{1}{\sqrt{5}}{ }_{2}\langle\uparrow|(|\uparrow \uparrow\rangle+2|\downarrow \downarrow\rangle)=\frac{1}{\sqrt{5}}|\uparrow\rangle_{1}=\frac{1}{\sqrt{5}}\left[\begin{array}{l}
1  \tag{1.262}\\
0
\end{array}\right],
$$

and

$$
{ }_{2}\langle\downarrow \mid \psi\rangle=\frac{1}{\sqrt{5}}{ }_{2}\langle\downarrow|(|\uparrow \uparrow\rangle+2|\downarrow \downarrow\rangle)=\frac{2}{\sqrt{5}}|\downarrow\rangle_{1}=\frac{2}{\sqrt{5}}\left[\begin{array}{l}
0  \tag{1.263}\\
1
\end{array}\right]
$$

so the reduced density operator matrix representation is

$$
\begin{align*}
\rho_{\mathrm{R}} & =\frac{1}{5}\left[\begin{array}{l}
1 \\
0
\end{array}\right]\left[\begin{array}{ll}
1 & 0
\end{array}\right]+\frac{4}{5}\left[\begin{array}{l}
0 \\
1
\end{array}\right]\left[\begin{array}{ll}
0 & 1
\end{array}\right] \\
& =\frac{1}{5}\left[\begin{array}{ll}
1 & 0 \\
0 & 4
\end{array}\right] . \tag{1.264}
\end{align*}
$$

von Neumann entropy. Here is the computation of the von Neumann entropy for each of these states.
(i) For this case, the eigenvalues of the reduced density matrix are $\{0,1\}$, so the von Neumann entropy is

$$
\begin{equation*}
S_{\mathrm{vN}}=-1 \ln 1-0 \ln 0=0 \tag{1.265}
\end{equation*}
$$

where the zero logarithm is interpreted in the $\operatorname{limit} \lim _{x \rightarrow 0}-x \ln x=$ 0 .
(ii) Since this reduced density matrix is diagonal, the von Neumann entropy can be calculated easily

$$
\begin{equation*}
S_{\mathrm{vN}}=2\left(-\frac{1}{2} \ln \frac{1}{2}\right)=\ln 2 \approx 0.69 \tag{1.266}
\end{equation*}
$$

(iii) This is also diagonal with von Neumann entropy of

$$
\begin{align*}
S_{\mathrm{vN}} & =-\frac{1}{5} \ln \frac{1}{5}-\frac{4}{5} \ln \frac{4}{5} \\
& =\frac{1}{5}(\ln 5+4 \ln (5 / 4))  \tag{1.267}\\
& \approx 0.50 .
\end{align*}
$$

In the diagonal representation the entropy of the reduced density operator for case (i) shows that it is the most ordered system, with a unit probability amplitude for one state of that representation, and zero amplitude for the other. Case (iii) of the reduced system is the next most ordered, with the more of the probability amplitude weighting associated with the first particle in the spin down state. This results in a greater than zero entropy. Case (ii) is the least ordered of the reduced systems with equal probability amplitudes for each first particle spin state, and has the maximum von Neumann entropy possible for a two dimensional Hilbert space.

Renyi entropy. Here is the computation of the Renyi entropy for $n=2$ for each of these states. That is

$$
\begin{equation*}
S_{n=2}=-\ln \operatorname{tr}\left(\rho_{\mathrm{R}}^{2}\right) \tag{1.268}
\end{equation*}
$$

(i) For this case the reduced density matrix is idempotent

$$
\rho_{\mathrm{R}}^{2}=\frac{1}{4}\left[\begin{array}{ll}
1 & 1  \tag{1.269}\\
1 & 1
\end{array}\right]\left[\begin{array}{ll}
1 & 1 \\
1 & 1
\end{array}\right]=\frac{1}{2}\left[\begin{array}{ll}
1 & 1 \\
1 & 1
\end{array}\right] .
$$

So the trace is the sum of the eigenvalues of $\rho_{\mathrm{R}}$, and the $n=2$ Renyi entropy is

$$
\begin{equation*}
S_{n=2}=-\ln (1+0)=0 \tag{1.270}
\end{equation*}
$$

(ii) For this case, the $n=2$ Renyi entropy is

$$
\begin{equation*}
S_{n=2}=-\ln \left(\frac{1}{2^{2}}+\frac{1}{2^{2}}\right)=\ln 2 \approx 0.69 \tag{1.271}
\end{equation*}
$$

(iii) Finally, for this case, the $n=2$ Renyi entropy is

$$
\begin{equation*}
S_{n=2}=-\ln \left(\frac{1}{25}+\frac{16}{25}\right)=\ln \frac{17}{25} \approx 0.39 \tag{1.272}
\end{equation*}
$$

Limit of Renyi entropy In a diagonal representation of the reduced density matrix, the $n \rightarrow 1$ summation in the numerator of the Renyi entropy

$$
\begin{equation*}
S_{n \rightarrow 1}=\lim _{n \rightarrow 1} \frac{\ln \sum_{k} \rho_{k k}^{n}}{1-n} \tag{1.273}
\end{equation*}
$$

tends to unity, so that logarithm tends to zero. This results in division of two nearly zero quantities, which can be evaluated using l'Hôpital's rule. That is

$$
\begin{align*}
S_{n \rightarrow 1} & =\frac{\left.\frac{d}{d n} \ln \sum_{k} \rho_{k k}^{n}\right|_{n=1}}{-1} \\
& =\frac{\left.\sum_{k} \frac{d}{d n} \rho_{k k}^{n}\right|_{n=1}}{-\left.\sum_{k} \rho_{k k}^{n}\right|_{n=1}}  \tag{1.274}\\
& =-\left.\sum_{k} \frac{d}{d n} \rho_{k k}^{n}\right|_{n=1} .
\end{align*}
$$

To evaluate the derivatives set $a^{n}=e^{y}$, or $y=n \ln a$. That gives

$$
\begin{align*}
\frac{d}{d n} a^{n} & =\frac{d}{d n} e^{n \ln a}  \tag{1.275}\\
& =a^{n} \ln a
\end{align*}
$$

which gives

$$
\begin{align*}
S_{n \rightarrow 1} & =-\left.\sum_{k} \rho_{k k}^{n} \ln \rho_{k k}\right|_{n=1}  \tag{1.276}\\
& =\sum_{k} \rho_{k k} \ln \rho_{k k},
\end{align*}
$$

which is precisely the von Neumann entropy.

## Exercise 1.21 Ensembles for spin one half. ([15] pr. 3.10)

a. Sakurai leaves it to the reader to verify that knowledge of the three ensemble averages [S_x], [S_y],[S_z] is sufficient to reconstruct the density operator for a spin one half system. Show this.
b. Show how the expectation values $\left\langle S_{x}\right\rangle,\left\langle S_{y}\right\rangle,\left\langle S_{x}\right\rangle$ fully determine the spin orientation for a pure ensemble.
Answer for Exercise 1.21

Part a. I'll do this in two parts, the first using a spin-up/down ensemble to see what form this has, then the general case. The general case is a bit messy algebraically. After first attempting it the hard way, I did the grunt work portion of that calculation in Mathematica, but then realized it's not so bad to do it manually.

Consider first an ensemble with density operator

$$
\begin{equation*}
\rho=w_{+}|+\rangle\langle+|+w_{-}|-\rangle\langle-|, \tag{1.277}
\end{equation*}
$$

where these are the $\mathbf{S} \cdot( \pm \hat{\mathbf{z}})$ eigenstates. The traces are

$$
\begin{align*}
\operatorname{tr}\left(\rho \sigma_{x}\right) & =\langle+| \rho \sigma_{x}|+\rangle+\langle-| \rho \sigma_{x}|-\rangle \\
& =\langle+| \rho\left[\begin{array}{cc}
0 & 1 \\
1 & 0
\end{array}\right]|+\rangle+\langle-| \rho\left[\begin{array}{cc}
0 & 1 \\
1 & 0
\end{array}\right]|-\rangle \\
& =\langle+|\left(w_{+}|+\rangle\langle+|+w_{-}|-\rangle\langle-|\right)|-\rangle+\langle-|\left(w_{+}|+\rangle\langle+|+w_{-}|-\rangle\langle-|\right)|+\rangle \\
& =\langle+| w_{-}|-\rangle+\langle-| w_{+}|+\rangle \\
& =0, \tag{1.278}
\end{align*}
$$

$$
\begin{align*}
\operatorname{tr}\left(\rho \sigma_{y}\right)= & \langle+| \rho \sigma_{y}|+\rangle+\langle-| \rho \sigma_{y}|-\rangle \\
= & \langle+| \rho\left[\begin{array}{cc}
0 & -i \\
i & 0
\end{array}\right]|+\rangle+\langle-| \rho\left[\begin{array}{cc}
0 & -i \\
i & 0
\end{array}\right]|-\rangle \\
= & i\langle+|\left(w_{+}|+\rangle\langle+|+w_{-}|-\rangle\langle-|\right)|-\rangle  \tag{1.279}\\
& -i\langle-|\left(w_{+}|+\rangle\langle+|+w_{-}|-\rangle\langle-|\right)|+\rangle \\
= & i\langle+| w_{-}|-\rangle-i\langle-| w_{+}|+\rangle \\
= & 0
\end{align*}
$$

and

$$
\begin{align*}
\operatorname{tr}\left(\rho \sigma_{z}\right) & =\langle+| \rho \sigma_{z}|+\rangle+\langle-| \rho \sigma_{z}|-\rangle \\
& =\langle+| \rho|+\rangle-\langle-| \rho|-\rangle \\
& =\langle+|\left(w_{+}|+\rangle\langle+|+w_{-}|-\rangle\langle-|\right)|+\rangle-\langle-|\left(w_{+}|+\rangle\langle+|+w_{-}|-\rangle\langle-|\right)|-\rangle \\
& =\langle+| w_{+}|+\rangle-\langle-| w_{-}|-\rangle \\
& =w_{+}-w_{-} \tag{1.280}
\end{align*}
$$

Since $w_{+}+w_{-}=1$, this gives

$$
\begin{align*}
& w_{+}=\frac{1+\operatorname{tr}\left(\rho \sigma_{z}\right)}{2} \\
& w_{-}=\frac{1-\operatorname{tr}\left(\rho \sigma_{z}\right)}{2} \tag{1.281}
\end{align*}
$$

Attempting to do a similar set of trace expansions this way for a more general spin basis turns out to be a really bad idea and horribly messy. So much so that I resorted to spinOneHalfSymbolicManipulation.nb to do this symbolic work. However, it's not so bad if the trace is done completely in matrix form.

Using the basis

$$
\begin{align*}
& |\mathbf{S} \cdot \hat{\mathbf{n}} ;+\rangle=\left[\begin{array}{c}
\cos (\theta / 2) \\
\sin (\theta / 2) e^{i \phi}
\end{array}\right] \\
& |\mathbf{S} \cdot \hat{\mathbf{n}} ;-\rangle=\left[\begin{array}{c}
\sin (\theta / 2) e^{-i \phi} \\
-\cos (\theta / 2)
\end{array}\right], \tag{1.282}
\end{align*}
$$

the projector matrices are

$$
\begin{align*}
|\mathbf{S} \cdot \hat{\mathbf{n}} ;+\rangle\langle\mathbf{S} \cdot \hat{\mathbf{n}} ;+| & =\left[\begin{array}{cc}
\cos (\theta / 2) \\
\sin (\theta / 2) e^{i \phi}
\end{array}\right]\left[\begin{array}{lc}
\cos (\theta / 2) & \sin (\theta / 2) e^{-i \phi}
\end{array}\right] \\
& =\left[\begin{array}{cc}
\cos ^{2}(\theta / 2) & \cos (\theta / 2) \sin (\theta / 2) e^{-i \phi} \\
\sin (\theta / 2) \cos (\theta / 2) e^{i \phi} & \sin ^{2}(\theta / 2)
\end{array}\right], \tag{1.283}
\end{align*}
$$

$$
\begin{align*}
|\mathbf{S} \cdot \hat{\mathbf{n}} ;-\rangle\langle\mathbf{S} \cdot \hat{\mathbf{n}} ;-| & =\left[\begin{array}{cc}
\sin (\theta / 2) e^{-i \phi} \\
-\cos (\theta / 2)
\end{array}\right]\left[\begin{array}{ll}
\sin (\theta / 2) e^{i \phi} & -\cos (\theta / 2)
\end{array}\right] \\
& =\left[\begin{array}{cc}
\sin ^{2}(\theta / 2) & -\cos (\theta / 2) \sin (\theta / 2) e^{-i \phi} \\
-\cos (\theta / 2) \sin (\theta / 2) e^{i \phi} & \cos ^{2}(\theta / 2)
\end{array}\right] . \tag{1.284}
\end{align*}
$$

With $C=\cos (\theta / 2), S=\sin (\theta / 2)$, a general density operator in this basis has the form

$$
\begin{align*}
\rho & =w_{+}\left[\begin{array}{cc}
C^{2} & C S e^{-i \phi} \\
S C e^{i \phi} & S^{2}
\end{array}\right]+w_{-}\left[\begin{array}{cc}
S^{2} & -C S e^{-i \phi} \\
-C S e^{i \phi} & C^{2}
\end{array}\right]  \tag{1.285}\\
& =\left[\begin{array}{cc}
w_{+} C^{2}+w_{-} S^{2} & \left(w_{+}-w_{-}\right) C S e^{-i \phi} \\
\left(w_{+}-w_{-}\right) S C e^{i \phi} & w_{+} S^{2}+w_{-} C^{2}
\end{array}\right]
\end{align*}
$$

The products with the Pauli matrices are

$$
\begin{align*}
\rho \sigma_{x} & =\left[\begin{array}{cc}
w_{+} C^{2}+w_{-} S^{2} & \left(w_{+}-w_{-}\right) C S e^{-i \phi} \\
\left(w_{+}-w_{-}\right) S C e^{i \phi} & w_{+} S^{2}+w_{-} C^{2}
\end{array}\right]\left[\begin{array}{ll}
0 & 1 \\
1 & 0
\end{array}\right]  \tag{1.286}\\
& =\left[\begin{array}{cc}
\left(w_{+}-w_{-}\right) C S e^{-i \phi} & w_{+} C^{2}+w_{-} S^{2} \\
w_{+} S^{2}+w_{-} C^{2} & \left(w_{+}-w_{-}\right) S C e^{i \phi}
\end{array}\right] \\
\rho \sigma_{y} & =\left[\begin{array}{cc}
w_{+} C^{2}+w_{-} S^{2} & \left(w_{+}-w_{-}\right) C S e^{-i \phi} \\
\left(w_{+}-w_{-}\right) S C e^{i \phi} & w_{+} S^{2}+w_{-} C^{2}
\end{array}\right]\left[\begin{array}{cc}
0 & -i \\
i & 0
\end{array}\right]  \tag{1.287}\\
& =i\left[\begin{array}{cc}
\left(w_{+}-w_{-}\right) C S e^{-i \phi} & -w_{+} C^{2}-w_{-} S^{2} \\
w_{+} S^{2}+w_{-} C^{2} & -\left(w_{+}-w_{-}\right) S C e^{i \phi}
\end{array}\right]
\end{align*}
$$

$$
\begin{align*}
\rho \sigma_{z} & =\left[\begin{array}{cc}
w_{+} C^{2}+w_{-} S^{2} & \left(w_{+}-w_{-}\right) C S e^{-i \phi} \\
\left(w_{+}-w_{-}\right) S C e^{i \phi} & w_{+} S^{2}+w_{-} C^{2}
\end{array}\right]\left[\begin{array}{cc}
1 & 0 \\
0 & -1
\end{array}\right]  \tag{1.288}\\
& =\left[\begin{array}{cc}
w_{+} C^{2}+w_{-} S^{2} & -\left(w_{+}-w_{-}\right) C S e^{-i \phi} \\
\left(w_{+}-w_{-}\right) S C e^{i \phi} & -\left(w_{+} S^{2}+w_{-} C^{2}\right)
\end{array}\right] .
\end{align*}
$$

The respective traces can be read right off the matrices

$$
\begin{align*}
& \operatorname{tr}\left(\rho \sigma_{x}\right)=\left(w_{+}-w_{-}\right) \sin \theta \cos \phi \\
& \operatorname{tr}\left(\rho \sigma_{y}\right)=\left(w_{+}-w_{-}\right) \sin \theta \sin \phi  \tag{1.289}\\
& \operatorname{tr}\left(\rho \sigma_{z}\right)=\left(w_{+}-w_{-}\right) \cos \theta .
\end{align*}
$$

This gives

$$
\begin{equation*}
\left(w_{+}-w_{-}\right) \hat{\mathbf{n}}=\left(\operatorname{tr}\left(\rho \sigma_{x}\right), \operatorname{tr}\left(\rho \sigma_{y}\right), \operatorname{tr}\left(\rho \sigma_{z}\right)\right) \tag{1.290}
\end{equation*}
$$

or

$$
\begin{equation*}
w_{ \pm}=\frac{1 \pm \sqrt{\operatorname{tr}^{2}\left(\rho \sigma_{x}\right)+\operatorname{tr}^{2}\left(\rho \sigma_{y}\right)+\operatorname{tr}^{2}\left(\rho \sigma_{z}\right)}}{2} \tag{1.291}
\end{equation*}
$$

So, as claimed, it's possible to completely describe the ensemble weight factors using the ensemble averages of $\left[S_{x}\right],\left[S_{y}\right],\left[S_{z}\right]$. I used the Pauli matrices instead, but the difference is just an $\hbar / 2$ scaling adjustment.

Alternate approach Another easier and trig free way to look at this problem is assume the density operator's representation is given by a $2 \times 2$ matrix with undetermined values

$$
\rho=\left[\begin{array}{ll}
a & b  \tag{1.292}\\
c & d
\end{array}\right]
$$

For such a representation we have

$$
\begin{align*}
& \rho \sigma_{x}=\left[\begin{array}{ll}
a & b \\
c & d
\end{array}\right]\left[\begin{array}{ll}
0 & 1 \\
1 & 0
\end{array}\right]=\left[\begin{array}{ll}
b & a \\
d & c
\end{array}\right] \\
& \rho \sigma_{y}=\left[\begin{array}{ll}
a & b \\
c & d
\end{array}\right]\left[\begin{array}{cc}
0 & -i \\
i & 0
\end{array}\right]=i\left[\begin{array}{ll}
b & -a \\
d & -c
\end{array}\right]  \tag{1.293}\\
& \rho \sigma_{z}=\left[\begin{array}{ll}
a & b \\
c & d
\end{array}\right]\left[\begin{array}{cc}
1 & 0 \\
0 & -1
\end{array}\right]=\left[\begin{array}{ll}
a & -b \\
c & -d
\end{array}\right] .
\end{align*}
$$

The ensemble averages can be read by inspection

$$
\begin{align*}
{\left[\sigma_{x}\right] } & =b+c \\
{\left[\sigma_{y}\right] } & =i(b-c)  \tag{1.294}\\
{\left[\sigma_{z}\right] } & =a-d
\end{align*}
$$

Noting that $\operatorname{tr}\left(E A E^{-1}\right)=\operatorname{tr}\left(A E^{-1} E\right)=\operatorname{tr}(A)$, and that there must be a diagonal basis for which $\langle+\mid-\rangle=0$ and

$$
\begin{equation*}
\rho=w_{+}|+\rangle\langle+|+w_{-}|-\rangle\langle-|, \tag{1.295}
\end{equation*}
$$

we must have

$$
\begin{equation*}
\operatorname{tr} \rho=a+d=w_{+}+w_{-}=1 \tag{1.296}
\end{equation*}
$$

This provides one set of equations for each of $b, c$ and $a, d$

$$
\begin{align*}
{\left[\sigma_{x}\right] } & =b+c  \tag{1.297}\\
{\left[\sigma_{y}\right] } & =i(b-c)
\end{align*}
$$

and

$$
\begin{align*}
{\left[\sigma_{z}\right] } & =a-d  \tag{1.298}\\
1 & =a+d
\end{align*}
$$

These have solutions

$$
\begin{align*}
& b=\frac{\left[\sigma_{x}\right]-i\left[\sigma_{y}\right]}{2} \\
& c=\frac{\left[\sigma_{x}\right]+i\left[\sigma_{y}\right]}{2}  \tag{1.299}\\
& a=\frac{1+\left[\sigma_{z}\right]}{2} \\
& d=\frac{1-\left[\sigma_{z}\right]}{2},
\end{align*}
$$

or

$$
\rho=\frac{1}{2}\left[\begin{array}{cc}
1+\left[\sigma_{z}\right] & {\left[\sigma_{x}\right]-i\left[\sigma_{y}\right]}  \tag{1.300}\\
{\left[\sigma_{x}\right]+i\left[\sigma_{y}\right]} & 1-\left[\sigma_{z}\right]
\end{array}\right]
$$

The characteristic equation for this operator is

$$
0=\left(\left(\frac{1}{2}-\lambda\right)+\frac{\left[\sigma_{z}\right]}{2}\right)\left(\left(\frac{1}{2}-\lambda\right)-\frac{\left[\sigma_{z}\right]}{2}\right)-\frac{1}{4}\left(\left[\sigma_{x}\right]+i\left[\sigma_{y}\right]\right)\left(\left[\sigma_{x}\right]-i\left[\sigma_{y}\right]\right)
$$

or

$$
\begin{equation*}
\lambda=\frac{1 \pm \sqrt{\left[\sigma_{x}\right]^{2}+\left[\sigma_{y}\right]^{2}+\left[\sigma_{z}\right]^{2}}}{2} \tag{1.302}
\end{equation*}
$$

as found above.

Part $b$. Suppose that the system is in the state $|\mathbf{S} \cdot \hat{\mathbf{n}} ;+\rangle$ as defined in eq. (1.282), then the expectation values of $\sigma_{x}, \sigma_{y}, \sigma_{z}$ with respect to this state are

$$
\begin{aligned}
\left\langle\sigma_{x}\right\rangle & =\left[\begin{array}{ll}
\cos (\theta / 2) & \sin (\theta / 2) e^{-i \phi}
\end{array}\right]\left[\begin{array}{ll}
0 & 1 \\
1 & 0
\end{array}\right]\left[\begin{array}{c}
\cos (\theta / 2) \\
\sin (\theta / 2) e^{i \phi}
\end{array}\right] \\
& =\left[\begin{array}{ll}
\cos (\theta / 2) & \sin (\theta / 2) e^{-i \phi}
\end{array}\right]\left[\begin{array}{c}
\sin (\theta / 2) e^{i \phi} \\
\cos (\theta / 2)
\end{array}\right] \\
& =\sin \theta \cos \phi
\end{aligned}
$$

$$
\left\langle\sigma_{y}\right\rangle
$$

$$
\begin{align*}
& =\left[\begin{array}{ll}
\cos (\theta / 2) & \sin (\theta / 2) e^{-i \phi}
\end{array}\right]\left[\begin{array}{cc}
0 & -i \\
i & 0
\end{array}\right]\left[\begin{array}{c}
\cos (\theta / 2) \\
\sin (\theta / 2) e^{i \phi}
\end{array}\right]  \tag{1.304}\\
& =i\left[\begin{array}{ll}
\cos (\theta / 2) & \sin (\theta / 2) e^{-i \phi}
\end{array}\right]\left[\begin{array}{c}
-\sin (\theta / 2) e^{i \phi} \\
\cos (\theta / 2)
\end{array}\right] \\
& =\sin \theta \sin \phi,
\end{align*}
$$

$$
\left\langle\sigma_{z}\right\rangle
$$

$$
=\left[\begin{array}{ll}
\cos (\theta / 2) & \sin (\theta / 2) e^{-i \phi}
\end{array}\right]\left[\begin{array}{cc}
1 & 0 \\
0 & -1
\end{array}\right]\left[\begin{array}{c}
\cos (\theta / 2) \\
\sin (\theta / 2) e^{i \phi}
\end{array}\right]
$$

$$
=\left[\begin{array}{ll}
\cos (\theta / 2) & \sin (\theta / 2) e^{-i \phi}
\end{array}\right]\left[\begin{array}{c}
\cos (\theta / 2)  \tag{1.305}\\
-\sin (\theta / 2) e^{i \phi}
\end{array}\right]
$$

$$
=\cos \theta
$$

So we have

$$
\begin{equation*}
\hat{\mathbf{n}}=\left(\left\langle\sigma_{x}\right\rangle,\left\langle\sigma_{y}\right\rangle,\left\langle\sigma_{z}\right\rangle\right) \tag{1.306}
\end{equation*}
$$

The spin direction is completely determined by this vector of expectation values (or equivalently, the expectation values of $S_{x}, S_{y}, S_{z}$ ).

### 2.1 CLASSICAL HARMONIC OSCILLATOR.

Recall the classical Harmonic oscillator equations in their Hamiltonian form

$$
\begin{equation*}
\frac{d x}{d t}=\frac{p}{m} \tag{2.1a}
\end{equation*}
$$

$$
\begin{equation*}
\frac{d p}{d t}=-k x \tag{2.1b}
\end{equation*}
$$

With

$$
\begin{align*}
x(t=0) & =x_{0} \\
p(t=0) & =p_{0}  \tag{2.2}\\
k & =m \omega^{2}
\end{align*}
$$

the solutions are ellipses in phase space

$$
\begin{equation*}
x(t)=x_{0} \cos (\omega t)+\frac{p_{0}}{m \omega} \sin (\omega t) \tag{2.3a}
\end{equation*}
$$

$$
\begin{equation*}
p(t)=p_{0} \cos (\omega t)-m \omega x_{0} \sin (\omega t) \tag{2.3b}
\end{equation*}
$$

After a suitable scaling of the variables, these elliptical orbits can be transformed into circular trajectories.

### 2.2 QUANTUM HARMONIC OSCILLATOR.

Starting with the Hamiltonian for the harmonic oscillator

$$
\begin{equation*}
\hat{H}=\frac{\hat{p}^{2}}{2 m}+\frac{1}{2} k \hat{x}^{2} \tag{2.4}
\end{equation*}
$$

set

$$
\begin{equation*}
\hat{X}=\sqrt{\frac{m \omega}{\hbar}} \hat{x}, \quad \hat{P}=\sqrt{\frac{1}{m \omega \hbar}} \hat{p} \tag{2.5}
\end{equation*}
$$

With this change of variables the commutators transform from

$$
\begin{equation*}
[\hat{x}, \hat{p}]=i \hbar, \tag{2.6}
\end{equation*}
$$

to

$$
\begin{equation*}
[\hat{X}, \hat{P}]=i \tag{2.7}
\end{equation*}
$$

and the Hamiltonian can be reexpressed in a factorized form

$$
\begin{align*}
\hat{H} & =\frac{\hbar \omega}{2}\left(\hat{X}^{2}+\hat{P}^{2}\right) \\
& =\hbar \omega\left(\left(\frac{\hat{X}-i \hat{P}}{\sqrt{2}}\right)\left(\frac{\hat{X}+i \hat{P}}{\sqrt{2}}\right)+\frac{1}{2}\right) . \tag{2.8}
\end{align*}
$$

Define ladder operators (raising and lowering operators respectively)

$$
\begin{align*}
& \hat{a}^{\dagger}=\frac{\hat{X}-i \hat{P}}{\sqrt{2}}  \tag{2.9a}\\
& \hat{a}=\frac{\hat{X}+i \hat{P}}{\sqrt{2}} \tag{2.9b}
\end{align*}
$$

SO

$$
\begin{equation*}
\hat{H}=\hbar \omega\left(\hat{a}^{\dagger} \hat{a}+\frac{1}{2}\right) . \tag{2.10}
\end{equation*}
$$

We can show

$$
\begin{equation*}
\left[\hat{a}, \hat{a}^{\dagger}\right]=1, \tag{2.11}
\end{equation*}
$$

and

$$
\begin{equation*}
N|n\rangle \equiv \hat{a}^{\dagger} a=n|n\rangle, \tag{2.12}
\end{equation*}
$$

where $n \geq 0$ is an integer.
Recall that

$$
\begin{equation*}
\hat{a}|0\rangle=0, \tag{2.13}
\end{equation*}
$$

and

$$
\begin{equation*}
\langle X| X+i P|0\rangle=0 \tag{2.14}
\end{equation*}
$$

With

$$
\begin{equation*}
\langle x \mid 0\rangle=\Psi_{0}(x) \tag{2.15}
\end{equation*}
$$

we can show

$$
\begin{equation*}
\frac{1}{\sqrt{2}}\left(X+\frac{\partial}{\partial X}\right) \Psi_{0}(X)=0 \tag{2.16}
\end{equation*}
$$

Also recall that

$$
\begin{equation*}
\hat{a}|n\rangle=\sqrt{n}|n-1\rangle \tag{2.17a}
\end{equation*}
$$

$$
\begin{equation*}
\hat{a}^{\dagger}|n\rangle=\sqrt{n+1}|n+1\rangle \tag{2.17b}
\end{equation*}
$$

### 2.3 COHERENT STATES.

Coherent states for the quantum harmonic oscillator are the eigenkets for the creation and annihilation operators

$$
\begin{equation*}
\hat{a}|z\rangle=z|z\rangle \tag{2.18a}
\end{equation*}
$$

$$
\begin{equation*}
\hat{a}^{\dagger}|\tilde{z}\rangle=\tilde{z}|\tilde{z}\rangle \tag{2.18b}
\end{equation*}
$$

where

$$
\begin{equation*}
|z\rangle=\sum_{n=0}^{\infty} c_{n}|n\rangle, \tag{2.19}
\end{equation*}
$$

and $z$ is allowed to be a complex number.
Looking for such a state, we compute

$$
\begin{align*}
\hat{a}|z\rangle & =\sum_{n=1}^{\infty} c_{n} \hat{a}|n\rangle \\
& =\sum_{n=1}^{\infty} c_{n} \sqrt{n}|n-1\rangle \tag{2.20}
\end{align*}
$$

Compare this to

$$
\begin{align*}
z|z\rangle & =z \sum_{n=0}^{\infty} c_{n}|n\rangle \\
& =\sum_{n=1}^{\infty} c_{n} \sqrt{n}|n-1\rangle  \tag{2.21}\\
& =\sum_{n=0}^{\infty} c_{n+1} \sqrt{n+1}|n\rangle,
\end{align*}
$$

SO

$$
\begin{equation*}
c_{n+1} \sqrt{n+1}=z c_{n} \tag{2.22}
\end{equation*}
$$

This gives

$$
\begin{align*}
& c_{n+1}=\frac{z c_{n}}{\sqrt{n+1}}  \tag{2.23}\\
& c_{1}=c_{0} z \\
& c_{2}=\frac{z c_{1}}{\sqrt{2}}=\frac{z^{2} c_{0}}{\sqrt{2}} \tag{2.24}
\end{align*}
$$

or

$$
\begin{equation*}
c_{n}=\frac{z^{n}}{\sqrt{n!}} . \tag{2.25}
\end{equation*}
$$

So the desired state is

$$
\begin{equation*}
|z\rangle=c_{0} \sum_{n=0}^{\infty} \frac{z^{n}}{\sqrt{n!}}|n\rangle . \tag{2.26}
\end{equation*}
$$

Also recall that

$$
\begin{equation*}
|n\rangle=\frac{\left(\hat{a}^{\dagger}\right)^{n}}{\sqrt{n!}}|0\rangle, \tag{2.27}
\end{equation*}
$$

which gives

$$
\begin{align*}
|z\rangle & =c_{0} \sum_{n=0}^{\infty} \frac{\left(z \hat{a}^{\dagger}\right)^{n}}{n!}|0\rangle  \tag{2.28}\\
& =c_{0} e^{z \hat{a}^{\dagger}}|0\rangle .
\end{align*}
$$

The normalization is

$$
\begin{equation*}
c_{0}=e^{-|z|^{2} / 2} \tag{2.29}
\end{equation*}
$$

While we have $\left\langle n_{1} \mid n_{2}\right\rangle=\delta_{n_{1}, n_{2}}$, these $|z\rangle$ states are not orthonormal. Figuring out that this overlap

$$
\begin{equation*}
\left\langle z_{1} \mid z_{2}\right\rangle \neq 0 \tag{2.30}
\end{equation*}
$$

will be left for homework.

### 2.4 COHERENT STATE TIME EVOLUTION.

We don't know much about these coherent states. For example does a coherent state at time zero evolve to a coherent state?

$$
\begin{equation*}
|z\rangle \xrightarrow{?}|z(t)\rangle . \tag{2.31}
\end{equation*}
$$

It turns out that these questions are best tackled in the Heisenberg picture, considering

$$
\begin{equation*}
e^{-i \hat{H} t / \hbar}|z\rangle . \tag{2.32}
\end{equation*}
$$

For example, what is the average of the position operator

$$
\begin{equation*}
\langle z| e^{i \hat{H} t / \hbar} \hat{x} e^{-i \hat{H} t / \hbar}|z\rangle=\sum_{n, n^{\prime}=0}^{\infty}\langle n| c_{n}^{*} e^{i E_{n} t / \hbar}\left(a+a^{\dagger}\right) \sqrt{\frac{\hbar}{m \omega}} c_{n^{\prime}} e^{i E_{n^{\prime}} t / \hbar}|n\rangle \tag{2.33}
\end{equation*}
$$

This is very messy to attempt. Instead if we know how the operator evolves we can calculate

$$
\begin{equation*}
\langle z| \hat{x}_{\mathrm{H}}(t)|z\rangle, \tag{2.34}
\end{equation*}
$$

that is

$$
\begin{equation*}
\langle\hat{x}\rangle(t)=\langle z| \hat{x}_{\mathrm{H}}(t)|z\rangle, \tag{2.35}
\end{equation*}
$$

and for momentum

$$
\begin{equation*}
\langle\hat{p}\rangle(t)=\langle z| \hat{p}_{\mathrm{H}}(t)|z\rangle . \tag{2.36}
\end{equation*}
$$

The question to ask is what are the expansions of

$$
\begin{align*}
& \hat{a}_{\mathrm{H}}(t)=e^{i \hat{H} t / \hbar} \hat{a} e^{-i \hat{H} t / \hbar}  \tag{2.37a}\\
& \hat{a}_{\mathrm{H}}^{\dagger}(t)=e^{i \hat{H} t / \hbar} \hat{a}^{\dagger} e^{-i \hat{H} t / \hbar} \tag{2.37b}
\end{align*}
$$

We want to understand how these operators act on the basis states

$$
\begin{align*}
\hat{a}_{\mathrm{H}}(t)|n\rangle & =e^{i \hat{H} t / \hbar} \hat{a} e^{-i \hat{H} t / \hbar}|n\rangle \\
& =e^{i \hat{H} t / \hbar} \hat{a} e^{-i t \omega(n+1 / 2)}|n\rangle \\
& =e^{-i t \omega(n+1 / 2)} e^{i \hat{H} t / \hbar} \sqrt{n}|n-1\rangle  \tag{2.38}\\
& =\sqrt{n} e^{-i t \omega(n+1 / 2)} e^{i t \omega(n-1 / 2)}|n-1\rangle \\
& =\sqrt{n} e^{-i \omega t}|n-1\rangle \\
& =e^{-i \omega t}|n\rangle
\end{align*}
$$

So we have found

$$
\begin{align*}
& \hat{a}_{\mathrm{H}}(t)=a e^{-i \omega t} \\
& \hat{a}_{\mathrm{H}}^{\dagger}(t)=a^{\dagger} e^{i \omega t} \tag{2.39}
\end{align*}
$$

### 2.5 EXPECTATION WITH RESPECT TO COHERENT STATES.

A coherent state for the SHO $H=\left(N+\frac{1}{2}\right) \hbar \omega$ was given by

$$
\begin{equation*}
a|z\rangle=z|z\rangle, \tag{2.40}
\end{equation*}
$$

where we showed that

$$
\begin{equation*}
|z\rangle=c_{0} e^{z a^{\dagger}}|0\rangle \tag{2.41}
\end{equation*}
$$

In the Heisenberg picture we found

$$
\begin{align*}
a_{\mathrm{H}}(t) & =e^{i H t / \hbar} a e^{-i H t / \hbar}=a e^{-i \omega t} \\
a_{\mathrm{H}}^{\dagger}(t) & =e^{i H t / \hbar} a^{\dagger} e^{-i H t / \hbar}=a^{\dagger} e^{i \omega t} \tag{2.42}
\end{align*}
$$

Recall that the position and momentum representation of the ladder operators was

$$
\begin{align*}
a & =\frac{1}{\sqrt{2}}\left(\hat{x} \sqrt{\frac{m \omega}{\hbar}}+i \hat{p} \sqrt{\frac{1}{m \hbar \omega}}\right) \\
a^{\dagger} & =\frac{1}{\sqrt{2}}\left(\hat{x} \sqrt{\frac{m \omega}{\hbar}}-i \hat{p} \sqrt{\frac{1}{m \hbar \omega}}\right), \tag{2.43}
\end{align*}
$$

or equivalently

$$
\begin{align*}
& \hat{x}=\left(a+a^{\dagger}\right) \sqrt{\frac{\hbar}{2 m \omega}}  \tag{2.44}\\
& \hat{p}=i\left(a^{\dagger}-a\right) \sqrt{\frac{m \hbar \omega}{2}}
\end{align*}
$$

Given this we can compute expectation value of position operator

$$
\begin{align*}
\langle z| \hat{x}|z\rangle & =\sqrt{\frac{\hbar}{2 m \omega}}\langle z|\left(a+a^{\dagger}\right)|z\rangle \\
& =\left(z+z^{*}\right) \sqrt{\frac{\hbar}{2 m \omega}}  \tag{2.45}\\
& =2 \operatorname{Re} z \sqrt{\frac{\hbar}{2 m \omega}}
\end{align*}
$$

Similarly

$$
\begin{align*}
\langle z| \hat{p}|z\rangle & =i \sqrt{\frac{m \hbar \omega}{2}}\langle z|\left(a^{\dagger}-a\right)|z\rangle  \tag{2.46}\\
& =\sqrt{\frac{m \hbar \omega}{2}} 2 \operatorname{Im} z .
\end{align*}
$$

How about the expectation of the Heisenberg position operator? That is

$$
\begin{align*}
\langle z| \hat{x}_{\mathrm{H}}(t)|z\rangle & =\sqrt{\frac{\hbar}{2 m \omega}}\langle z|\left(a+a^{\dagger}\right)|z\rangle \\
& =\sqrt{\frac{\hbar}{2 m \omega}}\left(z e^{-i \omega t}+z^{*} e^{i \omega t}\right) \\
& =\sqrt{\frac{\hbar}{2 m \omega}}\left(\left(z+z^{*}\right) \cos (\omega t)-i\left(z-z^{*}\right) \sin (\omega t)\right) \\
& =\sqrt{\frac{\hbar}{2 m \omega}}\left(\langle x(0)\rangle \sqrt{\frac{2 m \omega}{\hbar}} \cos (\omega t)-i\langle p(0)\rangle i \sqrt{\frac{2 m \omega}{\hbar}} \sin (\omega t)\right) \\
& =\langle x(0)\rangle \cos (\omega t)+\frac{\langle p(0)\rangle}{m \omega} \sin (\omega t) . \tag{2.47}
\end{align*}
$$

We find that the average of the Heisenberg position operator evolves in time in exactly the same fashion as position in the classical Harmonic oscillator. This phase space like trajectory is sketched in fig. 2.1.


Figure 2.1: Phase space like trajectory.

In the text it is shown that we have the same structure for the Heisenberg operator itself, before taking expectations

$$
\begin{equation*}
\hat{x}_{\mathrm{H}}(t)=x(0) \cos (\omega t)+\frac{p(0)}{m \omega} \sin (\omega t) \tag{2.48}
\end{equation*}
$$

Where the coherent states become useful is that we will see that the second moments of position and momentum are not time dependent with respect to the coherent states. Such states remain localized.

### 2.6 COHERENT STATE UNCERTAINTY.

First note that using the commutator relationship we have

$$
\begin{align*}
\langle z| a a^{\dagger}|z\rangle & =\langle z|\left(\left[a, a^{\dagger}\right]+a^{\dagger} a\right)|z\rangle  \tag{2.49}\\
& =\langle z|\left(1+a^{\dagger} a\right)|z\rangle .
\end{align*}
$$

For the second moment we have

$$
\begin{align*}
\langle z| \hat{x}^{2}|z\rangle & =\frac{\hbar}{2 m \omega}\langle z|\left(a+a^{\dagger}\right)\left(a+a^{\dagger}\right)|z\rangle \\
& =\frac{\hbar}{2 m \omega}\langle z|\left(a^{2}+\left(a^{\dagger}\right)^{2}+a a^{\dagger}+a^{\dagger} a\right)|z\rangle \\
& =\frac{\hbar}{2 m \omega}\langle z|\left(a^{2}+\left(a^{\dagger}\right)^{2}+2 a^{\dagger} a+1\right)|z\rangle  \tag{2.50}\\
& =\frac{\hbar}{2 m \omega}\left(z^{2}+\left(z^{*}\right)^{2}+2 z^{*} z+1\right)|z\rangle \\
& =\frac{\hbar}{2 m \omega}\left(z+z^{*}\right)^{2}+\frac{\hbar}{2 m \omega} .
\end{align*}
$$

We find

$$
\begin{equation*}
\sigma_{x}^{2}=\frac{\hbar}{2 m \omega} \tag{2.51}
\end{equation*}
$$

and

$$
\begin{equation*}
\sigma_{p}^{2}=\frac{m \hbar \omega}{2} \tag{2.52}
\end{equation*}
$$

so

$$
\begin{equation*}
\sigma_{x}^{2} \sigma_{p}^{2}=\frac{\hbar^{2}}{4} \tag{2.53}
\end{equation*}
$$

or

$$
\begin{equation*}
\sigma_{x} \sigma_{p}=\frac{\hbar}{2} \tag{2.54}
\end{equation*}
$$

This is the minimum uncertainty.

### 2.7 QUANTUM FIELD THEORY.

In Quantum Field theory the ideas of isolated oscillators is used to model particle creation. The lowest energy state (a no particle, vacuum state) is given the lowest energy level, with each additional quantum level modeling a new particle creation state as sketched in fig. 2.2.

We have to imagine many oscillators, each with a distinct vacuum energy $\sim \mathbf{k}^{2}$. The Harmonic oscillator can be used to model the creation of particles with $\hbar \omega$ energy differences from that "vacuum energy".


Figure 2.2: QFT energy levels.

### 2.8 CHARGED PARTICLE IN A MAGNETIC FIELD.

In the classical case ( with SI units or $c=1$ ) we have

$$
\begin{equation*}
\mathbf{F}=q \mathbf{E}+q \mathbf{v} \times \mathbf{B} . \tag{2.55}
\end{equation*}
$$

Alternately, we can look at the Hamiltonian view of the system, written in terms of potentials

$$
\begin{align*}
& \mathbf{B}=\boldsymbol{\nabla} \times \mathbf{A},  \tag{2.56}\\
& \mathbf{E}=-\boldsymbol{\nabla} \phi-\frac{\partial \mathbf{A}}{\partial t} . \tag{2.57}
\end{align*}
$$

Note that the curl form for the magnetic field implies one of the required Maxwell's equations $\boldsymbol{\nabla} \cdot \mathbf{B}=0$.

Ignoring time dependence of the potentials, the Hamiltonian can be expressed as

$$
\begin{equation*}
H=\frac{1}{2 m}(\mathbf{p}-q \mathbf{A})^{2}+q \phi . \tag{2.58}
\end{equation*}
$$

In this Hamiltonian the vector $\mathbf{p}$ is called the canonical momentum, the momentum conjugate to position in phase space.

It is left as an exercise to show that the Lorentz force equation results from application of the Hamiltonian equations of motion, and that the velocity is given by $\mathbf{v}=(\mathbf{p}-q \mathbf{A}) / m$.

For quantum mechanics, we use the same Hamiltonian, but promote our position, momentum and potentials to operators.

$$
\begin{equation*}
\hat{H}=\frac{1}{2 m}(\hat{\mathbf{p}}-q \hat{\mathbf{A}}(\mathbf{r}, t))^{2}+q \hat{\phi}(\mathbf{r}, t) . \tag{2.59}
\end{equation*}
$$

## 2.9 gauge invariance.

Can we say anything about this before looking at the question of a particle in a magnetic field?

Recall that the we can make a gauge transformation of the form

$$
\begin{align*}
& \mathbf{A} \rightarrow \mathbf{A}+\boldsymbol{\nabla} \chi,  \tag{2.60a}\\
& \phi \rightarrow \phi-\frac{\partial \chi}{\partial t} . \tag{2.60b}
\end{align*}
$$

Does this notion of gauge invariance also carry over to the Quantum Hamiltonian. After gauge transformation we have

$$
\begin{equation*}
\hat{H}^{\prime}=\frac{1}{2 m}(\hat{\mathbf{p}}-q \mathbf{A}-q \boldsymbol{\nabla} \chi)^{2}+q\left(\phi-\frac{\partial \chi}{\partial t}\right) . \tag{2.61}
\end{equation*}
$$

Now we are in a mess, since this function $\chi$ can make the Hamiltonian horribly complicated. We don't see how gauge invariance can easily be applied to the quantum problem. Next time we will introduce a transformation that resolves some of this mess.

Particle with E,B fields We express our fields with vector and scalar potentials

$$
\begin{equation*}
\mathbf{E}, \mathbf{B} \rightarrow \mathbf{A}, \phi \tag{2.62}
\end{equation*}
$$

and apply a gauge transformed Hamiltonian

$$
\begin{equation*}
H=\frac{1}{2 m}(\mathbf{p}-q \mathbf{A})^{2}+q \phi . \tag{2.63}
\end{equation*}
$$

Recall that in classical mechanics we have

$$
\begin{equation*}
\mathbf{p}-q \mathbf{A}=m \mathbf{v}, \tag{2.64}
\end{equation*}
$$

where $\mathbf{p}$ is not gauge invariant, but the classical momentum $m \mathbf{v}$ is.
If given a point in phase space we must also specify the gauge that we are working with. For the quantum case, temporarily considering a Hamiltonian without any scalar potential, but introducing a gauge transformation

$$
\begin{equation*}
\mathbf{A} \rightarrow \mathbf{A}+\nabla_{\chi} \tag{2.65}
\end{equation*}
$$

which takes the Hamiltonian from

$$
\begin{equation*}
H=\frac{1}{2 m}(\mathbf{p}-q \mathbf{A})^{2}, \tag{2.66}
\end{equation*}
$$

to

$$
\begin{equation*}
H=\frac{1}{2 m}(\mathbf{p}-q \mathbf{A}-q \boldsymbol{\nabla} \chi)^{2} . \tag{2.67}
\end{equation*}
$$

We care that the position and momentum operators obey

$$
\begin{equation*}
\left[\hat{r}_{i}, \hat{p}_{j}\right]=i \hbar \delta_{i j} . \tag{2.68}
\end{equation*}
$$

We can apply a transformation that keeps $\mathbf{r}$ the same, but changes the momentum

$$
\begin{align*}
\hat{\mathbf{r}}^{\prime} & =\hat{\mathbf{r}}, \\
\hat{\mathbf{p}}^{\prime} & =\hat{\mathbf{p}}-q \nabla \chi(\mathbf{r}) . \tag{2.69}
\end{align*}
$$

This maps the Hamiltonian to

$$
\begin{equation*}
H=\frac{1}{2 m}\left(\mathbf{p}^{\prime}-q \mathbf{A}-q \boldsymbol{\nabla} \chi\right)^{2}, \tag{2.70}
\end{equation*}
$$

We want to check if the commutator relationships have the desired structure, that is

$$
\begin{align*}
{\left[r_{i}^{\prime}, r_{j}^{\prime}\right] } & =0,  \tag{2.71}\\
{\left[p_{i}^{\prime}, p_{j}^{\prime}\right] } & =0
\end{align*}
$$

This is confirmed in exercise 2.3.
Another thing of interest is how are the wave functions altered by this change of variables? The wave functions must change in response to this transformation if the energies of the Hamiltonian are to remain the same. Consider a plane wave specified by

$$
\begin{equation*}
e^{i \mathbf{k} \cdot \mathbf{r}} \tag{2.72}
\end{equation*}
$$

where we alter the momentum by

$$
\begin{equation*}
\mathbf{k} \rightarrow \mathbf{k}-e \boldsymbol{\nabla} \chi \tag{2.73}
\end{equation*}
$$

This takes the plane wave to

$$
\begin{equation*}
e^{i(\mathbf{k}-q \boldsymbol{\nabla} \chi) \cdot \mathbf{r}} \tag{2.74}
\end{equation*}
$$

We want to try to find a wave function for the new Hamiltonian

$$
\begin{equation*}
H^{\prime}=\frac{1}{2 m}\left(\mathbf{p}^{\prime}-q \mathbf{A}-q \boldsymbol{\nabla} \chi\right)^{2} \tag{2.75}
\end{equation*}
$$

of the form

$$
\begin{equation*}
\psi^{\prime}(\mathbf{r}) \stackrel{?}{=} e^{i \theta(\mathbf{r})} \psi(\mathbf{r}) \tag{2.76}
\end{equation*}
$$

where the new wave function differs from a wave function for the original Hamiltonian by only a position dependent phase factor.

Let's look at the action of the Hamiltonian on the new wave function

$$
\begin{equation*}
H^{\prime} \psi^{\prime}(\mathbf{r}) \tag{2.77}
\end{equation*}
$$

Looking at just the first action

$$
\begin{align*}
(-i \hbar \boldsymbol{\nabla}-q \mathbf{A}-q \boldsymbol{\nabla} \chi) e^{i \theta(\mathbf{r})} \psi(\mathbf{r}) & =e^{i \theta}(-i \hbar \boldsymbol{\nabla}-q \mathbf{A}-q \boldsymbol{\nabla} \chi) \psi(\mathbf{r})+(-i \hbar i \boldsymbol{\nabla} \theta) e^{i \theta} \psi(\mathbf{r}) \\
& =e^{i \theta}(-i \hbar \boldsymbol{\nabla}-q \mathbf{A}-q \boldsymbol{\nabla} \chi+\hbar \boldsymbol{\nabla} \theta) \psi(\mathbf{r}) . \tag{2.78}
\end{align*}
$$

If we choose

$$
\begin{equation*}
\theta=\frac{q \chi}{\hbar} \tag{2.79}
\end{equation*}
$$

then we are left with

$$
\begin{equation*}
(-i \hbar \boldsymbol{\nabla}-q \mathbf{A}-q \boldsymbol{\nabla} \chi) e^{i \theta(\mathbf{r})} \psi(\mathbf{r})=e^{i \theta}(-i \hbar \boldsymbol{\nabla}-q \mathbf{A}) \psi(\mathbf{r}) \tag{2.80}
\end{equation*}
$$

Let $\mathbf{M}=-i \hbar \boldsymbol{\nabla}-q \mathbf{A}$, and act again with $(-i \hbar \boldsymbol{\nabla}-q \mathbf{A}-q \boldsymbol{\nabla} \chi)$

$$
\begin{align*}
(-i \hbar \boldsymbol{\nabla}-q \mathbf{A}-q \boldsymbol{\nabla} \chi) e^{i \theta} \mathbf{M} \psi & =e^{i \theta}(-i \hbar i \boldsymbol{\nabla} \theta-q \mathbf{A}-q \boldsymbol{\nabla} \chi) e^{i \theta} \mathbf{M} \psi+e^{i \theta}(-i \hbar \boldsymbol{\nabla}) \mathbf{M} \psi \\
& =e^{i \theta}(-i \hbar \boldsymbol{\nabla}-q \mathbf{A}+\boldsymbol{\nabla}(\hbar \theta-q \chi)) \mathbf{M} \psi \\
& =e^{i \theta} \mathbf{M}^{2} \psi \tag{2.81}
\end{align*}
$$

Restoring factors of $m$, we've shown that for a choice of $\hbar \theta-q \chi$, we have

$$
\begin{equation*}
\frac{1}{2 m}(-i \hbar \boldsymbol{\nabla}-q \mathbf{A}-q \boldsymbol{\nabla} \chi)^{2} e^{i \theta} \psi=e^{i \theta} \frac{1}{2 m}(-i \hbar \boldsymbol{\nabla}-q \mathbf{A})^{2} \psi . \tag{2.82}
\end{equation*}
$$

When $\psi$ is an energy eigenfunction, this means

$$
\begin{equation*}
H^{\prime} e^{i \theta} \psi=e^{i \theta} H \psi=e^{i \theta} E \psi=E\left(e^{i \theta} \psi\right) \tag{2.83}
\end{equation*}
$$

We've found a transformation of the wave function that has the same energy eigenvalues as the corresponding wave functions for the original untransformed Hamiltonian.

In summary

$$
\begin{align*}
H^{\prime} & =\frac{1}{2 m}(\mathbf{p}-q \mathbf{A}-q \boldsymbol{\nabla} \chi)^{2}  \tag{2.84}\\
\psi^{\prime}(\mathbf{r}) & =e^{i \theta(\mathbf{r})} \psi(\mathbf{r}), \quad \text { where } \theta(\mathbf{r})=q \chi(\mathbf{r}) / \hbar
\end{align*}
$$

Aharonov-Bohm effect Consider a periodic motion in a fixed ring as sketched in fig. 2.3. Here the displacement around the perimeter is $s=R \phi$


Figure 2.3: Particle confined to a ring.
and the Hamiltonian

$$
\begin{equation*}
H=-\frac{\hbar^{2}}{2 m} \frac{\partial^{2}}{\partial s^{2}}=-\frac{\hbar^{2}}{2 m R^{2}} \frac{\partial^{2}}{\partial \phi^{2}} . \tag{2.85}
\end{equation*}
$$

Now assume that there is a magnetic field squeezed into the point at the origin, by virtue of a flux at the origin

$$
\begin{equation*}
\mathbf{B}=\Phi_{0} \delta(\mathbf{r}) \hat{\mathbf{z}} . \tag{2.86}
\end{equation*}
$$

We know that

$$
\begin{equation*}
\oint \mathbf{A} \cdot d \mathbf{l}=\Phi_{0} \tag{2.87}
\end{equation*}
$$

so that

$$
\begin{equation*}
\mathbf{A}=\frac{\Phi_{0}}{2 \pi r} \hat{\boldsymbol{\phi}} \tag{2.88}
\end{equation*}
$$

The Hamiltonian for the new configuration is

$$
\begin{equation*}
H=-\left(-i \hbar \boldsymbol{\nabla}-q \frac{\Phi_{0}}{2 \pi r} \hat{\boldsymbol{\phi}}\right)^{2}=-\frac{1}{2 m}\left(-i \hbar \frac{1}{R} \frac{\partial}{\partial \phi}-q \frac{\Phi_{0}}{2 \pi R}\right)^{2} \tag{2.89}
\end{equation*}
$$

Here the replacement $r \rightarrow R$ makes use of the fact that this problem as been posed with the particle forced to move around the ring at the fixed radius $R$. For this transformed Hamiltonian, what are the wave functions?

$$
\begin{equation*}
\psi(\phi)^{\prime} \stackrel{?}{=} e^{i n \phi} \tag{2.90}
\end{equation*}
$$

$$
\begin{align*}
& H \psi=\frac{\frac{1}{2 m}\left(-i \hbar \frac{1}{R}(i n)-q \frac{\Phi_{0}}{2 \pi R}\right)^{2} e^{i n \phi}}{} \\
&=\frac{1}{2 m}\left(\frac{\hbar n}{R}-q \frac{\Phi_{0}}{2 \pi R}\right)^{2} e^{i n \phi}  \tag{2.91}\\
& E_{n}
\end{align*}
$$

This is very unclassical, since the energy changes in a way that depends on the flux, because particles are seeing magnetic fields that are not present at the point of the particle, as sketched in fig. 2.4. We see that there are multiple points that the energies hit the minimum levels.
problem set note. In the problem set we'll look at interference patterns for two slit electron interference like that of fig. 2.5, where a magnetic whisker that introduces flux is added to the configuration.

Aharonov-Bohm effect (cont.) Why do we have the zeros at integral multiples of $h / q$ ? Consider a particle in a circular trajectory as sketched in fig. 2.6 In classical mechanics we have


Figure 2.4: Energy variation with flux.


Figure 2.5: Two slit interference with magnetic whisker.


Figure 2.6: Circular trajectory.

$$
\begin{equation*}
\oint p d q \tag{2.92}
\end{equation*}
$$

The integral zero points are related to such a loop, but the $q \mathbf{A}$ portion of the momentum $\mathbf{p}-q \mathbf{A}$ needs to be considered.

Superconductors After cooling some materials sufficiently, superconductivity, a complete lack of resistance to electrical flow can be observed. A resistivity vs temperature plot of such a material is sketched in fig. 2.7. Just like $\mathrm{He}^{4}$ can undergo Bose condensation, superconductivity can be


Figure 2.7: Superconductivity with comparison to superfluidity.
explained by a hybrid Bosonic state where electrons are paired into one state containing integral spin. The Little-Parks experiment puts a superconducting ring around a magnetic whisker as sketched in fig. 2.8. This


Figure 2.8: Little-Parks superconducting ring.
experiment shows that the effective charge of the circulating charge was $2 e$, validating the concept of Cooper-pairing, the Bosonic combination (integral spin) of electrons in superconduction.

Motion around magnetic field For a constant uniform magnetic field, we will be able to identify an intrinsic angular frequency

$$
\begin{equation*}
\omega_{\mathrm{c}}=\frac{e B}{m} . \tag{2.93}
\end{equation*}
$$

To do so, we work with what is now called the Landau gauge

$$
\begin{equation*}
\mathbf{A}=(0, B x, 0) . \tag{2.94}
\end{equation*}
$$

The curl of such a vector potential yields a uniform constant magnetic field

$$
\begin{align*}
\mathbf{B} & =\boldsymbol{\nabla} \times \mathbf{A} \\
& =\left(\partial_{x} A_{y}-\partial_{y} A_{x}\right) \hat{\mathbf{z}}  \tag{2.95}\\
& =B \hat{\mathbf{z}} .
\end{align*}
$$

An alternate gauge choice, the symmetric gauge, is

$$
\begin{equation*}
\mathbf{A}=\left(-\frac{B y}{2}, \frac{B x}{2}, 0\right) \tag{2.96}
\end{equation*}
$$

that also has the same magnetic field

$$
\begin{align*}
\mathbf{B} & =\boldsymbol{\nabla} \mathbf{A} \\
& =\left(\partial_{x} A_{y}-\partial_{y} A_{x}\right) \hat{\mathbf{z}} \\
& =\left(\frac{B}{2}-\left(-\frac{B}{2}\right)\right) \hat{\mathbf{z}}  \tag{2.97}\\
& =B \hat{\mathbf{z}} .
\end{align*}
$$

We expect the physics for each to have the same results, although the wave functions in one gauge may be more complicated than in the other.

Our Hamiltonian is

$$
\begin{align*}
H & =\frac{1}{2 m}(\mathbf{p}-e \mathbf{A})^{2} \\
& =\frac{1}{2 m} \hat{p}_{x}^{2}+\frac{1}{2 m}\left(\hat{p}_{y}-e B \hat{x}\right)^{2} . \tag{2.98}
\end{align*}
$$

We can solve after noting that

$$
\begin{equation*}
\left[\hat{p}_{y}, H\right]=0, \tag{2.99}
\end{equation*}
$$

implies

$$
\begin{equation*}
\Psi(x, y)=e^{i k_{y} y} \phi(x) \tag{2.100}
\end{equation*}
$$

The eigensystem

$$
\begin{equation*}
H \psi(x, y)=E \phi(x, y) \tag{2.101}
\end{equation*}
$$

becomes

$$
\begin{equation*}
\left(\frac{1}{2 m} \hat{p}_{x}^{2}+\frac{1}{2 m}\left(\hbar k_{y}-e B \hat{x}\right)^{2}\right) \phi(x)=E \phi(x) \tag{2.102}
\end{equation*}
$$

This reduced Hamiltonian can be rewritten as

$$
\begin{equation*}
H_{x}=\frac{1}{2 m} p_{x}^{2}+\frac{1}{2 m} e^{2} B^{2}\left(\hat{x}-\frac{\hbar k_{y}}{e B}\right)^{2} \equiv \frac{1}{2 m} p_{x}^{2}+\frac{1}{2} m \omega^{2}\left(\hat{x}-x_{0}\right)^{2} \tag{2.103}
\end{equation*}
$$

where

$$
\begin{equation*}
\frac{1}{2 m} e^{2} B^{2}=\frac{1}{2} m \omega^{2} \tag{2.104}
\end{equation*}
$$

or

$$
\begin{equation*}
\omega=\frac{e B}{m} \equiv \omega_{\mathrm{c}} . \tag{2.105}
\end{equation*}
$$

and

$$
\begin{equation*}
x_{0}=\frac{\hbar}{k_{y}} e B \tag{2.106}
\end{equation*}
$$

But what is this $x_{0}$ ? Because $k_{y}$ is not really specified in this problem, we can consider that we have a zero point energy for every $k_{y}$, but the oscillator position is shifted for every such value of $k_{y}$. For each set of energy levels fig. 2.9 we can consider that there is a different zero point energy for each possible $k_{y}$.


Figure 2.9: Energy levels, and Energy vs flux.
This is an infinitely degenerate system with an infinite number of states for any given energy level. This tells us that there is a problem, and have to reconsider the assumption that any $k_{y}$ is acceptable. To resolve this we can introduce periodic boundary conditions, imagining that a square is rotated in space forming a cylinder as sketched in fig. 2.10. Requiring quantized momentum

$$
\begin{equation*}
k_{y} L_{y}=2 \pi n \tag{2.107}
\end{equation*}
$$



Figure 2.10: Landau degeneracy region.
or

$$
\begin{equation*}
k_{y}=\frac{2 \pi n}{L_{y}}, \quad n \in \mathbb{Z} \tag{2.108}
\end{equation*}
$$

gives

$$
\begin{equation*}
x_{0}(n)=\frac{\hbar}{e B} \frac{2 \pi n}{L_{y}} \tag{2.109}
\end{equation*}
$$

with $x_{0} \leq L_{x}$. The range is thus restricted to

$$
\begin{equation*}
\frac{\hbar}{e B} \frac{2 \pi n_{\max }}{L_{y}}=L_{x} \tag{2.110}
\end{equation*}
$$

or

$$
\begin{equation*}
n_{\max }=\underbrace{\text { area }}_{\substack{L_{x} L_{y}} \frac{e B}{2 \pi \hbar} .} . \tag{2.111}
\end{equation*}
$$

That is

$$
\begin{equation*}
n_{\max }=\frac{\Phi_{\text {total }}}{h / e}=\frac{\Phi_{\text {total }}}{\Phi_{0}} \tag{2.112}
\end{equation*}
$$

Attempting to measure Hall-effect systems, it was found that the Hall conductivity was quantized like

$$
\begin{equation*}
\sigma_{x y}=p \frac{e^{2}}{h} \tag{2.113}
\end{equation*}
$$

This quantization is explained by these Landau levels, and this experimental apparatus provides one of the more accurate ways to measure the fine structure constant.

### 2.10 DIAGONALIZATING THE QUANTUM HARMONIC OSCILLATOR.

Many authors pull the definitions of the raising and lowering (or ladder) operators out of their butt with no attempt at motivation. This is pointed out nicely in [7] by Eli along with one justification based on factoring the Hamiltonian.

In [16] is a small exception to the usual presentation. In that text, these operators are defined as usual with no motivation. However, after the utility of these operators has been shown, the raising and lowering operators show up in a context that does provide that missing motivation as a side effect. It doesn't look like the author was trying to provide a motivation, but it can be interpreted that way.

When seeking the time evolution of Heisenberg-picture position and momentum operators, we will see that those solutions can be trivially expressed using the raising and lowering operators. No special tools nor black magic is required to find the structure of these operators. Unfortunately, we must first switch to both the Heisenberg picture representation of the position and momentum operators, and also employ the Heisenberg equations of motion. Neither of these last two fit into standard narrative of most introductory quantum mechanics treatments. We will also see that these raising and lowering "operators" could also be introduced in classical mechanics, provided we were attempting to solve the SHO system using the Hamiltonian equations of motion.

I'll outline this route to finding the structure of the ladder operators below. Because these are encountered trying to solve the time evolution problem, I'll first show a simpler way to solve that problem. Because that simpler method depends a bit on lucky observation and is somewhat unstructured, I'll then outline a more structured procedure that leads to the ladder operators directly, also providing the solution to the time evolution problem as a side effect.

The starting point is the Heisenberg equations of motion. For a time independent Hamiltonian $H$, and a Heisenberg operator $A^{(H)}$, those equations are

$$
\begin{equation*}
\frac{d A^{(H)}}{d t}=\frac{1}{i \hbar}\left[A^{(H)}, H\right] . \tag{2.114}
\end{equation*}
$$

Here the Heisenberg operator $A^{(H)}$ is related to the Schrödinger operator $A^{(S)}$ by

$$
\begin{equation*}
A^{(H)}=U^{\dagger} A^{(S)} U, \tag{2.115}
\end{equation*}
$$

where $U$ is the time evolution operator. For this discussion, we need only know that $U$ commutes with $H$, and do not need to know the specific structure of that operator. In particular, the Heisenberg equations of motion take the form

$$
\begin{align*}
\frac{d A^{(H)}}{d t} & =\frac{1}{i \hbar}\left[A^{(H)}, H\right] \\
& =\frac{1}{i \hbar}\left[U^{\dagger} A^{(S)} U, H\right] \\
& =\frac{1}{i \hbar}\left(U^{\dagger} A^{(S)} U H-H U^{\dagger} A^{(S)} U\right)  \tag{2.116}\\
& =\frac{1}{i \hbar}\left(U^{\dagger} A^{(S)} H U-U^{\dagger} H A^{(S)} U\right) \\
& =\frac{1}{i \hbar} U^{\dagger}\left[A^{(S)}, H\right] U .
\end{align*}
$$

The Hamiltonian for the harmonic oscillator, with Schrödinger-picture position and momentum operators $x, p$ is

$$
\begin{equation*}
H=\frac{p^{2}}{2 m}+\frac{1}{2} m \omega^{2} x^{2} \tag{2.117}
\end{equation*}
$$

so the equations of motions are

$$
\begin{align*}
\frac{d x^{(H)}}{d t} & =\frac{1}{i \hbar} U^{\dagger}[x, H] U \\
& =\frac{1}{i \hbar} U^{\dagger}\left[x, \frac{p^{2}}{2 m}\right] U \\
& =\frac{1}{2 m i \hbar} U^{\dagger}\left(i \hbar \frac{\partial p^{2}}{\partial p}\right) U  \tag{2.118}\\
& =\frac{1}{m} U^{\dagger} p U \\
& =\frac{1}{m} p^{(H)},
\end{align*}
$$

and

$$
\begin{align*}
\frac{d p^{(H)}}{d t} & =\frac{1}{i \hbar} U^{\dagger}[p, H] U \\
& =\frac{1}{i \hbar} U^{\dagger}\left[p, \frac{1}{2} m \omega^{2} x^{2}\right] U \\
& =\frac{m \omega^{2}}{2 i \hbar} U^{\dagger}\left(-i \hbar \frac{\partial x^{2}}{\partial x}\right) U  \tag{2.119}\\
& =-m \omega^{2} U^{\dagger} x U \\
& =-m \omega^{2} x^{(H)}
\end{align*}
$$

In the Heisenberg picture the equations of motion are precisely those of classical Hamiltonian mechanics, except that we are dealing with operators instead of scalars

$$
\begin{align*}
\frac{d p^{(H)}}{d t} & =-m \omega^{2} x^{(H)}  \tag{2.120}\\
\frac{d x^{(H)}}{d t} & =\frac{1}{m} p^{(H)}
\end{align*}
$$

In the text the ladder operators are used to simplify the solution of these coupled equations, since they can decouple them. That's not really required since we can solve them directly in matrix form with little work

$$
\frac{d}{d t}\left[\begin{array}{l}
p^{(H)}  \tag{2.121}\\
x^{(H)}
\end{array}\right]=\left[\begin{array}{cc}
0 & -m \omega^{2} \\
\frac{1}{m} & 0
\end{array}\right]\left[\begin{array}{l}
p^{(H)} \\
x^{(H)}
\end{array}\right]
$$

or, with length scaled variables

$$
\begin{align*}
\frac{d}{d t}\left[\begin{array}{c}
\frac{p^{(H)}}{m \omega} \\
x^{(H)}
\end{array}\right] & =\left[\begin{array}{cc}
0 & -\omega \\
\omega & 0
\end{array}\right]\left[\begin{array}{c}
\frac{p^{(H)}}{m \omega} \\
x^{(H)}
\end{array}\right] \\
& =-i \omega\left[\begin{array}{cc}
0 & -i \\
i & 0
\end{array}\right]\left[\begin{array}{c}
\frac{p^{(H)}}{m \omega} \\
x^{(H)}
\end{array}\right]  \tag{2.122}\\
& =-i \omega \sigma_{y}\left[\begin{array}{c}
\frac{p^{(H)}}{m \omega} \\
x^{(H)}
\end{array}\right]
\end{align*}
$$

Writing $y=\left[\begin{array}{c}\frac{p^{(H)}}{m \omega} \\ x^{(H)}\end{array}\right]$, the solution can then be written immediately as

$$
\begin{align*}
y(t) & =\exp \left(-i \omega \sigma_{y} t\right) y(0) \\
& =\left(\cos (\omega t) I-i \sigma_{y} \sin (\omega t)\right) y(0) \\
& =\left[\begin{array}{cc}
\cos (\omega t) & \sin (\omega t) \\
-\sin (\omega t) & \cos (\omega t)
\end{array}\right] y(0), \tag{2.123}
\end{align*}
$$

or

$$
\begin{align*}
\frac{p^{(H)}(t)}{m \omega} & =\cos (\omega t) \frac{p^{(H)}(0)}{m \omega}+\sin (\omega t) x^{(H)}(0) \\
x^{(H)}(t) & =-\sin (\omega t) \frac{p^{(H)}(0)}{m \omega}+\cos (\omega t) x^{(H)}(0) \tag{2.124}
\end{align*}
$$

This solution depends on being lucky enough to recognize that the matrix has a Pauli matrix as a factor (which squares to unity, and allows the exponential to be evaluated easily.)

If we hadn't been that observant, then the first tool we'd have used instead would have been to diagonalize the matrix. For such diagonalization, it's natural to work in completely dimensionless variables. Such a non-dimensionalisation can be had by defining

$$
\begin{equation*}
x_{0}=\sqrt{\frac{\hbar}{m \omega}}, \tag{2.125}
\end{equation*}
$$

and dividing the working (operator) variables through by those values. Let $z=\frac{1}{x_{0}} y$, and $\tau=\omega t$ so that the equations of motion are

$$
\frac{d z}{d \tau}=\left[\begin{array}{cc}
0 & -1  \tag{2.126}\\
1 & 0
\end{array}\right] z
$$

This matrix can be diagonalized as

$$
\begin{align*}
A & =\left[\begin{array}{cc}
0 & -1 \\
1 & 0
\end{array}\right]  \tag{2.127}\\
& =V\left[\begin{array}{cc}
i & 0 \\
0 & -i
\end{array}\right] V^{-1},
\end{align*}
$$

where

$$
V=\frac{1}{\sqrt{2}}\left[\begin{array}{cc}
i & -i  \tag{2.128}\\
1 & 1
\end{array}\right]
$$

The equations of motion can now be written

$$
\frac{d}{d \tau}\left(V^{-1} z\right)=\left[\begin{array}{cc}
i & 0  \tag{2.129}\\
0 & -i
\end{array}\right]\left(V^{-1} z\right)
$$

This final change of variables $V^{-1} z$ decouples the system as desired. Expanding that gives

$$
\begin{align*}
V^{-1} z & =\frac{1}{\sqrt{2}}\left[\begin{array}{cc}
-i & 1 \\
i & 1
\end{array}\right]\left[\begin{array}{c}
\frac{p^{(H)}}{x_{0} m \omega} \\
\frac{x^{(H)}}{x_{0}}
\end{array}\right] \\
& =\frac{1}{\sqrt{2} x_{0}}\left[\begin{array}{c}
-i \frac{p^{(H)}}{m \omega}+x^{(H)} \\
i \frac{p^{(H)}}{m \omega}+x^{(H)}
\end{array}\right]  \tag{2.130}\\
& =\left[\begin{array}{c}
a^{\dagger} \\
a
\end{array}\right]
\end{align*}
$$

where

$$
\begin{align*}
a^{\dagger} & =\sqrt{\frac{m \omega}{2 \hbar}}\left(-i \frac{p^{(H)}}{m \omega}+x^{(H)}\right) \\
a & =\sqrt{\frac{m \omega}{2 \hbar}}\left(i \frac{p^{(H)}}{m \omega}+x^{(H)}\right) \tag{2.131}
\end{align*}
$$

Lo and behold, we have the standard form of the raising and lowering operators, and can write the system equations as

$$
\begin{align*}
\frac{d a^{\dagger}}{d t} & =i \omega a^{\dagger}  \tag{2.132}\\
\frac{d a}{d t} & =-i \omega a
\end{align*}
$$

It is actually a bit fluky that this matched exactly, since we could have chosen eigenvectors that differ by constant phase factors, like

$$
V=\frac{1}{\sqrt{2}}\left[\begin{array}{cc}
i e^{i \phi} & -i e^{i \psi}  \tag{2.133}\\
1 e^{i \phi} & e^{i \psi}
\end{array}\right]
$$

so

$$
\begin{align*}
V^{-1} z & =\frac{e^{-i(\phi+\psi)}}{\sqrt{2}}\left[\begin{array}{cc}
-i e^{i \psi} & e^{i \psi} \\
i e^{i \phi} & e^{i \phi}
\end{array}\right]\left[\begin{array}{c}
\frac{p^{(H)}}{x_{0} m \omega} \\
\frac{x^{(H)}}{x_{0}}
\end{array}\right] \\
& =\frac{1}{\sqrt{2} x_{0}}\left[\begin{array}{c}
-i e^{i \phi} \frac{p^{(H)}}{m \omega}+e^{i \phi} x^{(H)} \\
i e^{i \psi} \frac{p^{(H)}}{m \omega}+e^{i \psi} x^{(H)}
\end{array}\right]  \tag{2.134}\\
& =\left[\begin{array}{c}
e^{i \phi} a^{\dagger} \\
e^{i \psi} a
\end{array}\right] .
\end{align*}
$$

To make the resulting pairs of operators Hermitian conjugates, we'd want to constrain those constant phase factors by setting $\phi=-\psi$. If we were only interested in solving the time evolution problem no such additional constraints are required. The raising and lowering operators are seen to naturally occur when seeking the solution of the Heisenberg equations of motion. We non-dimensionalize and seeking a change of basis that diagonalizes the system matrix. Because the Heisenberg equations of motion are identical to the classical Hamiltonian equations of motion in this case, what we call the raising and lowering operators in quantum mechanics could also be utilized in the classical simple harmonic oscillator problem. However, in a classical context we wouldn't have a justification to call this more than a change of basis.

### 2.11 constant magnetic solenoid field.

In [15] the following vector potential

$$
\begin{equation*}
\mathbf{A}=\frac{B \rho_{a}^{2}}{2 \rho} \hat{\boldsymbol{\phi}} \tag{2.135}
\end{equation*}
$$

is introduced in a discussion on the Aharonov-Bohm effect, for configurations where the interior field of a solenoid is either a constant $\mathbf{B}$ or zero.

I wasn't able to make sense of this since the field I was calculating was zero for all $\rho \neq 0$

$$
\begin{align*}
\mathbf{B} & =\boldsymbol{\nabla} \times \mathbf{A} \\
& =\left(\hat{\boldsymbol{\rho}} \partial_{\rho}+\hat{\mathbf{z}} \partial_{z}+\frac{\hat{\boldsymbol{\phi}}}{\rho} \partial_{\phi}\right) \times \frac{B \rho_{a}^{2}}{2 \rho} \hat{\boldsymbol{\phi}} \\
& =\left(\hat{\boldsymbol{\rho}} \partial_{\rho}+\frac{\hat{\boldsymbol{\phi}}}{\rho} \partial_{\phi}\right) \times \frac{B \rho_{a}^{2}}{2 \rho} \hat{\boldsymbol{\phi}}  \tag{2.136}\\
& =\frac{B \rho_{a}^{2}}{2} \hat{\boldsymbol{\rho}} \times \hat{\boldsymbol{\phi}} \partial_{\rho}\left(\frac{1}{\rho}\right)+\frac{B \rho_{a}^{2}}{2 \rho} \frac{\hat{\boldsymbol{\phi}}}{\rho} \times \partial_{\phi} \hat{\boldsymbol{\phi}} \\
& =\frac{B \rho_{a}^{2}}{2 \rho^{2}}\left(-\hat{\mathbf{z}}+\hat{\boldsymbol{\phi}} \times \partial_{\phi} \hat{\boldsymbol{\phi}}\right) .
\end{align*}
$$

Note that the $\rho$ partial requires that $\rho \neq 0$. To expand the cross product in the second term let $j=\mathbf{e}_{1} \mathbf{e}_{2}$, and expand using a geometric algebra representation of the unit vector

$$
\begin{align*}
\hat{\boldsymbol{\phi}} \times \partial_{\phi} \hat{\boldsymbol{\phi}} & =\mathbf{e}_{2} e^{j \phi} \times\left(\mathbf{e}_{2} \mathbf{e}_{1} \mathbf{e}_{2} e^{j \phi}\right) \\
& =-\mathbf{e}_{1} \mathbf{e}_{2} \mathbf{e}_{3}\left\langle\mathbf{e}_{2} e^{j \phi}\left(-\mathbf{e}_{1}\right) e^{j \phi}\right\rangle_{2} \\
& =\mathbf{e}_{1} \mathbf{e}_{2} \mathbf{e}_{3} \mathbf{e}_{2} \mathbf{e}_{1}  \tag{2.137}\\
& =\mathbf{e}_{3} \\
& =\hat{\mathbf{z}}
\end{align*}
$$

So, provided $\rho \neq 0, \mathbf{B}=0$.
The errata [14] provides the clarification, showing that a $\rho>\rho_{a}$ constraint is required for this potential to produce the desired results. Continuity at $\rho=\rho_{a}$ means that in the interior (or at least on the boundary) we must have one of

$$
\begin{equation*}
\mathbf{A}=\frac{B \rho_{a}}{2} \hat{\boldsymbol{\phi}} \tag{2.138}
\end{equation*}
$$

or

$$
\begin{equation*}
\mathbf{A}=\frac{B \rho}{2} \hat{\boldsymbol{\phi}} \tag{2.139}
\end{equation*}
$$

The first doesn't work, but the second does

$$
\begin{align*}
\mathbf{B} & =\boldsymbol{\nabla} \times \mathbf{A} \\
& =\left(\hat{\boldsymbol{\rho}} \partial_{\rho}+\hat{\mathbf{z}} \partial_{z}+\frac{\hat{\boldsymbol{\phi}}}{\rho} \partial_{\phi}\right) \times \frac{B \rho}{2} \hat{\boldsymbol{\phi}}  \tag{2.140}\\
& =\frac{B}{2} \hat{\boldsymbol{\rho}} \times \hat{\boldsymbol{\phi}}+\frac{B \rho}{2} \frac{\hat{\boldsymbol{\phi}}}{\rho} \times \partial_{\phi} \hat{\boldsymbol{\phi}} \\
& =B \hat{\mathbf{z}} .
\end{align*}
$$

So the vector potential that we want for a constant $B \hat{\mathbf{z}}$ field in the interior $\rho<\rho_{a}$ of a cylindrical space, we need

$$
\mathbf{A}= \begin{cases}\frac{B \rho_{a}^{2}}{2 \rho} \hat{\boldsymbol{\phi}} & \text { if } \rho \geq \rho_{a}  \tag{2.141}\\ \frac{B \rho}{2} \hat{\boldsymbol{\phi}} & \text { if } \rho<\rho_{a}\end{cases}
$$

An example of the magnitude of potential is graphed in fig. 2.11.


Figure 2.11: Vector potential for constant field in cylindrical region.

### 2.12 LAGRANGIAN FOR MAGNETIC PORTION OF LORENTZ FORCE.

In [15] it is claimed in an Aharonov-Bohm discussion that a Lagrangian modification to include electromagnetism is

$$
\begin{equation*}
\mathcal{L} \rightarrow \mathcal{L}+\frac{e}{c} \mathbf{v} \cdot \mathbf{A} \tag{2.142}
\end{equation*}
$$

That can't be the full Lagrangian since there is no $\phi$ term, so what exactly do we get?

If you have somehow, like I did, forgot the exact form of the EulerLagrange equations (i.e. where do the dots go), then the derivation of those equations can come to your rescue. The starting point is the action

$$
\begin{equation*}
S=\int \mathcal{L}(x, \dot{x}, t) d t \tag{2.143}
\end{equation*}
$$

where the end points of the integral are fixed, and we assume we have no variation at the end points. The variational calculation is

$$
\begin{align*}
\delta S & =\int \delta \mathcal{L}(x, \dot{x}, t) d t \\
& =\int\left(\frac{\partial \mathcal{L}}{\partial x} \delta x+\frac{\partial \mathcal{L}}{\partial \dot{x}} \delta \dot{x}\right) d t \\
& =\int\left(\frac{\partial \mathcal{L}}{\partial x} \delta x+\frac{\partial \mathcal{L}}{\partial \dot{x}} \delta \frac{d x}{d t}\right) d t  \tag{2.144}\\
& =\int\left(\frac{\partial \mathcal{L}}{\partial x}-\frac{d}{d t}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}}\right)\right) \delta x d t+\delta x \frac{\partial \mathcal{L}}{\partial \dot{x}}
\end{align*}
$$

The boundary term is killed after evaluation at the end points where the variation is zero. For the result to hold for all variations $\delta x$, we must have

$$
\begin{equation*}
\frac{\partial \mathcal{L}}{\partial x}=\frac{d}{d t}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}}\right) . \tag{2.145}
\end{equation*}
$$

Now lets apply this to the Lagrangian at hand. For the position derivative we have

$$
\begin{equation*}
\frac{\partial \mathcal{L}}{\partial x_{i}}=\frac{e}{c} v_{j} \frac{\partial A_{j}}{\partial x_{i}} . \tag{2.146}
\end{equation*}
$$

For the canonical momentum term, assuming $\mathbf{A}=\mathbf{A}(\mathbf{x})$ we have

$$
\begin{align*}
\frac{d}{d t} \frac{\partial \mathcal{L}}{\partial \dot{x}_{i}} & =\frac{d}{d t}\left(m \dot{x}_{i}+\frac{e}{c} A_{i}\right) \\
& =m \ddot{x}_{i}+\frac{e}{c} \frac{d A_{i}}{d t}  \tag{2.147}\\
& =m \ddot{x}_{i}+\frac{e}{c} \frac{\partial A_{i}}{\partial x_{j}} \frac{d x_{j}}{d t} .
\end{align*}
$$

Assembling the results, we've got

$$
\begin{align*}
0 & =\frac{d}{d t} \frac{\partial \mathcal{L}}{\partial \dot{x}_{i}}-\frac{\partial \mathcal{L}}{\partial x_{i}} \\
& =m \ddot{x}_{i}+\frac{e}{c} \frac{\partial A_{i}}{\partial x_{j}} \frac{d x_{j}}{d t}-\frac{e}{c} v_{j} \frac{\partial A_{j}}{\partial x_{i}}, \tag{2.148}
\end{align*}
$$

or

$$
\begin{align*}
m \ddot{x}_{i} & =\frac{e}{c} v_{j} \frac{\partial A_{j}}{\partial x_{i}}-\frac{e}{c} \frac{\partial A_{i}}{\partial x_{j}} v_{j} \\
& =\frac{e}{c} v_{j}\left(\frac{\partial A_{j}}{\partial x_{i}}-\frac{\partial A_{i}}{\partial x_{j}}\right)  \tag{2.149}\\
& =\frac{e}{c} v_{j} B_{k} \epsilon_{i j k}
\end{align*}
$$

In vector form that is

$$
\begin{equation*}
m \ddot{\mathbf{x}}=\frac{e}{c} \mathbf{v} \times \mathbf{B} . \tag{2.150}
\end{equation*}
$$

So, we get the magnetic term of the Lorentz force. Also note that this shows the Lagrangian (and the end result), was not in SI units. The $1 / c$ term would have to be dropped for SI.

### 2.13 problems.

Exercise 2.1 Lorentz force, electrodynamic Hamiltonian.
Given the classical Hamiltonian

$$
\begin{equation*}
H=\frac{1}{2 m}(\mathbf{p}-q \mathbf{A})^{2}+q \phi . \tag{2.151}
\end{equation*}
$$

apply the Hamiltonian equations of motion

$$
\begin{align*}
& \frac{d \mathbf{p}}{d t}=-\frac{\partial H}{\partial \mathbf{q}} \\
& \frac{d \mathbf{q}}{d t}=\frac{\partial H}{\partial \mathbf{p}} \tag{2.152}
\end{align*}
$$

to show that this is the Hamiltonian that describes the Lorentz force equation, and to find the velocity in terms of the canonical momentum and vector potential.
Answer for Exercise 2.1
The particle velocity follows easily

$$
\begin{align*}
\mathbf{v} & =\frac{d \mathbf{r}}{d t} \\
& =\frac{\partial H}{\partial \mathbf{p}}  \tag{2.153}\\
& =\frac{1}{m}(\mathbf{p}-q \mathbf{A}) .
\end{align*}
$$

For the Lorentz force we can proceed in the coordinate representation

$$
\begin{align*}
\frac{d p_{k}}{d t} & =-\frac{\partial H}{\partial x_{k}} \\
& =-\frac{2}{2 m}\left(p_{m}-q A_{m}\right) \frac{\partial}{\partial x_{k}}\left(p_{m}-q A_{m}\right)-q \frac{\partial \phi}{\partial x_{k}}  \tag{2.154}\\
& =q v_{m} \frac{\partial A_{m}}{\partial x_{k}}-q \frac{\partial \phi}{\partial x_{k}}
\end{align*}
$$

We also have

$$
\begin{align*}
\frac{d p_{k}}{d t} & =\frac{d}{d t}\left(m x_{k}+q A_{k}\right) \\
& =m \frac{d^{2} x_{k}}{d t^{2}}+q \frac{\partial A_{k}}{\partial x_{m}} \frac{d x_{m}}{d t}+q \frac{\partial A_{k}}{\partial t} . \tag{2.155}
\end{align*}
$$

Putting these together we've got

$$
\begin{align*}
m \frac{d^{2} x_{k}}{d t^{2}} & =q v_{m} \frac{\partial A_{m}}{\partial x_{k}}-q \frac{\partial \phi}{\partial x_{k}},-q \frac{\partial A_{k}}{\partial x_{m}} \frac{d x_{m}}{d t}-q \frac{\partial A_{k}}{\partial t} \\
& =q v_{m}\left(\frac{\partial A_{m}}{\partial x_{k}}-\frac{\partial A_{k}}{\partial x_{m}}\right)+q E_{k}  \tag{2.156}\\
& =q v_{m} \epsilon_{k m s} B_{s}+q E_{k}
\end{align*}
$$

or

$$
\begin{align*}
m \frac{d^{2} \mathbf{x}}{d t^{2}} & =q \mathbf{e}_{k} v_{m} \epsilon_{k m s} B_{s}+q E_{k}  \tag{2.157}\\
& =q \mathbf{v} \times \mathbf{B}+q \mathbf{E}
\end{align*}
$$

Exercise 2.2 Show gauge invariance of the magnetic and electric fields.
After the gauge transformation of eq. (2.60) show that the electric and magnetic fields are unaltered.
Answer for Exercise 2.2
For the magnetic field the transformed field is

$$
\begin{align*}
\mathbf{B}^{\prime} & =\boldsymbol{\nabla} \times(\mathbf{A}+\boldsymbol{\nabla} \chi) \\
& =\boldsymbol{\nabla} \times \mathbf{A}+\boldsymbol{\nabla} \times(\boldsymbol{\nabla} \chi)  \tag{2.158}\\
& =\boldsymbol{\nabla} \times \mathbf{A} \\
& =\mathbf{B} .
\end{align*}
$$

$$
\begin{align*}
\mathbf{E}^{\prime} & =-\frac{\partial \mathbf{A}^{\prime}}{\partial t}-\boldsymbol{\nabla} \phi^{\prime} \\
& =-\frac{\partial}{\partial t}(\mathbf{A}+\boldsymbol{\nabla} \chi)-\boldsymbol{\nabla}\left(\phi-\frac{\partial \chi}{\partial t}\right)  \tag{2.159}\\
& =-\frac{\partial \mathbf{A}}{\partial t}-\boldsymbol{\nabla} \phi \\
& =\mathbf{E} .
\end{align*}
$$

## Exercise 2.3 Gauge transformation.

Show that after a transformation of position and momentum of the following form

$$
\begin{align*}
\hat{\mathbf{r}}^{\prime} & =\hat{\mathbf{r}} \\
\hat{\mathbf{p}}^{\prime} & =\hat{\mathbf{p}}-q \boldsymbol{\nabla} \chi(\mathbf{r}), \tag{2.160}
\end{align*}
$$

all the commutators have the expected values.

## Answer for Exercise 2.3

The position commutators don't need consideration. Of interest is the momentum-position commutators

$$
\begin{align*}
{\left[\hat{p}_{k}^{\prime}, \hat{x}_{k}^{\prime}\right] } & =\left[\hat{p}_{k}-q \partial_{k} \chi, \hat{x}_{k}\right] \\
& =\left[\hat{p}_{k}, \hat{x}_{k}\right]-q\left[\partial_{k} \chi, \hat{x}_{k}\right]  \tag{2.161}\\
& =\left[\hat{p}_{k}, \hat{x}_{k}\right],
\end{align*}
$$

and the momentum commutators

$$
\begin{align*}
{\left[\hat{p}_{k}^{\prime}, \hat{p}_{j}^{\prime}\right] } & =\left[\hat{p}_{k}-q \partial_{k} \chi, \hat{p}_{j}-q \partial_{j \chi}\right]  \tag{2.162}\\
& =\left[\hat{p}_{k}, \hat{p}_{j}\right]-q\left(\left[\partial_{k} \chi, \hat{p}_{j}\right]+\left[\hat{p}_{k}, \partial_{j \chi}\right]\right) .
\end{align*}
$$

That last sum of commutators is

$$
\begin{align*}
{\left[\partial_{k} \chi, \hat{p}_{j}\right]+\left[\hat{p}_{k}, \partial_{j \chi} \chi\right] } & =-i \hbar\left(\frac{\partial\left(\partial_{j} \chi\right)}{\partial k}-\frac{\partial\left(\partial_{k} \chi\right)}{\partial j}\right)  \tag{2.163}\\
& =0
\end{align*}
$$

We've shown that

$$
\begin{align*}
{\left[\hat{p}_{k}^{\prime}, \hat{x}_{k}^{\prime}\right] } & =\left[\hat{p}_{k}, \hat{x}_{k}\right] \\
{\left[\hat{p}_{k}^{\prime}, \hat{p}_{j}^{\prime}\right] } & =\left[\hat{p}_{k}, \hat{p}_{j}\right] . \tag{2.164}
\end{align*}
$$

All the other commutators clearly have the desired transformation properties.

## Exercise 2.4 Heisenberg picture spin precession. ([15] pr. 2.1)

For the spin Hamiltonian

$$
\begin{equation*}
H=-\frac{e B}{m c} S_{z}=\omega S_{z}, \tag{2.165}
\end{equation*}
$$

express and solve the Heisenberg equations of motion for $S_{x}(t), S_{y}(t)$, and $S_{z}(t)$.
Answer for Exercise 2.4

The equations of motion are of the form

$$
\begin{align*}
\frac{d S_{i}^{\mathrm{H}}}{d t} & =\frac{1}{i \hbar}\left[S_{i}^{\mathrm{H}}, H\right] \\
& =\frac{1}{i \hbar}\left[U^{\dagger} S_{i} U, H\right] \\
& =\frac{1}{i \hbar}\left(U^{\dagger} S_{i} U H-H U^{\dagger} S_{i} U\right)  \tag{2.166}\\
& =\frac{1}{i \hbar} U^{\dagger}\left(S_{i} H-H S_{i}\right) U \\
& =\frac{\omega}{i \hbar} U^{\dagger}\left[S_{i}, S_{z}\right] U .
\end{align*}
$$

These are

$$
\begin{align*}
\frac{d S_{x}^{\mathrm{H}}}{d t} & =-\omega U^{\dagger} S_{y} U \\
\frac{d S_{y}^{\mathrm{H}}}{d t} & =\omega U^{\dagger} S_{x} U  \tag{2.167}\\
\frac{d S_{z}^{\mathrm{H}}}{d t} & =0 .
\end{align*}
$$

To completely specify these equations, we need to expand $U(t)$, which is

$$
\begin{align*}
U(t) & =e^{-i H t / \hbar} \\
& =e^{-i \omega S_{z} t / \hbar} \\
& =e^{-i \omega \sigma_{z} t / 2} \\
& =\cos (\omega t / 2)-i \sigma_{z} \sin (\omega t / 2) \\
& =\left[\begin{array}{cc}
\cos (\omega t / 2)-i \sin (\omega t / 2) & 0 \\
0 & \cos (\omega t / 2)+i \sin (\omega t / 2)
\end{array}\right]  \tag{2.168}\\
& =\left[\begin{array}{cc}
e^{-i \omega t / 2} & 0 \\
0 & e^{i \omega t / 2}
\end{array}\right] .
\end{align*}
$$

The equations of motion can now be written out in full. To do so seems a bit silly since we also know that $S_{x}^{\mathrm{H}}=U^{\dagger} S_{x} U, S_{y}^{\mathrm{H}} U^{\dagger} S_{x} U$. However, if
that is temporarily forgotten, we can show that the Heisenberg equations of motion can be solved for these too.

$$
\begin{aligned}
U^{\dagger} S_{x} U & =\frac{\hbar}{2}\left[\begin{array}{cc}
e^{i \omega t / 2} & 0 \\
0 & e^{-i \omega t / 2}
\end{array}\right]\left[\begin{array}{ll}
0 & 1 \\
1 & 0
\end{array}\right]\left[\begin{array}{cc}
e^{-i \omega t / 2} & 0 \\
0 & e^{i \omega t / 2}
\end{array}\right] \\
& =\frac{\hbar}{2}\left[\begin{array}{cc}
0 & e^{i \omega t / 2} \\
e^{-i \omega t / 2} & 0
\end{array}\right]\left[\begin{array}{cc}
e^{-i \omega t / 2} & 0 \\
0 & e^{i \omega t / 2}
\end{array}\right] \\
& =\frac{\hbar}{2}\left[\begin{array}{cc}
0 & e^{i \omega t} \\
e^{-i \omega t} & 0
\end{array}\right]
\end{aligned}
$$

and

$$
\begin{align*}
U^{\dagger} S_{y} U & =\frac{\hbar}{2}\left[\begin{array}{cc}
e^{i \omega t / 2} & 0 \\
0 & e^{-i \omega t / 2}
\end{array}\right]\left[\begin{array}{cc}
0 & -i \\
i & 0
\end{array}\right]\left[\begin{array}{cc}
e^{-i \omega t / 2} & 0 \\
0 & e^{i \omega t / 2}
\end{array}\right] \\
& =\frac{i \hbar}{2}\left[\begin{array}{cc}
0 & -e^{i \omega t / 2} \\
e^{-i \omega t / 2} & 0
\end{array}\right]\left[\begin{array}{cc}
e^{-i \omega t / 2} & 0 \\
0 & e^{i \omega t / 2}
\end{array}\right]  \tag{2.170}\\
& =\frac{i \hbar}{2}\left[\begin{array}{cc}
0 & -e^{i \omega t} \\
e^{-i \omega t} & 0
\end{array}\right]
\end{align*}
$$

The equations of motion are now fully specified

$$
\begin{align*}
\frac{d S_{x}^{\mathrm{H}}}{d t} & =-\frac{i \hbar \omega}{2}\left[\begin{array}{cc}
0 & -e^{i \omega t} \\
e^{-i \omega t} & 0
\end{array}\right] \\
\frac{d S_{y}^{\mathrm{H}}}{d t} & =\frac{\hbar \omega}{2}\left[\begin{array}{cc}
0 & e^{i \omega t} \\
e^{-i \omega t} & 0
\end{array}\right]  \tag{2.171}\\
\frac{d S_{z}^{\mathrm{H}}}{d t} & =0 .
\end{align*}
$$

Integration gives

$$
\begin{align*}
S_{x}^{\mathrm{H}} & =\frac{\hbar}{2}\left[\begin{array}{cc}
0 & e^{i \omega t} \\
e^{-i \omega t} & 0
\end{array}\right]+C \\
S_{y}^{\mathrm{H}} & =\frac{\hbar}{2}\left[\begin{array}{cc}
0 & -i e^{i \omega t} \\
i e^{-i \omega t} & 0
\end{array}\right]+C  \tag{2.172}\\
S_{z}^{\mathrm{H}} & =C .
\end{align*}
$$

The integration constants are fixed by the boundary condition $S_{i}^{\mathrm{H}}(0)=S_{i}$, so

$$
\begin{align*}
& S_{x}^{\mathrm{H}}=\frac{\hbar}{2}\left[\begin{array}{cc}
0 & e^{i \omega t} \\
e^{-i \omega t} & 0
\end{array}\right] \\
& S_{y}^{\mathrm{H}}=\frac{i \hbar}{2}\left[\begin{array}{cc}
0 & -e^{i \omega t} \\
e^{-i \omega t} & 0
\end{array}\right]  \tag{2.173}\\
& S_{z}^{\mathrm{H}}=S_{z} .
\end{align*}
$$

Observe that these integrated values $S_{x}^{\mathrm{H}}, S_{y}^{\mathrm{H}}$ match eq. (2.169), and eq. (2.170) as expected.

Exercise 2.5 Dynamics of non-Hermitian Hamiltonian. ([15] pr. 2.2)
Revisiting an earlier Hamiltonian, but assuming it was entered incorrectly as

$$
\begin{equation*}
H=H_{11}|1\rangle\langle 1|+H_{22}|2\rangle\langle 2|+H_{12}|1\rangle\langle 2| . \tag{2.174}
\end{equation*}
$$

What principle is now violated? Illustrate your point explicitly by attempting to solve the most general time-dependent problem using an illegal Hamiltonian of this kind. You may assume that $H_{11}=H_{22}$ for simplicity. Answer for Exercise 2.5

In matrix form this Hamiltonian is

$$
\begin{align*}
H & =\left[\begin{array}{ll}
\langle 1| H|1\rangle & \langle 1| H|2\rangle \\
\langle 2| H|1\rangle & \langle 2| H|2\rangle
\end{array}\right]  \tag{2.175}\\
& =\left[\begin{array}{cc}
H_{11} & H_{12} \\
0 & H_{22}
\end{array}\right] .
\end{align*}
$$

This is not a Hermitian operator. What is the physical implication of this non-Hermicity? Consider the simpler case where $H_{11}=H_{22}$. Such a Hamiltonian has the form

$$
H=\left[\begin{array}{ll}
a & b  \tag{2.176}\\
0 & a
\end{array}\right]
$$

This has only one unique eigenvector ( $(1,0)$, but we can still solve the time evolution equation

$$
\begin{equation*}
i \hbar \frac{\partial U}{\partial t}=H U \tag{2.177}
\end{equation*}
$$

since for constant $H$, we have

$$
\begin{equation*}
U=e^{-i H t / \hbar} \tag{2.178}
\end{equation*}
$$

To exponentiate, note that we have

$$
\left[\begin{array}{ll}
a & b  \tag{2.179}\\
0 & a
\end{array}\right]^{n}=\left[\begin{array}{cc}
a^{n} & n a^{n-1} b \\
0 & a^{n}
\end{array}\right]
$$

To prove the induction, the $n=2$ case follows easily

$$
\left[\begin{array}{cc}
a & b  \tag{2.180}\\
0 & a
\end{array}\right]\left[\begin{array}{cc}
a & b \\
0 & a
\end{array}\right]=\left[\begin{array}{cc}
a^{2} & 2 a b \\
0 & a^{2}
\end{array}\right],
$$

as does the general case

$$
\left[\begin{array}{cc}
a^{n} & n a^{n-1} b  \tag{2.181}\\
0 & a^{n}
\end{array}\right]\left[\begin{array}{cc}
a & b \\
0 & a
\end{array}\right]=\left[\begin{array}{cc}
a^{n+1} & (n+1) a^{n} b \\
0 & a^{n+1}
\end{array}\right]
$$

The exponential sum is thus

$$
e^{H \tau}=\left[\begin{array}{cc}
e^{a \tau} & 0+\frac{b \tau}{1!}+\frac{2 a b \tau^{2}}{2!}+\frac{3 a^{2} b \tau^{3}}{3!}+\cdots  \tag{2.182}\\
0 & e^{a \tau}
\end{array}\right] .
$$

That sum simplifies to

$$
\begin{align*}
\frac{b \tau}{0!}+\frac{a b \tau^{2}}{1!}+\frac{a^{2} b \tau^{3}}{2!}+\cdots & \\
& =b \tau\left(1+\frac{a \tau}{1!}+\frac{(a \tau)^{2}}{2!}+\cdots\right)  \tag{2.183}\\
& =b \tau e^{a \tau}
\end{align*}
$$

The exponential is thus

$$
\begin{align*}
e^{H \tau} & =\left[\begin{array}{cc}
e^{a \tau} & b \tau e^{a \tau} \\
0 & e^{a \tau}
\end{array}\right]  \tag{2.184}\\
& =\left[\begin{array}{cc}
1 & b \tau \\
0 & 1
\end{array}\right] e^{a \tau} .
\end{align*}
$$

In particular

$$
\begin{align*}
U & =e^{-i H t / \hbar} \\
& =\left[\begin{array}{cc}
1 & -i b t / \hbar \\
0 & 1
\end{array}\right] e^{-i a t / \hbar .} \tag{2.185}
\end{align*}
$$

We can verify that this is a solution to eq. (2.177). The left hand side is

$$
\begin{align*}
i \hbar \frac{\partial U}{\partial t} & =i \hbar\left[\begin{array}{cc}
-i a / \hbar & -i b / \hbar+(-i b t / \hbar)(-i a / \hbar) \\
0 & -i a / \hbar
\end{array}\right] e^{-i a t / \hbar} \\
& =\left[\begin{array}{cc}
a & b-i a b t / \hbar \\
0 & a
\end{array}\right] e^{-i a t / \hbar} \tag{2.186}
\end{align*}
$$

and for the right hand side

$$
\begin{align*}
H U & =\left[\begin{array}{ll}
a & b \\
0 & a
\end{array}\right]\left[\begin{array}{cc}
1 & -i b t / \hbar \\
0 & 1
\end{array}\right] e^{-i a t / \hbar} \\
& =\left[\begin{array}{cc}
a & b-i a b t / \hbar \\
0 & a
\end{array}\right] e^{-i a t / \hbar}  \tag{2.187}\\
& =i \hbar \frac{\partial U}{\partial t} .
\end{align*}
$$

While the Schrödinger is satisfied, we don't have the unitary inversion physical property that is desired for the time evolution operator $U$. Namely

$$
\begin{align*}
U^{\dagger} U & =\left[\begin{array}{cc}
1 & 0 \\
i b t / \hbar & 1
\end{array}\right] e^{i a t / \hbar}\left[\begin{array}{cc}
1 & -i b t / \hbar \\
0 & 1
\end{array}\right] e^{-i a t / \hbar} \\
& =\left[\begin{array}{cc}
1 & -i b t / \hbar \\
i b t / \hbar & (b t)^{2} / \hbar^{2}
\end{array}\right] \neq I . \tag{2.188}
\end{align*}
$$

We required $U^{\dagger} U=I$ for the time evolution operator, but don't have that property for this non-Hermitian Hamiltonian.

Exercise 2.6 Spin 1/2 evolution and dispersion. ([15] pr. 2.3)
A spin $1 / 2 \operatorname{system} \mathbf{S} \cdot \hat{\mathbf{n}}$, with $\hat{\mathbf{n}}=\sin \beta \hat{\mathbf{x}}+\cos \beta \hat{\mathbf{z}}$, is in state with eigenvalue $\hbar / 2$, acted on by a magnetic field of strength $B$ in the $+z$ direction.
a. If $S_{x}$ is measured at time $t$, what is the probability of getting $+\hbar / 2$ ?
b. Evaluate the dispersion in $S_{x}$ as a function of t , that is,

$$
\begin{equation*}
\left\langle\left(S_{x}-\left\langle S_{x}\right\rangle\right)^{2}\right\rangle . \tag{2.189}
\end{equation*}
$$

c. Check your answers for $\beta \rightarrow 0, \pi / 2$ to see if they make sense.

Answer for Exercise 2.6

Part $a$. The spin operator in matrix form is

$$
\begin{align*}
S \cdot \hat{\mathbf{n}} & =\frac{\hbar}{2}\left(\sigma_{z} \cos \beta+\sigma_{x} \sin \beta\right) \\
& =\frac{\hbar}{2}\left(\left[\begin{array}{cc}
1 & 0 \\
0 & -1
\end{array}\right] \cos \beta+\left[\begin{array}{ll}
0 & 1 \\
1 & 0
\end{array}\right] \sin \beta\right)  \tag{2.190}\\
& =\frac{\hbar}{2}\left[\begin{array}{cc}
\cos \beta & \sin \beta \\
\sin \beta & -\cos \beta
\end{array}\right] .
\end{align*}
$$

The $|S \cdot \hat{\mathbf{n}} ;+\rangle$ eigenstate is found from

$$
(S \cdot \hat{\mathbf{n}}-\hbar / 2)\left[\begin{array}{l}
a  \tag{2.191}\\
b
\end{array}\right]=0
$$

or

$$
\begin{align*}
0 & =(\cos \beta-1) a+\sin \beta b \\
& =\left(-2 \sin ^{2}(\beta / 2)\right) a+2 \sin (\beta / 2) \cos (\beta / 2) b  \tag{2.192}\\
& =(-\sin (\beta / 2)) a+\cos (\beta / 2) b
\end{align*}
$$

or

$$
|S \cdot \hat{\mathbf{n}} ;+\rangle=\left[\begin{array}{c}
\cos (\beta / 2)  \tag{2.193}\\
\sin (\beta / 2)
\end{array}\right]
$$

The Hamiltonian is

$$
\begin{equation*}
H=-\frac{e B}{m c} S_{z}=-\frac{e B \hbar}{2 m c} \sigma_{z} \tag{2.194}
\end{equation*}
$$

so the time evolution operator is

$$
\begin{align*}
U & =e^{-i H t / \hbar} \\
& =e^{\frac{i e B t}{2 m c} \sigma_{z}} . \tag{2.195}
\end{align*}
$$

Let $\omega=e B /(2 m c)$, so

$$
\begin{align*}
U & =e^{i \sigma_{z} \omega t} \\
& =\cos (\omega t)+i \sigma_{z} \sin (\omega t) \\
& =\left[\begin{array}{ll}
1 & 0 \\
0 & 1
\end{array}\right] \cos (\omega t)+i\left[\begin{array}{cc}
1 & 0 \\
0 & -1
\end{array}\right] \sin (\omega t)  \tag{2.196}\\
& =\left[\begin{array}{cc}
e^{i \omega t} & 0 \\
0 & e^{-i \omega t}
\end{array}\right]
\end{align*}
$$

The time evolution of the initial state is

$$
\begin{align*}
|S \cdot \hat{\mathbf{n}} ;+\rangle(t) & =U|S \cdot \hat{\mathbf{n}} ;+\rangle(0) \\
& =\left[\begin{array}{cc}
e^{i \omega t} & 0 \\
0 & e^{-i \omega t}
\end{array}\right]\left[\begin{array}{l}
\cos (\beta / 2) \\
\sin (\beta / 2)
\end{array}\right]  \tag{2.197}\\
& =\left[\begin{array}{l}
\cos (\beta / 2) e^{i \omega t} \\
\sin (\beta / 2) e^{-i \omega t}
\end{array}\right]
\end{align*}
$$

The probability of finding the state in $|S \cdot \hat{\mathbf{x}} ;+\rangle$ at time $t$ (i.e. measuring $S_{x}$ and finding $\hbar / 2$ ) is

$$
\begin{align*}
|\langle S \cdot \hat{\mathbf{x}} ;+\mid S \cdot \hat{\mathbf{n}} ;+\rangle|^{2} & =\left|\frac{1}{\sqrt{2}}\left[\begin{array}{ll}
1 & 1
\end{array}\right]\left[\begin{array}{c}
\cos (\beta / 2) e^{i \omega t} \\
\sin (\beta / 2) e^{-i \omega t}
\end{array}\right]\right|^{2} \\
& =\frac{1}{2}\left|\cos (\beta / 2) e^{i \omega t}+\sin (\beta / 2) e^{-i \omega t}\right|^{2}  \tag{2.198}\\
& =\frac{1}{2}(1+2 \cos (\beta / 2) \sin (\beta / 2) \cos (2 \omega t)) \\
& =\frac{1}{2}(1+\sin (\beta) \cos (2 \omega t))
\end{align*}
$$

Part b. To calculate the dispersion first note that

$$
\begin{align*}
S_{x}^{2} & =\left(\frac{\hbar}{2}\right)^{2}\left[\begin{array}{ll}
0 & 1 \\
1 & 0
\end{array}\right]^{2}  \tag{2.199}\\
& =\left(\frac{\hbar}{2}\right)^{2}
\end{align*}
$$

so only the first order expectation is non-trivial to calculate. That is

$$
\begin{align*}
\left\langle S_{x}\right\rangle & =\frac{\hbar}{2}\left[\begin{array}{ll}
\cos (\beta / 2) e^{-i \omega t} & \sin (\beta / 2) e^{i \omega t}
\end{array}\right]\left[\begin{array}{ll}
0 & 1 \\
1 & 0
\end{array}\right]\left[\begin{array}{c}
\cos (\beta / 2) e^{i \omega t} \\
\sin (\beta / 2) e^{-i \omega t}
\end{array}\right] \\
& =\frac{\hbar}{2}\left[\begin{array}{ll}
\cos (\beta / 2) e^{-i \omega t} & \sin (\beta / 2) e^{i \omega t}
\end{array}\right]\left[\begin{array}{c}
\sin (\beta / 2) e^{-i \omega t} \\
\cos (\beta / 2) e^{i \omega t}
\end{array}\right]  \tag{2.200}\\
& =\frac{\hbar}{2} \sin (\beta / 2) \cos (\beta / 2)\left(e^{-2 i \omega t}+e^{2 i \omega t}\right) \\
& =\frac{\hbar}{2} \sin \beta \cos (2 \omega t)
\end{align*}
$$

This gives

$$
\begin{equation*}
\left\langle\left(\Delta S_{x}\right)^{2}\right\rangle=\left(\frac{\hbar}{2}\right)^{2}\left(1-\sin ^{2} \beta \cos ^{2}(2 \omega t)\right) \tag{2.201}
\end{equation*}
$$

Part c. For $\beta=0, \hat{\mathbf{n}}=\hat{\mathbf{z}}$, and $\beta=\pi / 2, \hat{\mathbf{n}}=\hat{\mathbf{x}}$. For the first case, the state is in an eigenstate of $S_{z}$, so must evolve as

$$
\begin{equation*}
|S \cdot \hat{\mathbf{n}} ;+\rangle(t)=|S \cdot \hat{\mathbf{n}} ;+\rangle(0) e^{i \omega t} \tag{2.202}
\end{equation*}
$$

The probability of finding it in state $|S \cdot \hat{\mathbf{x}} ;+\rangle$ is therefore

$$
\begin{align*}
\left|\frac{1}{\sqrt{2}}\left[\begin{array}{ll}
1 & 1
\end{array}\right]\left[\begin{array}{c}
e^{i \omega t} \\
0
\end{array}\right]\right|^{2} & =\frac{1}{2}\left|e^{i \omega t}\right|^{2} \\
& =\frac{1}{2}  \tag{2.203}\\
& =\frac{1}{2}(1+\sin (0) \cos (2 \omega t))
\end{align*}
$$

This matches eq. (2.198) as expected.
For $\beta=\pi / 2$ we have

$$
\begin{align*}
|S \cdot \hat{\mathbf{x}} ;+\rangle(t) & =\frac{1}{\sqrt{2}}\left[\begin{array}{cc}
e^{i \omega t} & 0 \\
0 & e^{-i \omega t}
\end{array}\right]\left[\begin{array}{l}
1 \\
1
\end{array}\right] \\
& =\frac{1}{\sqrt{2}}\left[\begin{array}{l}
e^{i \omega t} \\
e^{-i \omega t}
\end{array}\right] \tag{2.204}
\end{align*}
$$

The probability for the $\hbar / 2 S_{x}$ measurement at time $t$ is

$$
\begin{align*}
\left|\frac{1}{2}\left[\begin{array}{ll}
1 & 1
\end{array}\right]\left[\begin{array}{c}
e^{i \omega t} \\
e^{-i \omega t}
\end{array}\right]\right|^{2} & =\frac{1}{4}\left|e^{i \omega t}+e^{-i \omega t}\right|^{2} \\
& =\cos ^{2}(\omega t)  \tag{2.205}\\
& =\frac{1}{2}(1+\sin (\pi / 2) \cos (2 \omega t))
\end{align*}
$$

Again, this matches the expected value. For the dispersions, at $\beta=0$, the dispersion is

$$
\begin{equation*}
\left(\frac{\hbar}{2}\right)^{2} \tag{2.206}
\end{equation*}
$$

This is the maximum dispersion, which makes sense since we are measuring $S_{x}$ when the initial state is $|S \cdot \hat{\mathbf{z}} ;+\rangle$. For $\beta=\pi / 2$ the dispersion is

$$
\begin{equation*}
\left(\frac{\hbar}{2}\right)^{2} \sin ^{2}(2 \omega t) \tag{2.207}
\end{equation*}
$$

This starts off as zero dispersion (because the initial state is $|S \cdot \hat{\mathbf{x}} ;+\rangle$, but then oscillates.

Exercise 2.7 Heisenberg picture position commutator. ([15] pr. 2.5)
Evaluate

$$
\begin{equation*}
[x(t), x(0)], \tag{2.208}
\end{equation*}
$$

for a Heisenberg picture operator $x(t)$ for a free particle.
Answer for Exercise 2.7
The free particle Hamiltonian is

$$
\begin{equation*}
H=\frac{p^{2}}{2 m} \tag{2.209}
\end{equation*}
$$

so the time evolution operator is

$$
\begin{equation*}
U(t)=e^{-i p^{2} t /(2 m \hbar)} \tag{2.210}
\end{equation*}
$$

The Heisenberg picture position operator is

$$
\begin{align*}
x^{\mathrm{H}} & =U^{\dagger} x U \\
& =e^{i p^{2} t /(2 m \hbar)} x e^{-i p^{2} t /(2 m \hbar)} \\
& =\sum_{k=0}^{\infty} \frac{1}{k!}\left(\frac{i p^{2} t}{2 m \hbar}\right)^{k} x e^{-i p^{2} t /(2 m \hbar)} \\
& =\sum_{k=0}^{\infty} \frac{1}{k!}\left(\frac{i t}{2 m \hbar}\right)^{k} p^{2 k} x e^{-i p^{2} t /(2 m \hbar)} \\
& =\sum_{k=0}^{\infty} \frac{1}{k!}\left(\frac{i t}{2 m \hbar}\right)^{k}\left(\left[p^{2 k}, x\right]+x p^{2 k}\right) e^{-i p^{2} t /(2 m \hbar)} \\
& =x+\sum_{k=0}^{\infty} \frac{1}{k!}\left(\frac{i t}{2 m \hbar}\right)^{k}\left[p^{2 k}, x\right] e^{-i p^{2} t /(2 m \hbar)}  \tag{2.211}\\
& =x+\sum_{k=0}^{\infty} \frac{1}{k!}\left(\frac{i t}{2 m \hbar}\right)^{k}\left(-i \hbar \frac{\partial p^{2 k}}{\partial p}\right) e^{-i p^{2} t /(2 m \hbar)} \\
& =x+\sum_{k=0}^{\infty} \frac{1}{k!}\left(\frac{i t}{2 m \hbar}\right)^{k}\left(-i \hbar 2 k p^{2 k-1}\right) e^{-i p^{2} t /(2 m \hbar)} \\
& =x+-2 i \hbar p \frac{i t}{2 m \hbar} \sum_{k=1}^{\infty} \frac{1}{(k-1)!}\left(\frac{i t}{2 m \hbar}\right)^{k-1} p^{2(k-1)} e^{-i p^{2} t /(2 m \hbar)} \\
& =x+t \frac{p}{m} .
\end{align*}
$$

This has the structure of a classical free particle $x(t)=x+v t$, but in this case $x, p$ are operators.

The evolved position commutator is

$$
\begin{align*}
{[x(t), x(0)] } & =[x+t p / m, x] \\
& =\frac{t}{m}[p, x]  \tag{2.212}\\
& =-i \hbar \frac{t}{m}
\end{align*}
$$

Compare this to the classical Poisson bracket

$$
\begin{align*}
{[x(t), x(0)]_{\text {classical }} } & =\frac{\partial}{\partial x}(x+p t / m) \frac{\partial x}{\partial p}-\frac{\partial}{\partial p}(x+p t / m) \frac{\partial x}{\partial x}  \tag{2.213}\\
& =-\frac{t}{m} .
\end{align*}
$$

This has the expected relation $[x(t), x(0)]=i \hbar[x(t), x(0)]_{\text {classical }}$.

## Exercise $2.8 \quad$ Virial theorem. (2015 ps2.2)

Consider a three-dimensional Hamiltonian

$$
\begin{equation*}
H=\frac{\mathbf{p}^{2}}{2 m}+V(\mathbf{x}) \tag{2.214}
\end{equation*}
$$

a. Calculate $[\mathbf{x} \cdot \mathbf{p}, H]$ and show that

$$
\begin{equation*}
\frac{d}{d t}\langle\mathbf{x} \cdot \mathbf{p}\rangle=\left\langle\frac{\mathbf{p}^{2}}{m}\right\rangle-\langle\mathbf{x} \cdot \boldsymbol{\nabla} V\rangle \tag{2.215}
\end{equation*}
$$

When the l.h.s. vanishes, the result that the r.h.s. is zero is called the quantum virial theorem.
b. Consider the 3D isotropic harmonic oscillator, and show explicitly that its eigenstates obey the virial theorem.
c. Evaluate the r.h.s. for the superposition state $|0,0,0\rangle+|0,0,2\rangle$ where the notation stands for $\left|n_{x}, n_{y}, n_{z}\right\rangle$ occupation numbers.
d. Under what conditions does the left-hand side vanish?

Answer for Exercise 2.8

Part $a$. We'll need various commutators to evaluate the Heisenberg equation of motion.

$$
\begin{align*}
{[\mathbf{x} \cdot \mathbf{p}, H] } & =\frac{1}{2 m}\left[\mathbf{x} \cdot \mathbf{p}, \mathbf{p}^{2}\right]+[\mathbf{x} \cdot \mathbf{p}, V(\mathbf{x})] \\
& =\frac{1}{2 m}\left(x_{r} p_{r} \mathbf{p}^{2}-\mathbf{p}^{2} x_{r} p_{r}\right)+\left(x_{r} p_{r} V(\mathbf{x})-V(\mathbf{x}) x_{r} p_{r}\right)  \tag{2.216}\\
& =\frac{1}{2 m}\left[x_{r}, \mathbf{p}^{2}\right] p_{r}+x_{r}\left[p_{r}, V(\mathbf{x})\right]
\end{align*}
$$

Evaluating those commutators separately, gives

$$
\begin{align*}
{\left[x_{r}, \mathbf{p}^{2}\right] } & =\left[x_{r}, p_{r}^{2}\right] \quad \text { (no sum) }  \tag{2.217}\\
& =2 i \hbar p_{r}
\end{align*}
$$

and

$$
\begin{equation*}
\left[p_{r}, V(\mathbf{x})\right]=-i \hbar \frac{\partial V(\mathbf{x})}{\partial x_{r}} \tag{2.218}
\end{equation*}
$$

so

$$
\begin{align*}
\frac{d}{d t}(\mathbf{x} \cdot \mathbf{p}) & =\frac{1}{i \hbar}[\mathbf{x} \cdot \mathbf{p}, H] \\
& =\frac{1}{2 m} 2 p_{r} p_{r}-x_{r} \frac{\partial V(\mathbf{x})}{\partial x_{r}}  \tag{2.219}\\
& =\frac{\mathbf{p}^{2}}{m}-\mathbf{x} \cdot \boldsymbol{\nabla} V(\mathbf{x})
\end{align*}
$$

Evaluating the expectation of this identity with respect to a stationary state $|\psi\rangle$ (i.e. a state independent of time), we have

$$
\begin{align*}
\langle\psi| \frac{d}{d t} \mathbf{x} \cdot \mathbf{p}|\psi\rangle & =\frac{d}{d t}\langle\psi| \mathbf{x} \cdot \mathbf{p}|\psi\rangle \\
& =\frac{d}{d t}\langle\mathbf{x} \cdot \mathbf{p}\rangle  \tag{2.220}\\
& =\left\langle\frac{\mathbf{p}^{2}}{m}\right\rangle-\langle\mathbf{x} \cdot \nabla V(\mathbf{x})\rangle .
\end{align*}
$$

Because the expectation was with respect to a stationary state, the time derivative could be moved outside of the expectation operation.

Part b. When eq. (2.220) is zero, we have the quantum equivalent of the virial theorem, relating the average kinetic energy to the potential

$$
\begin{equation*}
2\langle T\rangle=\langle\mathbf{x} \cdot \nabla V(\mathbf{x})\rangle \tag{2.221}
\end{equation*}
$$

To evaluate these expectations operations with respect to the 3D SHO eigenstates, let

$$
\begin{align*}
& a_{x}(t)=a_{x} e^{-i \omega t} \\
& a_{y}(t)=a_{y} e^{-i \omega t}  \tag{2.222}\\
& a_{z}(t)=a_{z} e^{-i \omega t}
\end{align*}
$$

Note that

$$
\begin{align*}
\langle\mathbf{x} \cdot \boldsymbol{\nabla} V\rangle & =m \omega^{2}\left\langle\mathbf{x}^{2}(t)\right\rangle \\
& =m \omega^{2} \frac{x_{0}^{2}}{2} \sum_{k=1}^{3}\left\langle\left(a_{k}(t)+a_{k}^{\dagger}(t)\right)^{2}\right\rangle \\
& =\frac{\hbar \omega}{2} \sum_{k=1}^{3}\langle\psi|\left(a_{k}(t)+a_{k}^{\dagger}(t)\right)^{2}|\psi\rangle  \tag{2.223}\\
& =\frac{\hbar \omega}{2} \sum_{k=1}^{3}\langle\psi|\left(a_{k} e^{-i \omega t}+a_{k}^{\dagger} e^{i \omega t}\right)^{2}|\psi\rangle
\end{align*}
$$

and

$$
\begin{align*}
\left\langle\frac{\mathbf{p}^{2}}{m}\right\rangle & =\frac{-\hbar^{2}}{2 m x_{0}^{2}} \sum_{k=1}^{3}\left\langle\left(a_{k}^{\dagger}(t)-a_{k}(t)\right)^{2}\right\rangle \\
& =\frac{\hbar \omega}{2} \sum_{k=1}^{3}\langle\psi|\left(a_{k}(t)-a_{k}^{\dagger}(t)\right)\left(a_{k}^{\dagger}(t)-a_{k}(t)\right)|\psi\rangle  \tag{2.224}\\
& =\frac{\hbar \omega}{2} \sum_{k=1}^{3}\langle\psi|\left(a_{k} e^{-i \omega t}-a_{k}^{\dagger} e^{i \omega t}\right)\left(a_{k}^{\dagger} e^{i \omega t}-a_{k} e^{-i \omega t}\right)|\psi\rangle .
\end{align*}
$$

In both cases the pairs of raising and lowering operators have been factored into conjugate pairs so that only the action of the latter on $|\psi\rangle$ need be considered. Considering the $x$ component for example, we've got

$$
\begin{align*}
& \left(a_{x}^{\dagger} e^{i \omega t} \pm a_{x} e^{-i \omega t}\right)\left|n_{x}, n_{y}, n_{z}\right\rangle \\
& \quad=e^{i \omega t} \sqrt{n_{x}+1}\left|n_{x}+1, n_{y}, n_{z}\right\rangle \pm e^{-i \omega t} \sqrt{n_{x}}\left|n_{x}-1, n_{y}, n_{z}\right\rangle \tag{2.225}
\end{align*}
$$

This gives

$$
\begin{equation*}
\langle\mathbf{x} \cdot \boldsymbol{\nabla} V\rangle=\left\langle\frac{\mathbf{p}^{2}}{m}\right\rangle=\frac{\hbar \omega}{2}\left(\left(n_{x}+1\right)+n_{x}+\left(n_{y}+1\right)+n_{y}+\left(n_{z}+1\right)+n_{z}\right), \tag{2.226}
\end{equation*}
$$

or

$$
\begin{equation*}
\langle\mathbf{x} \cdot \boldsymbol{\nabla} V\rangle=\left\langle\frac{\mathbf{p}^{2}}{m}\right\rangle=\hbar \omega\left(n_{x}+n_{y}+n_{z}+\frac{3}{2}\right) . \tag{2.227}
\end{equation*}
$$

Neither expectation has any time dependence, and the virial theorem has been confirmed.

Part c. With $|\psi\rangle=|0,0,0\rangle+|0,0,2\rangle$, the gradient portion of the RHS is

$$
\begin{align*}
\langle\mathbf{x} \cdot \boldsymbol{\nabla} V\rangle & =m \omega^{2}\left\langle\mathbf{x}^{2}(t)\right\rangle \\
& =m \omega^{2} \frac{x_{0}^{2}}{2} \sum_{k=1}^{3}\langle\psi|\left(a_{k} e^{-i \omega t}+a_{k}^{\dagger} e^{i \omega t}\right)^{2}|\psi\rangle  \tag{2.228}\\
& =\frac{\hbar \omega}{2} \sum_{k=1}^{3}\langle\psi|\left(a_{k} e^{-i \omega t}+a_{k}^{\dagger} e^{i \omega t}\right)^{2}|\psi\rangle
\end{align*}
$$

We require the following for each $k$

$$
\begin{equation*}
\left(a_{k} e^{-i \omega t}+a_{k}^{\dagger} e^{i \omega t}\right)(|0,0,0\rangle+|0,0,2\rangle) \tag{2.229}
\end{equation*}
$$

For $k=1$ this is

$$
\begin{equation*}
|1,0,0\rangle+|1,0,2\rangle, \tag{2.230}
\end{equation*}
$$

for $k=2$ this is

$$
\begin{equation*}
|0,1,0\rangle+|0,1,2\rangle, \tag{2.231}
\end{equation*}
$$

and for $k=3$

$$
\begin{align*}
e^{-i \omega t} & \sqrt{2}|0,0,1\rangle+e^{i \omega t}(|0,0,1\rangle+\sqrt{3}|0,0,3\rangle) \\
& =|0,0,1\rangle\left(e^{-i \omega t} \sqrt{2}+e^{i \omega t}\right)+\sqrt{3} e^{i \omega t}|0,0,3\rangle \tag{2.232}
\end{align*}
$$

This gives

$$
\begin{align*}
\langle\mathbf{x} \cdot \boldsymbol{\nabla} V\rangle & =\frac{\hbar \omega}{2}\left(2+2+3+\left|e^{-i \omega t} \sqrt{2}+e^{i \omega t}\right|^{2}\right) \\
& =\frac{\hbar \omega}{2}(2+2+3+1+2+2 \sqrt{2} \cos (2 \omega t)), \tag{2.233}
\end{align*}
$$

or

$$
\begin{equation*}
\langle\mathbf{x} \cdot \nabla V\rangle=\hbar \omega(5+\sqrt{2} \cos (2 \omega t)) \tag{2.234}
\end{equation*}
$$

For the kinetic portion, we've got

$$
\begin{equation*}
\left\langle\frac{\mathbf{p}^{2}}{m}\right\rangle=\frac{\hbar \omega}{2} \sum_{k=1}^{3}\langle\psi|\left(a_{k} e^{-i \omega t}-a_{k}^{\dagger} e^{i \omega t}\right)\left(a_{k}^{\dagger} e^{i \omega t}-a_{k} e^{-i \omega t}\right)|\psi\rangle . \tag{2.235}
\end{equation*}
$$

The squared momentum operator has been factored into conjugate pairs, so only the action on $|\psi\rangle$ need be computed. For the $x$ component of that operation we have

$$
\begin{align*}
\left(a_{x}^{\dagger} e^{i \omega t}-a_{x} e^{-i \omega t}\right)(|0,0,0\rangle+|0,0,2\rangle) & =a_{x}^{\dagger} e^{i \omega t}(|0,0,0\rangle+|0,0,2\rangle) \\
& =e^{i \omega t}(|1,0,0\rangle+|1,0,2\rangle) . \tag{2.236}
\end{align*}
$$

The $y$ component, by inspection, must be

$$
\begin{align*}
\left(a_{x}^{\dagger} e^{i \omega t}-a_{x} e^{-i \omega t}\right) & (|0,0,0\rangle+|0,0,2\rangle)  \tag{2.237}\\
& =e^{i \omega t}(|0,1,0\rangle+|0,1,2\rangle)
\end{align*}
$$

The $z$ component is slightly messier to compute

$$
\begin{align*}
& \left(a_{z}^{\dagger} e^{i \omega t}-a_{z} e^{-i \omega t}\right)(|0,0,0\rangle+|0,0,2\rangle) \\
& =e^{i \omega t}(|0,0,1\rangle+\sqrt{3}|0,0,3\rangle)-e^{-i \omega t} \sqrt{2}|0,0,1\rangle  \tag{2.238}\\
& =\left(e^{i \omega t}-e^{-i \omega t} \sqrt{2}\right)|0,0,1\rangle+e^{i \omega t} \sqrt{3}|0,0,3\rangle
\end{align*}
$$

Summation of all the component contributions gives

$$
\begin{align*}
\left\langle\frac{\mathbf{p}^{2}}{m}\right\rangle & =\frac{\hbar \omega}{2}\left(2+2+3++\left|e^{i \omega t}-e^{-i \omega t} \sqrt{2}\right|^{2}\right)  \tag{2.239}\\
& =\frac{\hbar \omega}{2}(4+3+1+2-2 \sqrt{2} \cos (2 \omega t))
\end{align*}
$$

or

$$
\begin{equation*}
\left\langle\frac{\mathbf{p}^{2}}{m}\right\rangle=\hbar \omega(5-\sqrt{2} \cos (2 \omega t)) \tag{2.240}
\end{equation*}
$$

This does not match eq. (2.234).

Part d. To derive eq. (2.215) the expectation had to be calculated with respect to stationary (non-time dependent) states. As an example, we confirmed this by showing the r.h.s was zero with respect to the energy eigenstates. For the superposition state $|0,0,0\rangle+|0,0,2\rangle$ this was observed to be insufficient. It is clear that the perfect cancellation of the time dependence that was required so that $\left\langle\frac{\mathbf{p}^{2}}{m}\right\rangle=\langle\mathbf{x} \cdot \boldsymbol{\nabla} V\rangle$ will not, in general, be possible for such superposition states. The virial theorem requires not only expectations with respect to stationary states, but requires those stationary states to also be energy eigenstates.

Exercise $2.9 \quad$ A symmetric real Hamiltonian. ([15] pr. 2.9)
Find the time evolution for the state $\left|a^{\prime}\right\rangle$ for a Hamiltonian of the form

$$
\begin{equation*}
H=\delta\left(\left|a^{\prime}\right\rangle\left\langle a^{\prime}\right|+\left|a^{\prime \prime}\right\rangle\left\langle a^{\prime \prime}\right|\right) \tag{2.241}
\end{equation*}
$$

Answer for Exercise 2.9
This Hamiltonian has the matrix representation

$$
H=\left[\begin{array}{ll}
0 & \delta  \tag{2.242}\\
\delta & 0
\end{array}\right]
$$

which has a characteristic equation of

$$
\begin{equation*}
\lambda^{2}-\delta^{2}=0 \tag{2.243}
\end{equation*}
$$

so the energy eigenvalues are $\pm \delta$.
The diagonal basis states are respectively

$$
| \pm \delta\rangle=\frac{1}{\sqrt{2}}\left[\begin{array}{c} 
\pm 1  \tag{2.244}\\
1
\end{array}\right]
$$

The time evolution operator is

$$
\begin{align*}
U & =e^{-i H t / \hbar} \\
& =e^{-i \delta t / \hbar}|+\delta\rangle\langle+\delta|+e^{i \delta t / \hbar}|-\delta\rangle\langle-\delta| \\
& =\frac{e^{-i \delta t / \hbar}}{2}\left[\begin{array}{ll}
1 & 1
\end{array}\right]\left[\begin{array}{l}
1 \\
1
\end{array}\right]+\frac{e^{i \delta t / \hbar}}{2}\left[\begin{array}{ll}
-1 & 1
\end{array}\right]\left[\begin{array}{c}
-1 \\
1
\end{array}\right] \\
& =\frac{e^{-i \delta t / \hbar}}{2}\left[\begin{array}{ll}
1 & 1 \\
1 & 1
\end{array}\right]+\frac{e^{i \delta t / \hbar}}{2}\left[\begin{array}{cc}
1 & -1 \\
-1 & 1
\end{array}\right]  \tag{2.245}\\
& =\left[\begin{array}{cc}
\cos (\delta t / \hbar) & -i \sin (\delta t / \hbar) \\
-i \sin (\delta t / \hbar) & \cos (\delta t / \hbar)
\end{array}\right]
\end{align*}
$$

The desired time evolution in the original basis is

$$
\begin{align*}
\left|a^{\prime}, t\right\rangle & =e^{-i H t / \hbar}\left|a^{\prime}, 0\right\rangle \\
& =\left[\begin{array}{cc}
\cos (\delta t / \hbar) & -i \sin (\delta t / \hbar) \\
-i \sin (\delta t / \hbar) & \cos (\delta t / \hbar)
\end{array}\right]\left[\begin{array}{l}
1 \\
0
\end{array}\right] \\
& =\left[\begin{array}{c}
\cos (\delta t / \hbar) \\
-i \sin (\delta t / \hbar)
\end{array}\right]  \tag{2.246}\\
& =\cos (\delta t / \hbar)\left|a^{\prime}, 0\right\rangle-i \sin (\delta t / \hbar)\left|a^{\prime \prime}, 0\right\rangle
\end{align*}
$$

This evolution has the same structure as left circularly polarized light.
The probability of finding the system in state $\left|a^{\prime \prime}\right\rangle$ given an initial state of $\left|a^{\prime}, 0\right\rangle$ is

$$
\begin{equation*}
P=\left|\left\langle a^{\prime \prime} \mid a^{\prime}, t\right\rangle\right|^{2}=\sin ^{2}(\delta t / \hbar) \tag{2.247}
\end{equation*}
$$

## Exercise 2.10 SHO translation operator expectation. ([15] pr. 2.12)

Using the Heisenberg picture evaluate the expectation of the position operator $\langle x\rangle$ with respect to the initial time state

$$
\begin{equation*}
|\alpha, 0\rangle=e^{-i p_{0} a / \hbar}|0\rangle, \tag{2.248}
\end{equation*}
$$

where $p_{0}$ is the initial time position operator, and $a$ is a constant with dimensions of position.
Answer for Exercise 2.10
Recall that the Heisenberg picture position operator expands to

$$
\begin{align*}
x^{\mathrm{H}}(t) & =U^{\dagger} x U \\
& =x_{0} \cos (\omega t)+\frac{p_{0}}{m \omega} \sin (\omega t) \tag{2.249}
\end{align*}
$$

so the expectation of the position operator is

$$
\begin{align*}
\langle x\rangle & =\langle 0| e^{i p_{0} a / \hbar}\left(x_{0} \cos (\omega t)+\frac{p_{0}}{m \omega} \sin (\omega t)\right) e^{-i p_{0} a / \hbar}|0\rangle \\
& =\langle 0|\left(e^{i p_{0} a / \hbar} x_{0} \cos (\omega t) e^{-i p_{0} a / \hbar} \cos (\omega t)+\frac{p_{0}}{m \omega} \sin (\omega t)\right)|0\rangle \tag{2.250}
\end{align*}
$$

The exponential sandwich above can be expanded using the Baker-CampbellHausdorff [18] formula

$$
\begin{align*}
e^{i p_{0} a / \hbar} x_{0} e^{-i p_{0} a / \hbar} & =x_{0}+\frac{i a}{\hbar}\left[p_{0}, x_{0}\right]+\frac{1}{2!}\left(\frac{i a}{\hbar}\right)^{2}\left[p_{0},\left[p_{0}, x_{0}\right]\right]+\cdots \\
& =x_{0}+\frac{i a}{\hbar}(-i \hbar)+\frac{1}{2!}\left(\frac{i a}{\hbar}\right)^{2}\left[p_{0},-i \hbar\right]+\cdots  \tag{2.251}\\
& =x_{0}+a
\end{align*}
$$

The position expectation with respect to this translated state is

$$
\begin{align*}
\langle x\rangle & =\langle 0|\left(\left(x_{0}+a\right) \cos (\omega t)+\frac{p_{0}}{m \omega} \sin (\omega t)\right)|0\rangle  \tag{2.252}\\
& =a \cos (\omega t)
\end{align*}
$$

The final simplification above follows from $\langle n| x|n\rangle=\langle n| p|n\rangle=0$.
Exercise 2.11 SHO Hamiltonian, and virial theorem. ([15] pr. 2.14)
a. For a 1D SHO, compute $\langle m| x|n\rangle,\langle m| x^{2}|n\rangle,\langle m| p|n\rangle,\langle m| p^{2}|n\rangle$ and $\langle m|\{x, p\}|n\rangle$.
b. Verify the virial theorem is satisfied for energy eigenstates.

Part a. Using

$$
\begin{align*}
x & =\frac{x_{0}}{\sqrt{2}}\left(a+a^{\dagger}\right) \\
p & =\frac{i \hbar}{x_{0} \sqrt{2}}\left(a^{\dagger}-a\right) \\
a(t) & =a(0) e^{-i \omega t}  \tag{2.253}\\
a(0)|n\rangle & =\sqrt{n}|n-1\rangle \\
a^{\dagger}(0)|n\rangle & =\sqrt{n+1}|n+1\rangle \\
x_{0}^{2} & =\frac{\hbar}{\omega m},
\end{align*}
$$

we have

$$
\begin{align*}
\langle m| x|n\rangle= & \frac{x_{0}}{\sqrt{2}}\langle m|\left(a+a^{\dagger}\right)|n\rangle \\
= & \frac{x_{0}}{\sqrt{2}}\langle m|\left(e^{-i \omega t} \sqrt{n}|n-1\rangle+e^{i \omega t} \sqrt{n+1}|n+1\rangle\right)  \tag{2.254}\\
= & \frac{x_{0}}{\sqrt{2}}\left(\delta_{m, n-1} e^{-i \omega t} \sqrt{n}+\delta_{m, n+1} e^{i \omega t} \sqrt{n+1}\right) \\
\langle m| x^{2}|n\rangle= & \frac{x_{0}^{2}}{2}\langle m|\left(a+a^{\dagger}\right)^{2}|n\rangle \\
= & \frac{x_{0}^{2}}{2}\left(e^{i \omega t} \sqrt{m}\langle m-1|+e^{-i \omega t} \sqrt{m+1}\langle m+1|\right) \times \\
& \left(e^{-i \omega t} \sqrt{n}|n-1\rangle+e^{i \omega t} \sqrt{n+1}|n+1\rangle\right) \\
= & \frac{x_{0}^{2}}{2}\left(\delta_{m+1, n+1} \sqrt{(m+1)(n+1)}+\delta_{m+1, n-1} \sqrt{(m+1) n} e^{-2 i \omega t}\right. \\
& \left.+\delta_{m-1, n+1} \sqrt{m(n+1)} e^{2 i \omega t}+\delta_{m-1, n-1} \sqrt{m n}\right) \tag{2.255}
\end{align*}
$$

$$
\begin{align*}
\langle m| p|n\rangle & =\frac{i \hbar}{\sqrt{2} x_{0}}\langle m|\left(a^{\dagger}-a\right)|n\rangle \\
& =\frac{i \hbar}{\sqrt{2} x_{0}}\langle m|\left(e^{i \omega t} \sqrt{n+1}|n+1\rangle-e^{-i \omega t} \sqrt{n}|n-1\rangle\right)  \tag{2.256}\\
& =\frac{i \hbar}{\sqrt{2} x_{0}}\left(\delta_{m, n+1} e^{i \omega t} \sqrt{n+1}-\delta_{m, n-1} e^{-i \omega t} \sqrt{n}\right)
\end{align*}
$$

$$
\begin{align*}
\langle m| p^{2}|n\rangle= & \frac{\hbar^{2}}{2 x_{0}^{2}}|m\rangle\left(a-a^{\dagger}\right)\left(a^{\dagger}-a\right)|n\rangle \\
= & \frac{\hbar^{2}}{2 x_{0}^{2}}\left(-e^{-i \omega t} \sqrt{m+1}\langle m+1|+e^{i \omega t} \sqrt{m}\langle m-1|\right) \times \\
& \left(e^{i \omega t} \sqrt{n+1}|n+1\rangle-e^{-i \omega t} \sqrt{n}|n-1\rangle\right) \\
= & \frac{\hbar^{2}}{2 x_{0}^{2}}\left(\delta_{m+1, n+1} \sqrt{(m+1)(n+1)}+\delta_{m+1, n-1} \sqrt{(m+1) n} e^{-2 i \omega t}\right. \\
& \left.+\delta_{m-1, n+1} \sqrt{m(n+1)} e^{2 i \omega t}+\delta_{m-1, n-1} \sqrt{m n}\right) . \tag{2.257}
\end{align*}
$$

For the anticommutator $\{x, p\}$, we have

$$
\begin{align*}
\{x, p\}= & \frac{i \hbar}{2}\left(\left(a e^{-i \omega t}+a^{\dagger} e^{i \omega t}\right)\left(a^{\dagger} e^{i \omega t}-a e^{-i \omega t}\right)\right. \\
& \left.-\left(a^{\dagger} e^{i \omega t}-a e^{-i \omega t}\right)\left(a e^{-i \omega t}+a^{\dagger} e^{i \omega t}\right)\right) \\
= & \frac{i \hbar}{2}\left(-a^{2} e^{-2 i \omega t}+\left(a^{\dagger}\right)^{2} e^{2 i \omega t}+a a^{\dagger}-a^{\dagger} a\right.  \tag{2.258}\\
& \left.+a^{2} e^{-2 i \omega t}-\left(a^{\dagger}\right)^{2} e^{2 i \omega t}-a^{\dagger} a+a a^{\dagger}\right) \\
= & i \hbar\left(a a^{\dagger}-a^{\dagger} a\right)
\end{align*}
$$

So

$$
\begin{align*}
\langle m|\{x, p\}|n\rangle & =i \hbar\langle m|\left(a a^{\dagger}-a^{\dagger} a\right)|n\rangle \\
& =i \hbar\langle m|\left(\sqrt{(n+1)^{2}}|n\rangle-\sqrt{n^{2}}|n\rangle\right)  \tag{2.259}\\
& =i \hbar\langle m|(2 n+1)|n\rangle .
\end{align*}
$$

Part b. For the SHO , the virial theorem requires $\left\langle p^{2} / m\right\rangle=\left\langle m \omega x^{2}\right\rangle$. That momentum expectation with respect to the eigenstate $|n\rangle$ is

$$
\begin{align*}
\left\langle p^{2} / m\right\rangle & =\frac{\hbar^{2}}{2 x_{0}^{2} m}(\sqrt{(n+1)(n+1)}+\sqrt{n n}) \\
& =\frac{\hbar^{2} m \omega}{2 \hbar m}(2 n+1)  \tag{2.260}\\
& =\hbar \omega\left(n+\frac{1}{2}\right)
\end{align*}
$$

For the position expectation we've got

$$
\begin{align*}
\left\langle m \omega x^{2}\right\rangle & =\frac{m \omega^{2} x_{0}^{2}}{2}(\sqrt{(n+1)(n+1)}+\sqrt{n n}) \\
& =\frac{m \omega^{2} \hbar}{2 m \omega}(\sqrt{(n+1)(n+1)}+\sqrt{n n})  \tag{2.261}\\
& =\frac{\omega \hbar}{2}(2 n+1) \\
& =\omega \hbar\left(n+\frac{1}{2}\right) .
\end{align*}
$$

This shows that the virial theorem holds for the SHO Hamiltonian for eigenstates.

Exercise 2.12 Momentum space Schrödinger eq. ([15] pr. 2.15)
Using

$$
\begin{equation*}
\left\langle x^{\prime} \mid p^{\prime}\right\rangle=\frac{1}{\sqrt{2 \pi \hbar}} e^{i p^{\prime} x^{\prime} / \hbar} \tag{2.262}
\end{equation*}
$$

show that

$$
\begin{equation*}
\left\langle p^{\prime}\right| x|\alpha\rangle=i \hbar \frac{\partial}{\partial p^{\prime}}\left\langle p^{\prime} \mid \alpha\right\rangle \tag{2.263}
\end{equation*}
$$

Use this to find the momentum space representation of the Schrödinger equation for the one dimensional SHO and the energy eigenfunctions in their momentum representation.
Answer for Exercise 2.12

To expand the matrix element, introduce both momentum and position space identity operators

$$
\begin{aligned}
\left\langle p^{\prime}\right| x|\alpha\rangle & =\int d x^{\prime} d p^{\prime \prime}\left\langle p^{\prime} \mid x^{\prime}\right\rangle\left\langle x^{\prime}\right| x\left|p^{\prime \prime}\right\rangle\left\langle p^{\prime \prime} \mid \alpha\right\rangle \\
& =\int d x^{\prime} d p^{\prime \prime}\left\langle p^{\prime} \mid x^{\prime}\right\rangle x^{\prime}\left\langle x^{\prime} \mid p^{\prime \prime}\right\rangle\left\langle p^{\prime \prime} \mid \alpha\right\rangle \\
& =\frac{1}{2 \pi \hbar} \int d x^{\prime} d p^{\prime \prime} e^{-i p^{\prime} x^{\prime} / \hbar} x^{\prime} e^{i p^{\prime \prime} x^{\prime} / \hbar}\left\langle p^{\prime \prime} \mid \alpha\right\rangle \\
& =\frac{1}{2 \pi \hbar} \int d x^{\prime} d p^{\prime \prime} x^{\prime} e^{i\left(p^{\prime \prime}-p^{\prime}\right) x^{\prime} / \hbar}\left\langle p^{\prime \prime} \mid \alpha\right\rangle \\
& =\frac{1}{2 \pi \hbar} \int d x^{\prime} d p^{\prime \prime} \frac{d}{d p^{\prime \prime}}\left(\frac{-i \hbar^{i\left(p^{\prime \prime}-p^{\prime}\right) x^{\prime} / \hbar}}{e}\right)\left\langle p^{\prime \prime} \mid \alpha\right\rangle \\
& =i \hbar \int d p^{\prime \prime}\left(\frac{1}{2 \pi \hbar} \int d x^{\prime} e^{i\left(p^{\prime \prime}-p^{\prime}\right) x^{\prime} / \hbar}\right) \frac{d}{d p^{\prime \prime}}\left\langle p^{\prime \prime} \mid \alpha\right\rangle \\
& =i \hbar \int d p^{\prime \prime} \delta\left(p^{\prime \prime}-p^{\prime}\right) \frac{d}{d p^{\prime \prime}}\left\langle p^{\prime \prime} \mid \alpha\right\rangle \\
& =i \hbar \frac{d}{d p^{\prime}}\left\langle p^{\prime} \mid \alpha\right\rangle .
\end{aligned}
$$

Schrödinger's equation for a time dependent state $|\alpha\rangle=U(t)|\alpha, 0\rangle$ is

$$
\begin{equation*}
i \hbar \frac{\partial}{\partial t}|\alpha\rangle=H|\alpha\rangle \tag{2.265}
\end{equation*}
$$

with the momentum representation

$$
\begin{equation*}
i \hbar \frac{\partial}{\partial t}\left\langle p^{\prime} \mid \alpha\right\rangle=\left\langle p^{\prime}\right| H|\alpha\rangle \tag{2.266}
\end{equation*}
$$

Expansion of the Hamiltonian matrix element for a strictly spatial dependent potential $V(x)$ gives

$$
\begin{align*}
\left\langle p^{\prime}\right| H|\alpha\rangle & =\left\langle p^{\prime}\right|\left(\frac{p^{2}}{2 m}+V(x)\right)|\alpha\rangle  \tag{2.267}\\
& =\frac{\left(p^{\prime}\right)^{2}}{2 m}+\left\langle p^{\prime}\right| V(x)|\alpha\rangle
\end{align*}
$$

Assuming a Taylor representation of the potential $V(x)=\sum c_{k} x^{k}$, we want to calculate

$$
\begin{equation*}
\left\langle p^{\prime}\right| V(x)|\alpha\rangle=\sum c_{k}\left\langle p^{\prime}\right| x^{k}|\alpha\rangle \tag{2.268}
\end{equation*}
$$

With $|\alpha\rangle=\left|p^{\prime \prime}\right\rangle$ eq. (2.263) provides the $k=1$ term

$$
\begin{align*}
\left\langle p^{\prime}\right| x\left|p^{\prime \prime}\right\rangle & =i \hbar \frac{d}{d p^{\prime}}\left\langle p^{\prime} \mid p^{\prime \prime}\right\rangle  \tag{2.269}\\
& =i \hbar \frac{d}{d p^{\prime}} \delta\left(p^{\prime}-p^{\prime \prime}\right)
\end{align*}
$$

where it is implied here that the derivative is operating on not just the delta function, but on all else that follows.

Using this the higher powers of $\left\langle p^{\prime}\right| x^{k}|\alpha\rangle$ can be found easily. For example for $k=2$ we have

$$
\begin{align*}
\left\langle p^{\prime}\right| x^{2}|\alpha\rangle & =\int d p^{\prime \prime}\left\langle p^{\prime}\right| x\left|p^{\prime \prime}\right\rangle\left\langle p^{\prime \prime}\right| x|\alpha\rangle \\
& =\int d p^{\prime \prime} i \hbar \frac{d}{d p^{\prime}} \delta\left(p^{\prime}-p^{\prime \prime}\right) i \hbar \frac{d}{d p^{\prime \prime}}\left\langle p^{\prime \prime} \mid \alpha\right\rangle  \tag{2.270}\\
& =(i \hbar)^{2} \frac{d^{2}}{d\left(p^{\prime}\right)^{2}}\left\langle p^{\prime} \mid \alpha\right\rangle .
\end{align*}
$$

This means that the potential matrix element is

$$
\begin{align*}
\left\langle p^{\prime}\right| V(x)|\alpha\rangle & =\sum c_{k}\left(i \hbar \frac{d}{d p^{\prime}}\right)^{k}\left\langle p^{\prime} \mid \alpha\right\rangle  \tag{2.271}\\
& =V\left(i \hbar \frac{d}{d p^{\prime}}\right) .
\end{align*}
$$

With

$$
\begin{equation*}
\Psi_{\alpha}\left(p^{\prime}\right)=\left\langle p^{\prime} \mid \alpha\right\rangle \tag{2.272}
\end{equation*}
$$

the momentum space representation of Schrödinger's equation for a position dependent potential is

$$
\begin{equation*}
i \hbar \frac{\partial}{\partial t} \Psi_{\alpha}\left(p^{\prime}\right)=\left(\frac{\left(p^{\prime}\right)^{2}}{2 m}+V\left(i \hbar \partial / \partial p^{\prime}\right)\right) \Psi_{\alpha}\left(p^{\prime}\right) \tag{2.273}
\end{equation*}
$$

For the SHO Hamiltonian the potential is $V(x)=(1 / 2) m \omega^{2} x^{2}$, so the Schrödinger equation is

$$
\begin{align*}
i \hbar \frac{\partial}{\partial t} \Psi_{\alpha}\left(p^{\prime}\right) & =\left(\frac{\left(p^{\prime}\right)^{2}}{2 m}-\frac{1}{2} m \omega^{2} \hbar^{2} \frac{\partial^{2}}{\partial\left(p^{\prime}\right)^{2}}\right) \Psi_{\alpha}\left(p^{\prime}\right)  \tag{2.274}\\
& =\frac{1}{2 m}\left(\left(p^{\prime}\right)^{2}-m^{2} \omega^{2} \hbar^{2} \frac{\partial^{2}}{\partial\left(p^{\prime}\right)^{2}}\right) \Psi_{\alpha}\left(p^{\prime}\right) .
\end{align*}
$$

To determine the wave functions, let's non-dimensionalize this and compare to the position space Schrödinger equation. Let

$$
\begin{equation*}
p_{0}^{2}=m \omega \hbar, \tag{2.275}
\end{equation*}
$$

so

$$
\begin{align*}
i \hbar \frac{\partial}{\partial t} \Psi_{\alpha}\left(p^{\prime}\right) & =\frac{p_{0}^{2}}{2 m}\left(\left(\frac{p^{\prime}}{p_{0}}\right)^{2}-\frac{\partial^{2}}{\partial\left(p^{\prime} / p_{0}\right)^{2}}\right) \Psi_{\alpha}\left(p^{\prime}\right) \\
& =\frac{\omega \hbar}{2}\left(-\frac{\partial^{2}}{\partial\left(p^{\prime} / p_{0}\right)^{2}}+\left(\frac{p^{\prime}}{p_{0}}\right)^{2}\right) \Psi_{\alpha}\left(p^{\prime}\right) . \tag{2.276}
\end{align*}
$$

Compare this to the position space equation with $x_{0}^{2}=m \omega / \hbar$,

$$
\begin{align*}
i \hbar \frac{\partial}{\partial t} \Psi_{\alpha}\left(x^{\prime}\right) & =\left(-\frac{\hbar^{2}}{2 m} \frac{\partial^{2}}{\partial\left(x^{\prime}\right)^{2}}+\frac{1}{2} m \omega^{2}\left(x^{\prime}\right)^{2}\right) \Psi_{\alpha}\left(x^{\prime}\right) \\
& =\frac{\hbar^{2}}{2 m}\left(-\frac{\partial^{2}}{\partial\left(x^{\prime}\right)^{2}}+\frac{m^{2} \omega^{2}}{\hbar^{2}}\left(x^{\prime}\right)^{2}\right) \Psi_{\alpha}\left(x^{\prime}\right) \\
& =\frac{\hbar^{2} x_{0}^{2}}{2 m}\left(-\frac{\partial^{2}}{\partial\left(x^{\prime} / x_{0}\right)^{2}}+\left(\frac{x^{\prime}}{x_{0}}\right)^{2}\right) \Psi_{\alpha}\left(x^{\prime}\right)  \tag{2.277}\\
& =\frac{\hbar \omega}{2}\left(-\frac{\partial^{2}}{\partial\left(x^{\prime} / x_{0}\right)^{2}}+\left(\frac{x^{\prime}}{x_{0}}\right)^{2}\right) \Psi_{\alpha}\left(x^{\prime}\right)
\end{align*}
$$

It's clear that there is a straightforward duality relationship between the respective wave functions. Since

$$
\begin{equation*}
\left\langle x^{\prime} \mid n\right\rangle=\frac{1}{\pi^{1 / 4} \sqrt{2^{n} n!} x_{0}^{n+1 / 2}}\left(x^{\prime}-x_{0}^{2} \frac{d}{d x^{\prime}}\right)^{n} \exp \left(-\frac{1}{2}\left(\frac{x^{\prime}}{x_{0}}\right)^{2}\right), \tag{2.278}
\end{equation*}
$$

the momentum space wave functions are

$$
\begin{equation*}
\left\langle p^{\prime} \mid n\right\rangle=\frac{1}{\pi^{1 / 4} \sqrt{2^{n} n!} p_{0}^{n+1 / 2}}\left(p^{\prime}-p_{0}^{2} \frac{d}{d p^{\prime}}\right)^{n} \exp \left(-\frac{1}{2}\left(\frac{p^{\prime}}{p_{0}}\right)^{2}\right) \tag{2.279}
\end{equation*}
$$

## Exercise $2.13 \quad$ Correlation function. (2015 ps2.3)

Consider $\langle x(0) x(t)\rangle$ and $\langle p(0) p(t)\rangle$ where operators are in the Heisenberg representation. These are called correlation functions. Evaluate this for the 1D harmonic oscillator in an energy eigenstate $|n\rangle$.

## Answer for Exercise 2.13

In the Heisenberg representation the position and momentum operators evolve as

$$
\begin{align*}
& x(t)=x(0) \cos (\omega t)+\frac{p(0)}{m \omega} \sin (\omega t)  \tag{2.280}\\
& p(t)=p(0) \cos (\omega t)-m \omega x(0) \sin (\omega t)
\end{align*}
$$

To evaluate the expectation operations, we'll want the ladder operator representations of the position and momentum operators

$$
\begin{align*}
& x(0)=\left.\frac{x_{0}}{\sqrt{2}}\left(a+a^{\dagger}\right)\right|_{t=0} \\
& p(0)=\left.\frac{i \hbar}{x_{0} \sqrt{2}}\left(a^{\dagger}-a\right)\right|_{t=0}, \tag{2.281}
\end{align*}
$$

where

$$
\begin{equation*}
x_{0}=\sqrt{\frac{\hbar}{m \omega}} \tag{2.282}
\end{equation*}
$$

The expectations of interest, with the raising and lowering operators evaluated at $t=0$, are

$$
\begin{align*}
& \langle n| x(0) x(0)|n\rangle \\
& =\frac{x_{0}^{2}}{2}\langle n|\left(a+a^{\dagger}\right)^{2}|n\rangle \\
& =\frac{x_{0}^{2}}{2}(\sqrt{n+1}\langle n+1|+\sqrt{n}\langle n-1|)(\sqrt{n+1}|n+1\rangle+\sqrt{n}|n-1\rangle)  \tag{2.283}\\
& =\frac{x_{0}^{2}}{2}(2 n+1)
\end{align*}
$$

and

$$
\begin{aligned}
& \langle n| x(0) p(0)|n\rangle \\
& =\frac{i \hbar}{2}\langle n|\left(a+a^{\dagger}\right)\left(a^{\dagger}-a\right)|n\rangle \\
& =\frac{i \hbar}{2}(\sqrt{n+1}\langle n+1|+\sqrt{n}\langle n-1|)(\sqrt{n+1}|n+1\rangle-\sqrt{n}|n-1\rangle) \\
& =\frac{i \hbar}{2}
\end{aligned}
$$

$$
\begin{align*}
\langle n| p(0) p(0)|n\rangle & =\frac{-\hbar^{2}}{2 x_{0}^{2}}\langle n|\left(a^{\dagger}-a\right)\left(a^{\dagger}-a\right)|n\rangle \\
& =\frac{-\hbar^{2}}{2 x_{0}^{2}}(\sqrt{n+1}\langle n+1|-\sqrt{n}\langle n-1|)(\sqrt{n+1}|n+1\rangle-\sqrt{n}|n-1\rangle) \\
& =\frac{\left(-\hbar^{2}\right)}{2 x_{0}^{2}}(2 n+1) \tag{2.285}
\end{align*}
$$

and finally

$$
\begin{align*}
\langle n| p(0) x(0)|n\rangle & =\langle n|([p(0), x(0)]+x(0) p(0))|n\rangle \\
& =-i \hbar+\frac{i \hbar}{2}  \tag{2.286}\\
& =\frac{i \hbar}{2}
\end{align*}
$$

Now we are ready to compute the correlations. The position correlation is

$$
\begin{align*}
\langle n| x(0) x(t)|n\rangle & =\langle n| x(0)\left(x(0) \cos (\omega t)+\frac{p(0)}{m \omega} \sin (\omega t)\right)|n\rangle \\
& =\cos (\omega t)\langle n| x(0) x(0)|n\rangle+\frac{1}{m \omega} \sin (\omega t)\langle n| x(0) p(0)|n\rangle \\
& =\cos (\omega t) \frac{x_{0}^{2}}{2}(2 n+1)+\frac{1}{m \omega} \sin (\omega t) \frac{i \hbar}{2}, \tag{2.287}
\end{align*}
$$

which is

$$
\begin{equation*}
\langle n| x(0) x(t)|n\rangle=\frac{x_{0}^{2}}{2}((2 n+1) \cos (\omega t)+i \sin (\omega t)) . \tag{2.288}
\end{equation*}
$$

The momentum correlation is

$$
\begin{align*}
\langle n| p(0) p(t)|n\rangle & =\langle n| p(0)(p(0) \cos (\omega t)-m \omega x(0) \sin (\omega t))|n\rangle \\
& =\cos (\omega t)(2 n+1) \frac{\left(-\hbar^{2}\right)}{2 x_{0}^{2}}-m \omega \sin (\omega t) \frac{i \hbar}{2} . \tag{2.289}
\end{align*}
$$

With $p_{0}^{2}=m \omega \hbar$, this is

$$
\begin{equation*}
\langle n| p(0) p(t)|n\rangle=-\frac{p_{0}^{2}}{2}((2 n+1) \cos (\omega t)+i \sin (\omega t)) \tag{2.290}
\end{equation*}
$$

## Exercise 2.14 Superposition expectation max. ([15] pr. 2.17)

For a 1D SHO
a. Construct a linear combination of $|0\rangle,|1\rangle$ that maximizes $\langle x\rangle$ without using wave functions.
b. How does this state evolve with time?
c. Evaluate $\langle x\rangle$ using the Schrödinger picture.
d. Evaluate $\langle x\rangle$ using the Heisenberg picture.
e. Evaluate $\left\langle(\Delta x)^{2}\right\rangle$.

## Answer for Exercise 2.14

Part a. Forming

$$
\begin{equation*}
|\psi\rangle=\frac{|0\rangle+\sigma|1\rangle}{\sqrt{1+|\sigma|^{2}}} \tag{2.291}
\end{equation*}
$$

the position expectation is

$$
\begin{equation*}
\langle\psi| x|\psi\rangle=\frac{1}{1+|\sigma|^{2}}\left(\langle 0|+\sigma^{*}\langle 1|\right) \frac{x_{0}}{\sqrt{2}}\left(a^{\dagger}+a\right)(|0\rangle+\sigma|1\rangle) . \tag{2.292}
\end{equation*}
$$

Evaluating the action of the operators on the kets, we've got

$$
\begin{equation*}
\left(a^{\dagger}+a\right)(|0\rangle+\sigma|1\rangle)=|1\rangle+\sqrt{2} \sigma|2\rangle+\sigma|0\rangle \tag{2.293}
\end{equation*}
$$

The $|2\rangle$ term is killed by the bras, leaving

$$
\begin{align*}
\langle x\rangle & =\frac{1}{1+|\sigma|^{2}} \frac{x_{0}}{\sqrt{2}}\left(\sigma+\sigma^{*}\right) \\
& =\frac{\sqrt{2} x_{0} \operatorname{Re} \sigma}{1+|\sigma|^{2}} \tag{2.294}
\end{align*}
$$

Any imaginary component in $\sigma$ will reduce the expectation, so we are constrained to picking a real value.

The derivative of

$$
\begin{equation*}
f(\sigma)=\frac{\sigma}{1+\sigma^{2}} \tag{2.295}
\end{equation*}
$$

is

$$
\begin{equation*}
f^{\prime}(\sigma)=\frac{1-\sigma^{2}}{\left(1+\sigma^{2}\right)^{2}} \tag{2.296}
\end{equation*}
$$

That has zeros at $\sigma= \pm 1$. The second derivative is

$$
\begin{equation*}
f^{\prime \prime}(\sigma)=\frac{-2 \sigma\left(3-\sigma^{2}\right)}{\left(1+\sigma^{2}\right)^{3}} \tag{2.297}
\end{equation*}
$$

That will be negative (maximum for the extreme value) at $\sigma=1$, so the linear superposition of these first two energy eigenkets that maximizes the position expectation is

$$
\begin{equation*}
\psi=\frac{1}{\sqrt{2}}(|0\rangle+|1\rangle) . \tag{2.298}
\end{equation*}
$$

That maximized position expectation is

$$
\begin{equation*}
\langle x\rangle=\frac{x_{0}}{\sqrt{2}} . \tag{2.299}
\end{equation*}
$$

Part b. The time evolution is given by

$$
\begin{align*}
|\Psi(t)\rangle & =e^{-i H t / \hbar} \frac{1}{\sqrt{2}}(|0\rangle+|1\rangle) \\
& =\frac{1}{\sqrt{2}}\left(e^{-i(0+1 / 2) \hbar \omega t / \hbar}|0\rangle+e^{-i(1+1 / 2) \hbar \omega t / \hbar}|1\rangle\right)  \tag{2.300}\\
& =\frac{1}{\sqrt{2}}\left(e^{-i \omega t / 2}|0\rangle+e^{-3 i \omega t / 2}|1\rangle\right) .
\end{align*}
$$

Part c. The position expectation in the Schrödinger representation is

$$
\begin{align*}
\langle x(t)\rangle & =\frac{1}{2}\left(e^{i \omega t / 2}\langle 0|+e^{3 i \omega t / 2}\langle 1|\right) \frac{x_{0}}{\sqrt{2}}\left(a^{\dagger}+a\right)\left(e^{-i \omega t / 2}|0\rangle+e^{-3 i \omega t / 2}|1\rangle\right) \\
& =\frac{x_{0}}{2 \sqrt{2}}\left(e^{i \omega t / 2}\langle 0|+e^{3 i \omega t / 2}\langle 1|\right)\left(e^{-i \omega t / 2}|1\rangle+e^{-3 i \omega t / 2} \sqrt{2}|2\rangle+e^{-3 i \omega t / 2}|0\rangle\right) \\
& =\frac{x_{0}}{\sqrt{2}} \cos (\omega t) \tag{2.301}
\end{align*}
$$

Part d.

$$
\begin{align*}
\langle x(t)\rangle & =\frac{1}{2}(\langle 0|+\langle 1|) \frac{x_{0}}{\sqrt{2}}\left(a^{\dagger} e^{i \omega t}+a e^{-i \omega t}\right)(|0\rangle+|1\rangle) \\
& =\frac{x_{0}}{2 \sqrt{2}}(\langle 0|+\langle 1|)\left(e^{i \omega t}|1\rangle+\sqrt{2} e^{i \omega t}|2\rangle+e^{-i \omega t}|0\rangle\right)  \tag{2.302}\\
& =\frac{x_{0}}{\sqrt{2}} \cos (\omega t),
\end{align*}
$$

matching the calculation using the Schrödinger picture.

Part e. Let's use the Heisenberg picture for the uncertainty calculation. Using the calculation above we have

$$
\begin{align*}
& \left\langle x^{2}\right\rangle \\
& =\frac{1}{2} \frac{x_{0}^{2}}{2}\left(e^{-i \omega t}\langle 1|+\sqrt{2} e^{-i \omega t}\langle 2|+e^{i \omega t}\langle 0|\right)\left(e^{i \omega t}|1\rangle+\sqrt{2} e^{i \omega t}|2\rangle+e^{-i \omega t}|0\rangle\right) \\
& =\frac{x_{0}^{2}}{4}(1+2+1) \\
& =x_{0}^{2} \tag{2.303}
\end{align*}
$$

The uncertainty is

$$
\begin{align*}
\left\langle(\Delta x)^{2}\right\rangle & =\left\langle x^{2}\right\rangle-\langle x\rangle^{2} \\
& =x_{0}^{2}-\frac{x_{0}^{2}}{2} \cos ^{2}(\omega t) \\
& =\frac{x_{0}^{2}}{2}\left(2-\cos ^{2}(\omega t)\right)  \tag{2.304}\\
& =\frac{x_{0}^{2}}{2}\left(1+\sin ^{2}(\omega t)\right) .
\end{align*}
$$

Exercise 2.15 Plane wave ground state expectation. ([15] pr. 2.18)
For a 1D SHO, show that

$$
\begin{equation*}
\langle 0| e^{i k x}|0\rangle=\exp \left(-k^{2}\langle 0| x^{2}|0\rangle / 2\right) \tag{2.305}
\end{equation*}
$$

Answer for Exercise 2.15
Despite the simple appearance of this problem, I found this quite involved to show. To do so, start with a series expansion of the expectation

$$
\begin{equation*}
\langle 0| e^{i k x}|0\rangle=\sum_{m=0}^{\infty} \frac{(i k)^{m}}{m!}\langle 0| x^{m}|0\rangle \tag{2.306}
\end{equation*}
$$

Let

$$
\begin{equation*}
X=\left(a+a^{\dagger}\right) \tag{2.307}
\end{equation*}
$$

so that

$$
\begin{equation*}
x=\sqrt{\frac{\hbar}{2 \omega m}} X=\frac{x_{0}}{\sqrt{2}} X \tag{2.308}
\end{equation*}
$$

Consider the first few values of $\langle 0| X^{n}|0\rangle$

$$
\begin{align*}
\langle 0| X|0\rangle & =\langle 0|\left(a+a^{\dagger}\right)|0\rangle \\
& =\langle 0 \mid 1\rangle  \tag{2.309}\\
& =0,
\end{align*}
$$

$$
\langle 0| X^{2}|0\rangle=\langle 0|\left(a+a^{\dagger}\right)^{2}|0\rangle
$$

$$
\begin{equation*}
=\langle 1 \mid 1\rangle \tag{2.310}
\end{equation*}
$$

$$
=1
$$

$$
\begin{align*}
\langle 0| X^{3}|0\rangle & =\langle 0|\left(a+a^{\dagger}\right)^{3}|0\rangle \\
& =\langle 1|(\sqrt{2}|2\rangle+|0\rangle)  \tag{2.311}\\
& =0 .
\end{align*}
$$

Whenever the power $n$ in $X^{n}$ is even, the braket can be split into a bra that has only contributions from odd eigenstates and a ket with even eigenstates. We conclude that $\langle 0| X^{n}|0\rangle=0$ when $n$ is odd.

Noting that $\langle 0| x^{2}|0\rangle=x_{0}^{2} / 2$, this leaves

$$
\begin{align*}
\langle 0| e^{i k x}|0\rangle & =\sum_{m=0}^{\infty} \frac{(i k)^{2 m}}{(2 m)!}\langle 0| x^{2 m}|0\rangle \\
& =\sum_{m=0}^{\infty} \frac{(i k)^{2 m}}{(2 m)!}\left(\frac{x_{0}^{2}}{2}\right)^{m}\langle 0| X^{2 m}|0\rangle  \tag{2.312}\\
& =\sum_{m=0}^{\infty} \frac{1}{(2 m)!}\left(-k^{2}\langle 0| x^{2}|0\rangle\right)^{m}\langle 0| X^{2 m}|0\rangle .
\end{align*}
$$

This problem is now reduced to showing that

$$
\begin{equation*}
\frac{1}{(2 m)!}\langle 0| X^{2 m}|0\rangle=\frac{1}{m!2^{m}} \tag{2.313}
\end{equation*}
$$

or

$$
\begin{align*}
\langle 0| X^{2 m}|0\rangle & =\frac{(2 m)!}{m!2^{m}} \\
& =\frac{(2 m)(2 m-1)(2 m-2) \cdots(2)(1)}{2^{m} m!} \\
& =\frac{2^{m}(m)(2 m-1)(m-1)(2 m-3)(m-2) \cdots(2)(3)(1)(1)}{2^{m} m!} \\
& =(2 m-1)!!, \tag{2.314}
\end{align*}
$$

where $n!!=n(n-2)(n-4) \cdots$.
It looks like $\langle 0| X^{2 m}|0\rangle$ can be expanded by inserting an identity operator and proceeding recursively, like

$$
\begin{align*}
\langle 0| X^{2 m}|0\rangle & =\langle 0| X^{2}\left(\sum_{n=0}^{\infty}|n\rangle\langle n|\right) X^{2 m-2}|0\rangle  \tag{2.315}\\
& =\langle 0| X^{2}(|0\rangle\langle 0|+|2\rangle\langle 2|) X^{2 m-2}|0\rangle \\
& =\langle 0| X^{2 m-2}|0\rangle+\langle 0| X^{2}|2\rangle\langle 2| X^{2 m-2}|0\rangle .
\end{align*}
$$

This has made use of the observation that $\langle 0| X^{2}|n\rangle=0$ for all $n \neq 0,2$. The remaining term includes the factor

$$
\begin{align*}
\langle 0| X^{2}|2\rangle & =\langle 0|\left(a+a^{\dagger}\right)^{2}|2\rangle \\
& =(\langle 0|+\sqrt{2}\langle 2|)|2\rangle  \tag{2.316}\\
& =\sqrt{2}
\end{align*}
$$

Since $\sqrt{2}|2\rangle=\left(a^{\dagger}\right)^{2}|0\rangle$, the expectation of interest can be written

$$
\begin{equation*}
\langle 0| X^{2 m}|0\rangle=\langle 0| X^{2 m-2}|0\rangle+\langle 0| a^{2} X^{2 m-2}|0\rangle \tag{2.317}
\end{equation*}
$$

How do we expand the second term. Let's look at how $a$ and $X$ commute

$$
\begin{align*}
a X & =[a, X]+X a \\
& =\left[a, a+a^{\dagger}\right]+X a  \tag{2.318}\\
& =\left[a, a^{\dagger}\right]+X a \\
& =1+X a,
\end{align*}
$$

$$
\begin{align*}
a^{2} X & =a(a X) \\
& =a(1+X a) \\
& =a+a X a  \tag{2.319}\\
& =a+(1+X a) a \\
& =2 a+X a^{2} .
\end{align*}
$$

Proceeding to expand $a^{2} X^{n}$ we find

$$
\begin{align*}
& a^{2} X^{3}=6 X+6 X^{2} a+X^{3} a^{2} \\
& a^{2} X^{4}=12 X^{2}+8 X^{3} a+X^{4} a^{2} \\
& a^{2} X^{5}=20 X^{3}+10 X^{4} a+X^{5} a^{2}  \tag{2.320}\\
& a^{2} X^{6}=30 X^{4}+12 X^{5} a+X^{6} a^{2} .
\end{align*}
$$

It appears that we have

$$
\begin{equation*}
\left[a^{2} X^{n}, X^{n} a^{2}\right]=\beta_{n} X^{n-2}+2 n X^{n-1} a \tag{2.321}
\end{equation*}
$$

where

$$
\begin{equation*}
\beta_{n}=\beta_{n-1}+2(n-1) \tag{2.322}
\end{equation*}
$$

and $\beta_{2}=2$. Some goofing around shows that $\beta_{n}=n(n-1)$, so the induction hypothesis is

$$
\begin{equation*}
\left[a^{2} X^{n}, X^{n} a^{2}\right]=n(n-1) X^{n-2}+2 n X^{n-1} a \tag{2.323}
\end{equation*}
$$

Let's check the induction

$$
\begin{align*}
a^{2} X^{n+1} & =a^{2} X^{n} X \\
& =\left(n(n-1) X^{n-2}+2 n X^{n-1} a+X^{n} a^{2}\right) X \\
& =n(n-1) X^{n-1}+2 n X^{n-1} a X+X^{n} a^{2} X \\
& =n(n-1) X^{n-1}+2 n X^{n-1}(1+X a)+X^{n}\left(2 a+X a^{2}\right)  \tag{2.324}\\
& =n(n-1) X^{n-1}+2 n X^{n-1}+2 n X^{n} a+2 X^{n} a+X^{n+1} a^{2} \\
& =X^{n+1} a^{2}+(2+2 n) X^{n} a+(2 n+n(n-1)) X^{n-1} \\
& =X^{n+1} a^{2}+2(n+1) X^{n} a+(n+1) n X^{n-1},
\end{align*}
$$

which concludes the induction, giving

$$
\begin{equation*}
\langle 0| a^{2} X^{n}|0\rangle=n(n-1)\langle 0| X^{n-2}|0\rangle \tag{2.325}
\end{equation*}
$$

and

$$
\begin{equation*}
\langle 0| X^{2 m}|0\rangle=\langle 0| X^{2 m-2}|0\rangle+(2 m-2)(2 m-3)\langle 0| X^{2 m-4}|0\rangle \tag{2.326}
\end{equation*}
$$

Let

$$
\begin{equation*}
\sigma_{n}=\langle 0| X^{n}|0\rangle \tag{2.327}
\end{equation*}
$$

so that the recurrence relation, for $2 n \geq 4$ is

$$
\begin{equation*}
\sigma_{2 n}=\sigma_{2 n-2}+(2 n-2)(2 n-3) \sigma_{2 n-4} \tag{2.328}
\end{equation*}
$$

We want to show that this simplifies to

$$
\begin{equation*}
\sigma_{2 n}=(2 n-1)!! \tag{2.329}
\end{equation*}
$$

The first values are

$$
\begin{equation*}
\sigma_{0}=\langle 0| X^{0}|0\rangle=1 \tag{2.330a}
\end{equation*}
$$

$$
\begin{equation*}
\sigma_{2}=\langle 0| X^{2}|0\rangle=1, \tag{2.330b}
\end{equation*}
$$

which gives us the right result for the first term in the induction

$$
\begin{align*}
\sigma_{4} & =\sigma_{2}+2 \times 1 \times \sigma_{0} \\
& =1+2  \tag{2.331}\\
& =3!!
\end{align*}
$$

For the general induction term, consider

$$
\begin{align*}
\sigma_{2 n+2} & =\sigma_{2 n}+2 n(2 n-1) \sigma_{2 n-2} \\
& =(2 n-1)!!+2 n(2 n-1)(2 n-3)!! \\
& =(2 n+1)(2 n-1)!!  \tag{2.332}\\
& =(2 n+1)!!
\end{align*}
$$

which completes the final induction. That was also the last thing required to complete the proof, so we are done!

Exercise $2.16 \quad$ Relation of probability flux to momentum.

Show that the probability flux

$$
\begin{equation*}
\mathbf{j}(\mathbf{x}, t)=-\frac{i \hbar}{2 m}\left(\psi^{*} \boldsymbol{\nabla} \psi-\psi \boldsymbol{\nabla} \psi^{*}\right), \tag{2.333}
\end{equation*}
$$

is related to the momentum expectation at a given time by the integral of the flux over all space

$$
\begin{equation*}
\int d^{3} x \mathbf{j}(\mathbf{x}, t)=\frac{\langle\mathbf{p}\rangle_{t}}{m} \tag{2.334}
\end{equation*}
$$

Answer for Exercise 2.16
This can be seen by recasting the integral in bra-ket form. Let

$$
\begin{equation*}
\psi(\mathbf{x}, t)=\langle\mathbf{x} \mid \psi(t)\rangle \tag{2.335}
\end{equation*}
$$

and note that the momentum portions of the flux can be written as

$$
\begin{equation*}
-i \hbar \boldsymbol{\nabla} \psi(\mathbf{x}, t)=\langle\mathbf{x}| \mathbf{p}|\psi(t)\rangle \tag{2.336}
\end{equation*}
$$

The current is therefore

$$
\begin{align*}
\mathbf{j}(\mathbf{x}, t) & =\frac{1}{2 m}\left(\psi^{*}\langle\mathbf{x}| \mathbf{p}|\psi(t)\rangle+\psi\langle\mathbf{x}| \mathbf{p}|\psi(t)\rangle^{*}\right) \\
& =\frac{1}{2 m}\left(\langle\mathbf{x} \mid \psi(t)\rangle^{*}\langle\mathbf{x}| \mathbf{p}|\psi(t)\rangle+\langle\mathbf{x} \mid \psi(t)\rangle\langle\mathbf{x}| \mathbf{p}|\psi(t)\rangle^{*}\right)  \tag{2.337}\\
& =\frac{1}{2 m}(\langle\psi(t) \mid \mathbf{x}\rangle\langle\mathbf{x}| \mathbf{p}|\psi(t)\rangle+\langle\psi(t)| \mathbf{p}|\mathbf{x}\rangle\langle\mathbf{x} \mid \psi(t)\rangle) .
\end{align*}
$$

Integrating and noting that the spatial identity is $1=\int d^{3} x|\mathbf{x}\rangle\langle\mathbf{x}|$, we have

$$
\begin{equation*}
\int d^{3} x \mathbf{j}(\mathbf{x}, t)=\langle\psi(t)| \mathbf{p}|\psi(t)\rangle \tag{2.338}
\end{equation*}
$$

This is just the expectation of $\mathbf{p}$ with respect to a specific time-instance state, demonstrating the desired relationship.

Exercise 2.17 Hermite polynomial normalization. ([15] pr. 2.21)
Derive the normalization constant $c_{n}$ for the Harmonic oscillator solution

$$
\begin{equation*}
u_{n}(x)=c_{n} H_{n}\left(x \sqrt{\frac{m \omega}{\hbar}}\right) e^{-m \omega x^{2} / 2 \hbar} \tag{2.339}
\end{equation*}
$$

by deriving the orthogonality relationship using generating functions

$$
\begin{equation*}
g(x, t)=e^{-t^{2}+2 t x}=\sum_{n=0}^{\infty} H_{n}(x) \frac{t^{n}}{n!} \tag{2.340}
\end{equation*}
$$

Start by working out the integral

$$
\begin{equation*}
I=\int_{-\infty}^{\infty} g(x, t) g(x, s) e^{-x^{2}} d x \tag{2.341}
\end{equation*}
$$

consider the integral twice with each side definition of the generating function.
Answer for Exercise 2.17
First using the exponential definition of the generating function

$$
\begin{align*}
\int_{-\infty}^{\infty} g(x, t) g(x, s) e^{-x^{2}} d x & =\int_{-\infty}^{\infty} e^{-t^{2}+2 t x} e^{-s^{2}+2 s x} e^{-x^{2}} d x \\
& =e^{-t^{2}-s^{2}} \int_{-\infty}^{\infty} e^{-\left(x^{2}-2 t x-2 s x\right)} d x \\
& =e^{-t^{2}-s^{2}+(s+t)^{2}} \int_{-\infty}^{\infty} e^{-(x-t-s)^{2}} d x  \tag{2.342}\\
& =e^{2 s t} \int_{-\infty}^{\infty} e^{-u^{2}} d u \\
& =\sqrt{\pi} e^{2 s t}
\end{align*}
$$

With the Hermite polynomial definition of the generating function, this integral is

$$
\begin{align*}
\int_{-\infty}^{\infty} g(x, t) g(x, s) e^{-x^{2}} d x & =\int_{-\infty}^{\infty} \sum_{n=0}^{\infty} H_{n}(x) \frac{t^{n}}{n!} \sum_{m=0}^{\infty} H_{m}(x) \frac{s^{m}}{m!} e^{-x^{2}} d x \\
& =\sum_{n=0}^{\infty} \frac{t^{n}}{n!} \sum_{m=0}^{\infty} \frac{s^{m}}{m!} \int_{-\infty}^{\infty} H_{n}(x) H_{m}(x) e^{-x^{2}} d x \tag{2.343}
\end{align*}
$$

Let

$$
\begin{equation*}
\alpha_{n m}=\int_{-\infty}^{\infty} H_{n}(x) H_{m}(x) e^{-x^{2}} d x \tag{2.344}
\end{equation*}
$$

and equate the two expansions of this integral

$$
\begin{equation*}
\sqrt{\pi} \sum_{n=0}^{\infty} \frac{(2 s t)^{n}}{n!}=\sum_{n=0}^{\infty} \frac{t^{n}}{n!} \sum_{m=0}^{\infty} \frac{s^{m}}{m!} \alpha_{n m} \tag{2.345}
\end{equation*}
$$

or, after equating powers of $t^{n}$

$$
\begin{equation*}
\sqrt{\pi}(2 s)^{n}=\sum_{m=0}^{\infty} \frac{s^{m}}{m!} \alpha_{n m} \tag{2.346}
\end{equation*}
$$

This requires $\alpha_{n m}$ to be zero for $n \neq m$, so

$$
\begin{equation*}
\sqrt{\pi} 2^{n}=\frac{1}{n!} \alpha_{n n}, \tag{2.347}
\end{equation*}
$$

and

$$
\begin{equation*}
\int_{-\infty}^{\infty} H_{n}(x) H_{m}(x) e^{-x^{2}} d x=\delta_{n m} \sqrt{\pi} 2^{n} n! \tag{2.348}
\end{equation*}
$$

The SHO normalization is fixed by

$$
\begin{align*}
\int_{-\infty}^{\infty} u_{n}^{2}(x) d x & =c_{n}^{2} \int_{-\infty}^{\infty} H_{n}^{2}\left(x / x_{0}\right) e^{-\left(x / x_{0}\right)^{2}} d x  \tag{2.349}\\
& =c_{n}^{2} x_{0} \sqrt{\pi} 2^{n} n!
\end{align*}
$$

or

$$
\begin{align*}
c_{n} & =\frac{1}{\sqrt{\sqrt{\pi} 2^{n} n!\sqrt{\frac{\hbar}{m \omega}}}}  \tag{2.350}\\
& =\left(\frac{m \omega}{\hbar \pi}\right)^{1 / 4} 2^{-n / 2} \frac{1}{\sqrt{n!}} .
\end{align*}
$$

Exercise 2.18 Dirac delta function potential. ([15] pr. 2.24,2.25)
Given a Dirac delta function potential

$$
\begin{equation*}
H=\frac{p^{2}}{2 m}-V_{0} \delta(x) \tag{2.351}
\end{equation*}
$$

which vanishes after $t=0$.
a. Solve for the bound state for $t<0$,
b. Solve for the time evolution after that.

Answer for Exercise 2.18

Part $a$. This problem can be solved directly by considering the $|x|>0$ and $x=0$ regions separately.

For $|x|>0$ Schrödinger's equation takes the form

$$
\begin{equation*}
E \psi=-\frac{\hbar^{2}}{2 m} \frac{d^{2} \psi}{d x^{2}} \tag{2.352}
\end{equation*}
$$

With

$$
\begin{equation*}
\kappa=\frac{\sqrt{-2 m E}}{\hbar} \tag{2.353}
\end{equation*}
$$

this has solutions

$$
\begin{equation*}
\psi=e^{ \pm \kappa x} \tag{2.354}
\end{equation*}
$$

For $x>0$ we must have

$$
\begin{equation*}
\psi=a e^{-\kappa x} \tag{2.355}
\end{equation*}
$$

and for $x<0$

$$
\begin{equation*}
\psi=b e^{\kappa x} \tag{2.356}
\end{equation*}
$$

requiring that $\psi$ is continuous at $x=0$ means $a=b$, or

$$
\begin{equation*}
\psi=\psi(0) e^{-\kappa|x|} \tag{2.357}
\end{equation*}
$$

For the $x=0$ region, consider an interval $[-\epsilon, \epsilon]$ region around the origin. We must have

$$
\begin{equation*}
E \int_{-\epsilon}^{\epsilon} \psi(x) d x=\frac{-\hbar^{2}}{2 m} \int_{-\epsilon}^{\epsilon} \frac{d^{2} \psi}{d x^{2}} d x-V_{0} \int_{-\epsilon}^{\epsilon} \delta(x) \psi(x) d x \tag{2.358}
\end{equation*}
$$

The RHS is zero

$$
\begin{align*}
E \int_{-\epsilon}^{\epsilon} \psi(x) d x & =E \frac{e^{-\kappa(\epsilon)}-1}{-\kappa}-E \frac{1-e^{\kappa(-\epsilon)}}{\kappa}  \tag{2.359}\\
& \rightarrow 0
\end{align*}
$$

That leaves

$$
\begin{align*}
V_{0} \int_{-\epsilon}^{\epsilon} \delta(x) \psi(x) d x & =\frac{-\hbar^{2}}{2 m} \int_{-\epsilon}^{\epsilon} \frac{d^{2} \psi}{d x^{2}} d x \\
& =\left.\frac{-\hbar^{2}}{2 m} \frac{d \psi}{d x}\right|_{-\epsilon} ^{\epsilon}  \tag{2.360}\\
& =\frac{-\hbar^{2}}{2 m} \psi(0)\left(-\kappa e^{-\kappa(\epsilon)}-\kappa e^{\kappa(-\epsilon)}\right)
\end{align*}
$$

In the $\epsilon \rightarrow 0$ limit this gives

$$
\begin{equation*}
V_{0}=\frac{\hbar^{2} \kappa}{m} \tag{2.361}
\end{equation*}
$$

Equating relations for $\kappa$ we have

$$
\begin{equation*}
\kappa=\frac{m V_{0}}{\hbar^{2}}=\frac{\sqrt{-2 m E}}{\hbar}, \tag{2.362}
\end{equation*}
$$

or

$$
\begin{equation*}
E=-\frac{1}{2 m}\left(\frac{m V_{0}}{\hbar}\right)^{2} \tag{2.363}
\end{equation*}
$$

with

$$
\begin{equation*}
\psi(x, t<0)=C \exp (-i E t / \hbar-\kappa|x|) . \tag{2.364}
\end{equation*}
$$

The normalization requires

$$
\begin{align*}
1 & =2|C|^{2} \int_{0}^{\infty} e^{-2 \kappa x} d x \\
& =\left.2|C|^{2} \frac{e^{-2 \kappa x}}{-2 \kappa}\right|_{0} ^{\infty}  \tag{2.365}\\
& =\frac{|C|^{2}}{\kappa}
\end{align*}
$$

SO

$$
\begin{equation*}
\psi(x, t<0)=\sqrt{\kappa} \exp (-i E t / \hbar-\kappa|x|) . \tag{2.366}
\end{equation*}
$$

There is only one bound state for such a potential.

Part b. After turning off the potential, any plane wave

$$
\begin{equation*}
\psi(x, t)=e^{i k x-i E(k) t / \hbar} \tag{2.367}
\end{equation*}
$$

where

$$
\begin{equation*}
k=\frac{\sqrt{2 m E}}{\hbar} \tag{2.368}
\end{equation*}
$$

is a solution. In particular, at $t=0$, the wave packet

$$
\begin{equation*}
\psi(x, 0)=\frac{1}{\sqrt{2 \pi}} \int_{-\infty}^{\infty} e^{i k x} A(k) d k \tag{2.369}
\end{equation*}
$$

is a solution. To solve for $A(k)$, we require

$$
\begin{equation*}
\frac{1}{\sqrt{2 \pi}} \int_{-\infty}^{\infty} e^{i k x} A(k) d k=\sqrt{\kappa} e^{-\kappa|x|} \tag{2.370}
\end{equation*}
$$

or

$$
\begin{equation*}
A(k)=\sqrt{\frac{\kappa}{2 \pi}} \int_{-\infty}^{\infty} e^{-i k x} e^{-\kappa|x|} d x=\sqrt{\frac{2}{\pi}} \frac{\kappa^{3 / 2}}{\kappa^{2}+k^{2}} \tag{2.371}
\end{equation*}
$$

The initial time state established by the delta function potential evolves as

$$
\begin{equation*}
\psi(x, t>0)=\frac{1}{\sqrt{2 \pi}} \int_{-\infty}^{\infty} e^{i k x-i \hbar k^{2} t / 2 m} A(k) d k \tag{2.372}
\end{equation*}
$$

In terms of $m, V_{0}$ that is

$$
\begin{equation*}
\psi(x, t>0)=\frac{\left(m V_{0}\right)^{3 / 2}}{\pi \hbar} \int_{-\infty}^{\infty} \frac{e^{i k x-i \hbar k^{2} t / 2 m}}{k^{2} \hbar^{2}+m^{2} V_{0}^{2} / \hbar^{2}} d k \tag{2.373}
\end{equation*}
$$

This integral resists an attempt to evaluate with Mathematica.

## Exercise 2.19 Free particle propagator. ([15] pr. 2.31)

Derive the free particle propagator in one and three dimensions.
Answer for Exercise 2.19
I found the description in the text confusing, so let's start from scratch with the definition of the propagator. This is the kernel of the spatial convolution integral that encodes time evolution, and can be expressed by expanding a general time state with two sets of identity operators. Let the position relative state at time $t$, relative to an initial time $t_{0}$ be given by
$\left\langle\mathbf{x} \mid \alpha, t ; t_{0}\right\rangle$, and expand this in terms of a complete basis of energy eigenstates $\left|a^{\prime}\right\rangle$ and the time evolution operator

$$
\begin{align*}
\left\langle\mathbf{x}^{\prime \prime} \mid \alpha, t ; t_{0}\right\rangle & =\left\langle\mathbf{x}^{\prime \prime}\right| U\left|\alpha, t_{0}\right\rangle \\
& =\left\langle\mathbf{x}^{\prime \prime}\right| e^{-i H\left(t-t_{0}\right) / \hbar}\left|\alpha, t_{0}\right\rangle \\
& =\left\langle\mathbf{x}^{\prime \prime}\right| e^{-i H\left(t-t_{0}\right) / \hbar}\left(\sum_{a^{\prime}}\left|a^{\prime}\right\rangle\left\langle a^{\prime}\right|\right)\left|\alpha, t_{0}\right\rangle \\
& =\left\langle\mathbf{x}^{\prime \prime}\right| \sum_{a^{\prime}} e^{-i E_{a^{\prime}}\left(t-t_{0}\right) / \hbar}\left|a^{\prime}\right\rangle\left\langle a^{\prime} \mid \alpha, t_{0}\right\rangle \\
& =\left\langle\mathbf{x}^{\prime \prime}\right| \sum_{a^{\prime}} e^{-i E_{a^{\prime}}\left(t-t_{0}\right) / \hbar}\left|a^{\prime}\right\rangle\left\langle a^{\prime}\right|\left(\int d^{3} \mathbf{x}^{\prime}\left|\mathbf{x}^{\prime}\right\rangle\left\langle\mathbf{x}^{\prime}\right|\right)\left|\alpha, t_{0}\right\rangle \\
& =\int d^{3} \mathbf{x}^{\prime}\left(\left\langle\mathbf{x}^{\prime \prime}\right| \sum_{a^{\prime}} e^{-i E_{a^{\prime}}\left(t-t_{0}\right) / \hbar}\left|a^{\prime}\right\rangle\left\langle a^{\prime} \mid \mathbf{x}^{\prime}\right\rangle\right)\left\langle\mathbf{x}^{\prime} \mid \alpha, t_{0}\right\rangle \\
& =\int d^{3} \mathbf{x}^{\prime} K\left(\mathbf{x}^{\prime \prime}, t ; \mathbf{x}^{\prime}, t_{0}\right)\left\langle\mathbf{x}^{\prime} \mid \alpha, t_{0}\right\rangle, \tag{2.374}
\end{align*}
$$

where

$$
\begin{equation*}
K\left(\mathbf{x}^{\prime \prime}, t ; \mathbf{x}^{\prime}, t_{0}\right)=\sum_{a^{\prime}}\left\langle\mathbf{x}^{\prime \prime} \mid a^{\prime}\right\rangle\left\langle a^{\prime} \mid \mathbf{x}^{\prime}\right\rangle e^{-i E_{a^{\prime}}\left(t-t_{0}\right) / \hbar} \tag{2.375}
\end{equation*}
$$

the propagator, is the kernel of the convolution integral that takes the state $\left|\alpha, t_{0}\right\rangle$ to state $\left|\alpha, t ; t_{0}\right\rangle$. Evaluating this over the momentum states (where integration and not plain summation is required), we have

$$
\begin{align*}
K\left(\mathbf{x}^{\prime \prime}, t ; \mathbf{x}^{\prime}, t_{0}\right)= & \int d^{3} \mathbf{p}^{\prime}\left\langle\mathbf{x}^{\prime \prime} \mid \mathbf{p}^{\prime}\right\rangle\left\langle\mathbf{p}^{\prime} \mid \mathbf{x}^{\prime}\right\rangle e^{-i E_{\mathbf{p}^{\prime}}\left(t-t_{0}\right) / \hbar} \\
= & \int d^{3} \mathbf{p}^{\prime}\left\langle\mathbf{x}^{\prime \prime} \mid \mathbf{p}^{\prime}\right\rangle\left\langle\mathbf{p}^{\prime} \mid \mathbf{x}^{\prime}\right\rangle \exp \left(-i \frac{\left(\mathbf{p}^{\prime}\right)^{2}\left(t-t_{0}\right)}{2 m \hbar}\right) \\
= & \int d^{3} \mathbf{p}^{\prime} \frac{e^{i \mathbf{x}^{\prime \prime} \cdot \mathbf{p}^{\prime} / \hbar}}{(\sqrt{2 \pi \hbar})^{3}} \frac{e^{-i \mathbf{x}^{\prime} \cdot \mathbf{p}^{\prime} / \hbar}}{\left(\sqrt{2 \pi \hbar)^{3}}\right.} \exp \left(-i \frac{\left(\mathbf{p}^{\prime}\right)^{2}\left(t-t_{0}\right)}{2 m \hbar}\right) \\
= & \frac{1}{(2 \pi \hbar)^{3}} \int d^{3} \mathbf{p}^{\prime} e^{i\left(\mathbf{x}^{\prime \prime}-\mathbf{x}^{\prime}\right) \cdot \mathbf{p}^{\prime} / \hbar} \exp \left(-i \frac{\left(\mathbf{p}^{\prime}\right)^{2}\left(t-t_{0}\right)}{2 m \hbar}\right) \\
= & \frac{1}{2 \pi \hbar} \int_{-\infty}^{\infty} d p_{1}^{\prime} e^{i\left(x_{1}^{\prime \prime}-x_{1}^{\prime}\right) p_{1}^{\prime} / \hbar} \exp \left(-i \frac{\left(p_{1}^{\prime}\right)^{2}\left(t-t_{0}\right)}{2 m \hbar}\right) \times \\
& \frac{1}{2 \pi \hbar} \int_{-\infty}^{\infty} d p_{2}^{\prime} e^{i\left(x_{2}^{\prime \prime}-x_{2}^{\prime}\right) p_{2}^{\prime} / \hbar} \exp \left(-i \frac{\left(p_{2}^{\prime}\right)^{2}\left(t-t_{0}\right)}{2 m \hbar}\right) \times \\
& \frac{1}{2 \pi \hbar} \int_{-\infty}^{\infty} d p_{3}^{\prime} e^{i\left(x_{3}^{\prime \prime}-x_{3}^{\prime}\right) p_{3}^{\prime} / \hbar} \exp \left(-i \frac{\left(p_{3}^{\prime}\right)^{2}\left(t-t_{0}\right)}{2 m \hbar}\right) \tag{2.376}
\end{align*}
$$

With $a=\left(t-t_{0}\right) / 2 m \hbar$, each of these three integral factors is of the form

$$
\begin{align*}
\frac{1}{2 \pi \hbar} & \int_{-\infty}^{\infty} d p e^{i \Delta x p / \hbar} \exp \left(-i a p^{2}\right) \\
& =\frac{1}{2 \pi \hbar \sqrt{a}} \int_{-\infty}^{\infty} d u e^{i \Delta x u /(\sqrt{a} \hbar)} \exp \left(-i u^{2}\right) \\
& =\frac{1}{2 \pi \hbar \sqrt{a}} \int_{-\infty}^{\infty} d u e^{i \Delta x u /(\sqrt{a} \hbar)} \times \\
& =\frac{1}{2 \pi \hbar \sqrt{a}} \exp \left(-i(u-\Delta x /(2 \sqrt{a} \hbar))^{2}+i(\Delta x /(2 \sqrt{a} \hbar))^{2}\right)  \tag{2.377}\\
& =\sqrt{\frac{-i \pi 2 m \hbar}{4 \pi^{2} \hbar^{2}\left(t-t_{0}\right)}} \exp \left(\frac{i(\Delta x)^{2} m}{2\left(t-t_{0}\right) \hbar}\right) \\
& =\sqrt{\frac{m}{2 \pi i \hbar\left(t-t_{0}\right)}} \exp \left(\frac{i(\Delta x)^{2} m}{2\left(t-t_{0}\right) \hbar}\right)
\end{align*}
$$

Note that the integral above has value $\sqrt{-i \pi}$ which can be found by integrating over the contour of fig. 2.12, letting $R \rightarrow \infty$.


Figure 2.12: Integration contour for $\int e^{-i z^{2}}$.
Multiplying out each of the spatial direction factors gives the propagator in its closed form

$$
\begin{equation*}
K\left(\mathbf{x}^{\prime \prime}, t ; \mathbf{x}^{\prime}, t_{0}\right)=\left(\sqrt{\frac{m}{2 \pi i \hbar\left(t-t_{0}\right)}}\right)^{3} \exp \left(\frac{i\left(\mathbf{x}^{\prime \prime}-\mathbf{x}^{\prime}\right)^{2} m}{2\left(t-t_{0}\right) \hbar}\right) \tag{2.378}
\end{equation*}
$$

In one or two dimensions the exponential power 3 need only be adjusted appropriately.

Exercise 2.20 Partition function, ground state energy. ([15] pr. 2.32)
Define the partition function as

$$
\begin{equation*}
Z=\left.\int d^{3} x^{\prime} K\left(\mathbf{x}^{\prime}, t ; \mathbf{x}^{\prime}, 0\right)\right|_{\beta=i t / \hbar} \tag{2.379}
\end{equation*}
$$

Show that the ground state energy is given by

$$
\begin{equation*}
-\frac{1}{Z} \frac{\partial Z}{\partial \beta}, \quad \beta \rightarrow \infty \tag{2.380}
\end{equation*}
$$

Answer for Exercise 2.20
The propagator evaluated at the same point is

$$
\begin{align*}
K\left(\mathbf{x}^{\prime}, t ; \mathbf{x}^{\prime}, 0\right) & =\sum_{a^{\prime}}\left\langle\mathbf{x}^{\prime} \mid a^{\prime}\right\rangle\left|a^{\prime}\right\rangle \mathbf{x}^{\prime} \exp \left(-\frac{i E_{a^{\prime}} t}{\hbar}\right) \\
& =\sum_{a^{\prime}}\left|\left\langle\mathbf{x}^{\prime} \mid a^{\prime}\right\rangle\right|^{2} \exp \left(-\frac{i E_{a^{\prime}} t}{\hbar}\right)  \tag{2.381}\\
& =\sum_{a^{\prime}}\left|\left\langle\mathbf{x}^{\prime} \mid a^{\prime}\right\rangle\right|^{2} \exp \left(-E_{a^{\prime}} \beta\right)
\end{align*}
$$

The derivative is

$$
\begin{equation*}
\frac{\partial Z}{\partial \beta}=-\int d^{3} x^{\prime} \sum_{a^{\prime}} E_{a^{\prime}}\left|\left\langle\mathbf{x}^{\prime} \mid a^{\prime}\right\rangle\right|^{2} \exp \left(-E_{a^{\prime}} \beta\right) . \tag{2.382}
\end{equation*}
$$

In the $\beta \rightarrow \infty$ this sum will be dominated by the term with the lowest value of $E_{a^{\prime}}$. Suppose that state is $a^{\prime}=0$, then

$$
\begin{align*}
\lim _{\beta \rightarrow \infty}-\frac{1}{Z} \frac{\partial Z}{\partial \beta} & =\frac{\int d^{3} x^{\prime} E_{0}\left|\left\langle\mathbf{x}^{\prime} \mid 0\right\rangle\right|^{2} \exp \left(-E_{0} \beta\right)}{\int d^{3} x^{\prime}\left|\left\langle\mathbf{x}^{\prime} \mid 0\right\rangle\right|^{2} \exp \left(-E_{0} \beta\right)}  \tag{2.383}\\
& =E_{0}
\end{align*}
$$

This stat mech like result seems very striking and profound, and makes me want to go off and study the QM formulation of stat mech that I recall seeing in [13], but not covered back in phy 452 .

Exercise 2.21 Momentum space free propagator. ([15] pr. 2.33)
Derive the free particle propagator in momentum space.
Answer for Exercise 2.21
The momentum space propagator follows in the same fashion as the spatial propagator

$$
\begin{align*}
\left\langle\mathbf{p}^{\prime \prime} \mid \alpha, t ; t_{0}\right\rangle & =\left\langle\mathbf{p}^{\prime \prime}\right| U\left|\alpha, t_{0}\right\rangle \\
& =\left\langle\mathbf{p}^{\prime \prime}\right| e^{-i H\left(t-t_{0}\right) / \hbar}\left|\alpha, t_{0}\right\rangle \\
& =\left\langle\mathbf{p}^{\prime \prime}\right| e^{-i H\left(t-t_{0}\right) / \hbar}\left(\sum_{a^{\prime}}\left|a^{\prime}\right\rangle\left\langle a^{\prime}\right|\right)\left|\alpha, t_{0}\right\rangle \\
& =\left\langle\mathbf{p}^{\prime \prime}\right| \sum_{a^{\prime}} e^{-i E_{a^{\prime}}\left(t-t_{0}\right) / \hbar}\left|a^{\prime}\right\rangle\left\langle a^{\prime} \mid \alpha, t_{0}\right\rangle \\
& =\left\langle\mathbf{p}^{\prime \prime}\right| \sum_{a^{\prime}} e^{-i E_{a^{\prime}}\left(t-t_{0}\right) / \hbar}\left|a^{\prime}\right\rangle\left\langle a^{\prime}\right|\left(\int d^{3} \mathbf{p}^{\prime}\left|\mathbf{p}^{\prime}\right\rangle\left\langle\mathbf{p}^{\prime}\right|\right)\left|\alpha, t_{0}\right\rangle \\
& =\int d^{3} \mathbf{p}^{\prime}\left(\left\langle\mathbf{p}^{\prime \prime}\right| \sum_{a^{\prime}} e^{-i E_{a^{\prime}}\left(t-t_{0}\right) / \hbar}\left|a^{\prime}\right\rangle\left\langle a^{\prime} \mid \mathbf{p}^{\prime}\right\rangle\right)\left\langle\mathbf{p}^{\prime} \mid \alpha, t_{0}\right\rangle \\
& =\int d^{3} \mathbf{p}^{\prime} K\left(\mathbf{p}^{\prime \prime}, t ; \mathbf{p}^{\prime}, t_{0}\right)\left\langle\mathbf{p}^{\prime} \mid \alpha, t_{0}\right\rangle, \tag{2.384}
\end{align*}
$$

$$
\begin{equation*}
K\left(\mathbf{p}^{\prime \prime}, t ; \mathbf{p}^{\prime}, t_{0}\right)=\sum_{a^{\prime}}\left\langle\mathbf{p}^{\prime \prime} \mid a^{\prime}\right\rangle\left\langle a^{\prime} \mid \mathbf{p}^{\prime}\right\rangle e^{-i E_{a^{\prime}}\left(t-t_{0}\right) / \hbar} \tag{2.385}
\end{equation*}
$$

For the free particle Hamiltonian, this can be evaluated over a momentum space basis

$$
\begin{align*}
K\left(\mathbf{p}^{\prime \prime}, t ; \mathbf{p}^{\prime}, t_{0}\right) & =\int d^{3} \mathbf{p}^{\prime \prime \prime}\left\langle\mathbf{p}^{\prime \prime} \mid \mathbf{p}^{\prime \prime \prime}\right\rangle\left\langle\mathbf{p}^{\prime \prime \prime} \mid \mathbf{p}^{\prime}\right\rangle e^{-i E_{\mathbf{p}^{\prime \prime \prime}\left(t-t_{0}\right) / \hbar}} \\
& =\int d^{3} \mathbf{p}^{\prime \prime \prime}\left\langle\mathbf{p}^{\prime \prime} \mid \mathbf{p}^{\prime \prime \prime}\right\rangle \delta\left(\mathbf{p}^{\prime \prime \prime}-\mathbf{p}^{\prime}\right) \exp \left(-i \frac{\left(\mathbf{p}^{\prime \prime \prime}\right)^{2}\left(t-t_{0}\right)}{2 m \hbar}\right) \\
& =\left\langle\mathbf{p}^{\prime \prime} \mid \mathbf{p}^{\prime}\right\rangle \exp \left(-i \frac{\left(\mathbf{p}^{\prime}\right)^{2}\left(t-t_{0}\right)}{2 m \hbar}\right), \tag{2.386}
\end{align*}
$$

or

$$
\begin{equation*}
K\left(\mathbf{p}^{\prime \prime}, t ; \mathbf{p}^{\prime}, t_{0}\right)=\delta\left(\mathbf{p}^{\prime \prime}-\mathbf{p}^{\prime}\right) \exp \left(-i \frac{\left(\mathbf{p}^{\prime}\right)^{2}\left(t-t_{0}\right)}{2 m \hbar}\right) \tag{2.387}
\end{equation*}
$$

This is what we expect since the time evolution is given by just this exponential factor

$$
\begin{align*}
\left\langle\mathbf{p}^{\prime} \mid \alpha, t_{0} ; t\right\rangle & =\left\langle\mathbf{p}^{\prime}\right| \exp \left(-i \frac{\left(\mathbf{p}^{\prime}\right)^{2}\left(t-t_{0}\right)}{2 m \hbar}\right)\left|\alpha, t_{0}\right\rangle \\
& =\exp \left(-i \frac{\left(\mathbf{p}^{\prime}\right)^{2}\left(t-t_{0}\right)}{2 m \hbar}\right)\left\langle\mathbf{p}^{\prime} \mid \alpha, t_{0}\right\rangle . \tag{2.388}
\end{align*}
$$

## Exercise 2.22 Gauge transformation, free Hamiltonian. ([15] pr. 37(a))

Given a gauge transformation of the free particle Hamiltonian to

$$
\begin{equation*}
H=\frac{1}{2 m} \boldsymbol{\Pi} \cdot \boldsymbol{\Pi}+e \phi, \tag{2.389}
\end{equation*}
$$

where

$$
\begin{equation*}
\boldsymbol{\Pi}=\mathbf{p}-\frac{e}{c} \mathbf{A}, \tag{2.390}
\end{equation*}
$$

calculate $m d \mathbf{x} / d t,\left[\Pi_{i}, \Pi_{j}\right]$, and $m d^{2} \mathbf{x} / d t^{2}$, where $\mathbf{x}$ is the Heisenberg picture position operator, and the fields are functions only of position $\phi=\phi(\mathbf{x}), \mathbf{A}=\mathbf{A}(\mathbf{x})$.
Answer for Exercise 2.22
The final results for these calculations are found in [15], but seem worth deriving to exercise our commutator muscles.

Heisenberg picture velocity operator The first order of business is the Heisenberg picture velocity operator, but first note

$$
\begin{align*}
\boldsymbol{\Pi} \cdot \boldsymbol{\Pi} & =\left(\mathbf{p}-\frac{e}{c} \mathbf{A}\right) \cdot\left(\mathbf{p}-\frac{e}{c} \mathbf{A}\right) \\
& =\mathbf{p}^{2}-\frac{e}{c}(\mathbf{A} \cdot \mathbf{p}+\mathbf{p} \cdot \mathbf{A})+\frac{e^{2}}{c^{2}} \mathbf{A}^{2} . \tag{2.391}
\end{align*}
$$

The time evolution of the Heisenberg picture position operator is therefore

$$
\begin{align*}
\frac{d \mathbf{x}}{d t} & =\frac{1}{i \hbar}[\mathbf{x}, H] \\
& =\frac{1}{i \hbar 2 m}\left[\mathbf{x}, \Pi^{2}\right] \\
& =\frac{1}{i \hbar 2 m}\left[\mathbf{x}, \mathbf{p}^{2}-\frac{e}{c}(\mathbf{A} \cdot \mathbf{p}+\mathbf{p} \cdot \mathbf{A})+\frac{e^{2}}{c^{2}} \mathbf{A}^{2}\right]  \tag{2.392}\\
& =\frac{1}{i \hbar 2 m}\left(\left[\mathbf{x}, \mathbf{p}^{2}\right]-\frac{e}{c}[\mathbf{x}, \mathbf{A} \cdot \mathbf{p}+\mathbf{p} \cdot \mathbf{A}]\right) .
\end{align*}
$$

For the $\mathbf{p}^{2}$ commutator we have

$$
\begin{align*}
{\left[x_{r}, \mathbf{p}^{2}\right] } & =i \hbar \frac{\partial \mathbf{p}^{2}}{\partial p_{r}}  \tag{2.393}\\
& =2 i \hbar p_{r}
\end{align*}
$$

or

$$
\begin{equation*}
\left[\mathbf{x}, \mathbf{p}^{2}\right]=2 i \hbar \mathbf{p} \tag{2.394}
\end{equation*}
$$

Computing the remaining commutator, we've got

$$
\begin{align*}
{\left[x_{r}, \mathbf{p} \cdot \mathbf{A}+\mathbf{A} \cdot \mathbf{p}\right]=} & x_{r} p_{s} A_{s}-p_{s} A_{s} x_{r} \\
& +x_{r} A_{s} p_{s}-A_{s} p_{s} x_{r} \\
= & \left(\left[x_{r}, p_{s}\right]+p_{s} x_{r}\right) A_{s}-p_{s} A_{s} x_{r} \\
& +x_{r} A_{s} p_{s}-A_{s}\left(\left[p_{s}, x_{r}\right]+x_{r} p_{s}\right) \\
= & {\left[x_{r}, p_{s}\right] A_{s}+\underline{p_{s} A_{s} x_{r}-p_{s} A_{s} x_{r}} }  \tag{2.395}\\
& +x_{r} A_{s} p_{s}-x_{r} A_{s} p_{s}+A_{s}\left[x_{r}, p_{s}\right] \\
= & 2 i \hbar \delta_{r s} A_{s} \\
= & 2 i \hbar A_{r},
\end{align*}
$$

so

$$
\begin{equation*}
[\mathbf{x}, \mathbf{p} \cdot \mathbf{A}+\mathbf{A} \cdot \mathbf{p}]=2 i \hbar \mathbf{A} . \tag{2.396}
\end{equation*}
$$

Assembling these results gives

$$
\begin{equation*}
\frac{d \mathbf{x}}{d t}=\frac{1}{m}\left(\mathbf{p}-\frac{e}{c} \mathbf{A}\right)=\frac{1}{m} \boldsymbol{\Pi} \tag{2.397}
\end{equation*}
$$

as asserted in the text.

## Kinetic Momentum commutators

$$
\begin{align*}
{\left[\Pi_{r}, \Pi_{s}\right] } & =\left[p_{r}-e A_{r} / c, p_{s}-e A_{s} / c\right] \\
& =\left[p_{r}, p_{s}\right]-\frac{e}{c}\left(\left[p_{r}, A_{s}\right]+\left[A_{r}, p_{s}\right]\right)+\frac{e^{2}}{c^{2}}\left[A_{r}, A_{s}\right] \\
& =-\frac{e}{c}\left((-i \hbar) \frac{\partial A_{s}}{\partial x_{r}}+(i \hbar) \frac{\partial A_{r}}{\partial x_{s}}\right)  \tag{2.398}\\
& =-\frac{i e \hbar}{c}\left(-\frac{\partial A_{s}}{\partial x_{r}}+\frac{\partial A_{r}}{\partial x_{s}}\right) \\
& =-\frac{i e \hbar}{c} \epsilon_{t s r} B_{t},
\end{align*}
$$

or

$$
\begin{equation*}
\left[\Pi_{r}, \Pi_{s}\right]=\frac{i e \hbar}{c} \epsilon_{r s t} B_{t} \tag{2.399}
\end{equation*}
$$

Quantum Lorentz force For the force equation we have

$$
\begin{align*}
m \frac{d^{2} \mathbf{x}}{d t^{2}} & =\frac{d \boldsymbol{\Pi}}{d t} \\
& =\frac{1}{i \hbar}[\boldsymbol{\Pi}, H]  \tag{2.400}\\
& =\frac{1}{i \hbar 2 m}\left[\boldsymbol{\Pi}, \boldsymbol{\Pi}^{2}\right]+\frac{1}{i \hbar}[\boldsymbol{\Pi}, e \phi]
\end{align*}
$$

For the $\phi$ commutator consider one component

$$
\begin{align*}
{\left[\Pi_{r}, e \phi\right] } & =e\left[p_{r}-\frac{e}{c} A_{r}, \phi\right] \\
& =e\left[p_{r}, \phi\right]  \tag{2.401}\\
& =e(-i \hbar) \frac{\partial \phi}{\partial x_{r}}
\end{align*}
$$

or

$$
\begin{equation*}
\frac{1}{i \hbar}[\boldsymbol{\Pi}, e \phi]=-e \boldsymbol{\nabla} \phi=e \mathbf{E} . \tag{2.402}
\end{equation*}
$$

For the $\Pi^{2}$ commutator I initially did this the hard way (it took four notebook pages, plus two for a false start.) Realizing that I didn't use eq. (2.399) for that expansion was the clue to doing this more expediently.

Considering a single component

$$
\begin{align*}
{\left[\Pi_{r}, \Pi^{2}\right] } & =\left[\Pi_{r}, \Pi_{s} \Pi_{s}\right] \\
& =\Pi_{r} \Pi_{s} \Pi_{s}-\Pi_{s} \Pi_{s} \Pi_{r} \\
& =\left(\left[\Pi_{r}, \Pi_{s}\right]+\Pi_{s} \Pi_{r}\right) \Pi_{s}-\Pi_{s}\left(\left[\Pi_{s}, \Pi_{r}\right]+\Pi_{r} \Pi_{s}\right)  \tag{2.403}\\
& =i \hbar \frac{e}{c} \epsilon_{r s t}\left(B_{t} \Pi_{s}+\Pi_{s} B_{t}\right)
\end{align*}
$$

or

$$
\begin{align*}
\frac{1}{i \hbar 2 m}\left[\boldsymbol{\Pi}, \Pi^{2}\right] & =\frac{e}{2 m c} \epsilon_{r s t} \mathbf{e}_{r}\left(B_{t} \Pi_{s}+\Pi_{s} B_{t}\right)  \tag{2.404}\\
& =\frac{e}{2 m c}(\boldsymbol{\Pi} \times \mathbf{B}-\mathbf{B} \times \boldsymbol{\Pi}) .
\end{align*}
$$

Putting all the pieces together we've got the quantum equivalent of the Lorentz force equation

$$
\begin{equation*}
m \frac{d^{2} \mathbf{x}}{d t^{2}}=e \mathbf{E}+\frac{e}{2 c}\left(\frac{d \mathbf{x}}{d t} \times \mathbf{B}-\mathbf{B} \times \frac{d \mathbf{x}}{d t}\right) \tag{2.405}
\end{equation*}
$$

While this looks equivalent to the classical result, all the vectors here are Heisenberg picture operators dependent on position.

## Exercise 2.23 Gauge transformed current. ([15] pr. 2.37 (b))

a. For the gauge transformed Schrödinger equation

$$
\begin{equation*}
\frac{1}{2 m} \Pi(\mathbf{x}) \cdot \boldsymbol{\Pi}(\mathbf{x}) \psi(\mathbf{x}, t)+e \phi(\mathbf{x}) \psi(\mathbf{x}, t)=i \hbar \frac{\partial}{\partial t} \psi(\mathbf{x}, t), \tag{2.406}
\end{equation*}
$$

where

$$
\begin{equation*}
\boldsymbol{\Pi}(\mathbf{x})=-i \hbar \boldsymbol{\nabla}-\frac{e}{c} \mathbf{A}(\mathbf{x}) \tag{2.407}
\end{equation*}
$$

find the probability current defined by

$$
\begin{equation*}
\frac{\partial \psi}{\partial t}+\boldsymbol{\nabla} \cdot \mathbf{j} \tag{2.408}
\end{equation*}
$$

b. Once obtained, let use a $\psi=\sqrt{\rho} e^{i S / \hbar}$ wavefunction representation, and find the corresponding form for the probability current.
c. Evaluate $\int d^{3} x \mathbf{j}$.

Answer for Exercise 2.23

Part a. Equation eq. (2.406) and its conjugate are

$$
\begin{align*}
\frac{1}{2 m} \boldsymbol{\Pi} \cdot \Pi \psi+e \phi \psi & =i \hbar \frac{\partial \psi}{\partial t}  \tag{2.409}\\
\frac{1}{2 m} \boldsymbol{\Pi}^{*} \cdot \Pi^{*} \psi^{*}+e \phi \psi^{*} & =-i \hbar \frac{\partial \psi^{*}}{\partial t}
\end{align*}
$$

which can be used immediately in a chain rule expansion of the probability time derivative

$$
\begin{align*}
i \hbar \frac{\partial \rho}{\partial t}= & i \hbar \psi^{*} \frac{\partial \psi}{\partial t}+i \hbar \psi \frac{\partial \psi^{*}}{\partial t} \\
= & \psi^{*}\left(\frac{1}{2 m} \boldsymbol{\Pi} \cdot \boldsymbol{\Pi} \psi+e \phi \psi\right)-  \tag{2.410}\\
& \psi\left(\frac{1}{2 m} \boldsymbol{\Pi}^{*} \cdot \Pi^{*} \psi^{*}+e \phi \psi^{*}\right) \\
= & \frac{1}{2 m}\left(\psi^{*} \boldsymbol{\Pi} \cdot \boldsymbol{\Pi} \psi-\psi \Pi^{*} \cdot \Pi^{*} \psi^{*}\right) .
\end{align*}
$$

We have a difference of conjugates, so can get away with expanding just the first term

$$
\begin{align*}
\psi^{*} \boldsymbol{\Pi} \cdot \boldsymbol{\Pi} \psi & =\psi^{*} \psi \\
& =\psi^{*}\left(-i \hbar \boldsymbol{\nabla}-\frac{e}{c} \mathbf{A}\right) \cdot\left(-i \hbar \boldsymbol{\nabla}-\frac{e}{c} \mathbf{A}\right) \psi  \tag{2.411}\\
& =\psi^{*}\left(-\hbar^{2} \boldsymbol{\nabla}^{2}+\frac{i \hbar e}{c}(\mathbf{A} \cdot \boldsymbol{\nabla}+\boldsymbol{\nabla} \cdot \mathbf{A})+\frac{e^{2}}{c^{2}} \mathbf{A}^{2}\right) \psi
\end{align*}
$$

Note that in the directional derivative terms, the gradient operates on everything to its right, including $\mathbf{A}$. Also note that the last term has no imag-
inary component, so it will not contribute to the difference of conjugates. This gives

$$
\begin{align*}
& \psi^{*} \boldsymbol{\Pi} \cdot \boldsymbol{\Pi} \psi-\psi \boldsymbol{\Pi}^{*} \cdot \boldsymbol{\Pi}^{*} \psi^{*} \\
&= \psi^{*}\left(-\hbar^{2} \boldsymbol{\nabla}^{2} \psi+\frac{i \hbar e}{c}(\mathbf{A} \cdot \boldsymbol{\nabla} \psi+\boldsymbol{\nabla} \cdot(\mathbf{A} \psi))\right) \\
&-\psi\left(-\hbar^{2} \boldsymbol{\nabla}^{2} \psi^{*}-\frac{i \hbar e}{c}\left(\mathbf{A} \cdot \boldsymbol{\nabla} \psi^{*}+\boldsymbol{\nabla} \cdot\left(\mathbf{A} \psi^{*}\right)\right)\right)  \tag{2.412}\\
&=-\hbar^{2}\left(\psi^{*} \boldsymbol{\nabla}^{2} \psi-\psi \boldsymbol{\nabla}^{2} \psi^{*}\right) \\
&+\frac{i \hbar e}{c}\left(\psi^{*} \mathbf{A} \cdot \boldsymbol{\nabla} \psi+\psi^{*} \boldsymbol{\nabla} \cdot(\mathbf{A} \psi)+\psi \mathbf{A} \cdot \boldsymbol{\nabla} \psi^{*}+\psi \boldsymbol{\nabla} \cdot\left(\mathbf{A} \psi^{*}\right)\right)
\end{align*}
$$

The first term is recognized as a divergence

$$
\begin{align*}
\boldsymbol{\nabla} & \cdot\left(\psi^{*} \boldsymbol{\nabla} \psi-\psi \boldsymbol{\nabla} \psi^{*}\right) \\
& =\psi^{*} \boldsymbol{\nabla} \cdot \boldsymbol{\nabla} \psi+\boldsymbol{\nabla} \psi \cdot \boldsymbol{\nabla} \psi^{*}-\psi \boldsymbol{\nabla} \cdot \boldsymbol{\nabla} \psi^{*}-\boldsymbol{\nabla} \psi^{*} \cdot \boldsymbol{\nabla} \psi  \tag{2.413}\\
& =\psi^{*} \boldsymbol{\nabla}^{2} \psi-\psi \boldsymbol{\nabla}^{2} \psi^{*}
\end{align*}
$$

The second term can also be factored into a divergence operation

$$
\begin{align*}
\psi^{*} \mathbf{A} \cdot \boldsymbol{\nabla} \psi & +\psi^{*} \boldsymbol{\nabla} \cdot(\mathbf{A} \psi)+\psi \mathbf{A} \cdot \boldsymbol{\nabla} \psi^{*}+\psi \boldsymbol{\nabla} \cdot\left(\mathbf{A} \psi^{*}\right) \\
& =\left(\psi^{*} \mathbf{A} \cdot \boldsymbol{\nabla} \psi+\psi \boldsymbol{\nabla} \cdot\left(\mathbf{A} \psi^{*}\right)\right)+\left(\psi \mathbf{A} \cdot \boldsymbol{\nabla} \psi^{*}+\psi^{*} \boldsymbol{\nabla} \cdot(\mathbf{A} \psi)\right)  \tag{2.414}\\
& =2 \boldsymbol{\nabla} \cdot\left(\mathbf{A} \psi \psi^{*}\right)
\end{align*}
$$

Putting all the pieces back together we have

$$
\begin{align*}
\frac{\partial \rho}{\partial t} & =\frac{1}{2 m i \hbar}\left(\psi^{*} \boldsymbol{\Pi} \cdot \boldsymbol{\Pi} \psi-\psi \boldsymbol{\Pi}^{*} \cdot \boldsymbol{\Pi}^{*} \psi^{*}\right) \\
& =\boldsymbol{\nabla} \cdot \frac{1}{2 m i \hbar}\left(-\hbar^{2}\left(\psi^{*} \boldsymbol{\nabla} \psi-\psi \boldsymbol{\nabla} \psi^{*}\right)+\frac{i \hbar e}{c} 2 \mathbf{A} \psi \psi^{*}\right)  \tag{2.415}\\
& =\boldsymbol{\nabla} \cdot\left(\frac{i \hbar}{2 m}\left(\psi^{*} \boldsymbol{\nabla} \psi-\psi \boldsymbol{\nabla} \psi^{*}\right)+\frac{e}{m c} \mathbf{A} \psi \psi^{*}\right)
\end{align*}
$$

From eq. (2.408), the probability current must be

$$
\begin{equation*}
\mathbf{j}=\frac{\hbar}{2 i m}\left(\psi^{*} \boldsymbol{\nabla} \psi-\psi \boldsymbol{\nabla} \psi^{*}\right)-\frac{e}{m c} \mathbf{A} \psi \psi^{*}, \tag{2.416}
\end{equation*}
$$

or

$$
\begin{equation*}
\mathbf{j}=\frac{\hbar}{m} \operatorname{Im}\left(\psi^{*} \boldsymbol{\nabla} \psi\right)-\frac{e}{m c} \mathbf{A} \psi \psi^{*} \tag{2.417}
\end{equation*}
$$

Part b. To find the $\psi=\sqrt{\rho} e^{i S / \hbar}$ form of the current, note that

$$
\begin{equation*}
\boldsymbol{\nabla} \psi=e^{i S / \hbar} \boldsymbol{\nabla} \sqrt{\rho}+\sqrt{\rho} e^{i S / \hbar} \boldsymbol{\nabla}(i S / \hbar), \tag{2.418}
\end{equation*}
$$

so

$$
\begin{equation*}
\psi^{*} \boldsymbol{\nabla} \psi=\sqrt{\rho} \boldsymbol{\nabla} \sqrt{\rho}+\frac{i \rho}{\hbar} \boldsymbol{\nabla} S . \tag{2.419}
\end{equation*}
$$

Discarding the real part of this product, we have

$$
\begin{equation*}
\mathbf{j}=\frac{\hbar}{m} \rho \boldsymbol{\nabla} S-\frac{e}{m c} \mathbf{A} \rho, \tag{2.420}
\end{equation*}
$$

or

$$
\begin{equation*}
\mathbf{j}=\frac{\rho}{m}\left(\nabla S-\frac{e}{c} \mathbf{A}\right) . \tag{2.421}
\end{equation*}
$$

Part c. Finally, note that

$$
\begin{equation*}
-i \hbar \nabla \psi=\langle\mathbf{x}| \mathbf{p}|\psi\rangle, \tag{2.422}
\end{equation*}
$$

so

$$
\begin{equation*}
\mathbf{j}=\frac{\hbar}{m} \operatorname{Im}\left(\langle\Psi \mid \mathbf{x}\rangle\left(\frac{i}{\hbar}\right)\langle\mathbf{x}| \mathbf{p}|\psi\rangle\right)-\frac{e}{m c} \mathbf{A}\langle\psi \mid \mathbf{x}\rangle\langle\mathbf{x} \mid \psi\rangle . \tag{2.423}
\end{equation*}
$$

Integrating over all space to eliminate the identity operators, this is

$$
\begin{align*}
\int d^{3} x \mathbf{j} & =\frac{1}{m} \operatorname{Im}(i\langle\Psi| \mathbf{p}|\psi\rangle)-\frac{e}{m c} \mathbf{A}\langle\psi \mid \psi\rangle \\
& =\frac{1}{m}\langle\psi|\left(\mathbf{p}-\frac{e}{c} \mathbf{A}\right)|\psi\rangle  \tag{2.424}\\
& =\frac{1}{m}\langle\boldsymbol{\Pi}\rangle .
\end{align*}
$$

## Exercise 2.24 Coherent States. (2015 ps2.1, and [15] pr. 2.19(c))

Consider the harmonic oscillator Hamiltonian $H=p^{2} / 2 m+m \omega^{2} x^{2} / 2$. Define the coherent state $|z\rangle$ as the eigenfunction of the annihilation operator, via $a|z\rangle=z|z\rangle$, where $a$ is the oscillator annihilation operator and $z$ is some complex number which characterizes the coherent state.
a. Expanding $|z\rangle$ in terms of oscillator energy eigenstates $|n\rangle$, show that $|z\rangle=C e^{z a^{\dagger}}|0\rangle$. Find the normalization constant C .
b. Calculate the overlap $\left\langle z \mid z^{\prime}\right\rangle$ for normalized coherent states $|z\rangle$.
c. Using the wavefunction $|z\rangle$, compute $\langle x\rangle,\langle p\rangle,\left\langle x^{2}\right\rangle$, and $\left\langle p^{2}\right\rangle$ by defining $x, p$ in terms of $a, a^{\dagger}$.
d. The time evolution of any observed quantity in quantum mechanics can be described in two ways:
(a) Schrödinger: the wavefunction evolves as $|\psi(t)\rangle=e^{-i H t / \hbar}|\psi(0)\rangle$ and the operator $A$ is time-independent, or
(b) Heisenberg: the wavefunction is fixed to its value at $t=0$, say $|\psi\rangle$, and operators evolve as $A(t)=e^{i H t / \hbar} A e^{-i H t / \hbar}$.

Show that both prescriptions yield the same result for any matrix elements or measured quantities.
e. Using the Heisenberg picture, compute the time evolution of

$$
\begin{equation*}
\langle x(t)\rangle,\langle p(t)\rangle,\left\langle x^{2}(t)\right\rangle, \tag{2.425}
\end{equation*}
$$

and $\left\langle p^{2}(t)\right\rangle$ in the coherent state $|z\rangle$. Comment on connections to classical dynamics of the oscillator in phase space.
f. Show that $|f(n)|^{2}$ for a coherent state written as

$$
\begin{equation*}
|z\rangle=\sum_{n=0}^{\infty} f(n)|n\rangle, \tag{2.426}
\end{equation*}
$$

has the form of a Poisson distribution, and find the most probable value of $n$, and thus the most probable energy.

Answer for Exercise 2.24

Part a. Let

$$
\begin{equation*}
|z\rangle=\sum_{n=0}^{\infty} c_{n}|n\rangle, \tag{2.427}
\end{equation*}
$$

The defining identity for $|z\rangle$ becomes

$$
\begin{align*}
a|z\rangle & =\sum_{n=0}^{\infty} c_{n} a|n\rangle \\
& =\sum_{n=1}^{\infty} c_{n} \sqrt{n}|n-1\rangle \\
& =\sum_{n=0}^{\infty} c_{n+1} \sqrt{n+1}|n\rangle  \tag{2.428}\\
& =\sum_{n=0}^{\infty} c_{n} z|n\rangle
\end{align*}
$$

Equating like terms provides a recurrence relation for $c_{n}$

$$
\begin{equation*}
c_{n}=\frac{c_{n-1} z}{\sqrt{n}} \tag{2.429}
\end{equation*}
$$

or

$$
\begin{align*}
& c_{1}=\frac{c_{0} z}{\sqrt{1}} \\
& c_{2}=\frac{c_{1} z}{\sqrt{2}}=\frac{c_{0} z^{2}}{\sqrt{2 \times 1}}  \tag{2.430}\\
& c_{3}=\frac{c_{2} z}{\sqrt{3}}=\frac{c_{0} z^{3}}{\sqrt{3!}}
\end{align*}
$$

or more generally

$$
\begin{equation*}
c_{n}=\frac{c_{0} z^{n}}{\sqrt{n!}} \tag{2.431}
\end{equation*}
$$

or

$$
\begin{equation*}
|z\rangle=c_{0} \sum_{n=0}^{\infty} \frac{z^{n}}{\sqrt{n!}}|n\rangle \tag{2.432}
\end{equation*}
$$

A similar recurrence relation can be constructed for $|n\rangle$

$$
\begin{align*}
|n\rangle & =a^{\dagger} \frac{|n-1\rangle}{\sqrt{n}} \\
& =\left(a^{\dagger}\right)^{2} \frac{|n-2\rangle}{\sqrt{(n)(n-1)}} \\
& =\left(a^{\dagger}\right)^{n-1} \frac{|n-(n-1)\rangle}{\sqrt{n(n-1)(n-2) \ldots(n-(n-2))}}  \tag{2.433}\\
& =\left(a^{\dagger}\right)^{n} \frac{|0\rangle}{\sqrt{n!}}
\end{align*}
$$

so that

$$
\begin{equation*}
|z\rangle=c_{0} \sum_{n=0}^{\infty} \frac{\left(a^{\dagger} z\right)^{n}}{n!}|0\rangle \tag{2.434}
\end{equation*}
$$

or

$$
\begin{equation*}
|z\rangle=c_{0} e^{a^{\dagger} z}|0\rangle \tag{2.435}
\end{equation*}
$$

The normalization follows nicely from the eq. (2.432) representation

$$
\begin{align*}
\langle z \mid z\rangle & =\left|c_{0}\right|^{2} \sum_{n, m=0}^{\infty}\langle m| \frac{\left(z^{*}\right)^{m}}{\sqrt{m!}} \frac{z^{n}}{\sqrt{n!}}|n\rangle \\
& =\left|c_{0}\right|^{2} \sum_{n=0}^{\infty}\langle n| \frac{|z|^{2 n}}{n!}|n\rangle \\
& =\left|c_{0}\right|^{2} \sum_{n=0}^{\infty} \frac{|z|^{2 n}}{n!}  \tag{2.436}\\
& =\left|c_{0}\right|^{2} e^{|z|^{2}} \\
& =1
\end{align*}
$$

Picking a real value for the constant provides a z-dependent normalization for the state

$$
\begin{equation*}
c_{0}=e^{-|z|^{2} / 2} \tag{2.437}
\end{equation*}
$$

or

$$
\begin{equation*}
|z\rangle=e^{-|z|^{2} / 2+a^{\dagger} z}|0\rangle \tag{2.438}
\end{equation*}
$$

Part b.

$$
\begin{align*}
\left\langle z \mid z^{\prime}\right\rangle & =e^{-|z|^{2} / 2} e^{-\left|z^{\prime}\right|^{2} / 2} \sum_{n, m=0}^{\infty}\langle m| \frac{\left(z^{*}\right)^{m}}{\sqrt{m!}} \frac{\left(z^{\prime}\right)^{n}}{\sqrt{n!}}|n\rangle \\
& =e^{-|z|^{2} / 2-\left|z^{\prime}\right|^{2} / 2} \sum_{n=0}^{\infty}\langle n| \frac{\left(z^{*}\right)^{n}}{\sqrt{n!}} \frac{\left(z^{\prime}\right)^{n}}{\sqrt{n!}}|n\rangle  \tag{2.439}\\
& =\exp \left(-|z|^{2} / 2-\left|z^{\prime}\right|^{2} / 2+z^{*} z^{\prime}\right)
\end{align*}
$$

This can be rewritten in terms of the absolute difference between the two z values

$$
\begin{equation*}
\left\langle z \mid z^{\prime}\right\rangle=\exp \left(-\frac{1}{2}\left(\left|z-z^{\prime}\right|^{2}-2 i \operatorname{Im}\left(z^{\prime} z^{*}\right)\right)\right) \tag{2.440}
\end{equation*}
$$

however I'm not sure that's any prettier.

Part c. First note that

$$
\begin{align*}
\langle z| a^{\dagger} & =(a|z\rangle)^{\dagger} \\
& =(z|z\rangle)^{\dagger}  \tag{2.441}\\
& =\langle z| z^{*},
\end{align*}
$$

SO

$$
\begin{align*}
\langle x\rangle & =\frac{x_{0}}{\sqrt{2}}\langle z| a+a^{\dagger}|z\rangle \\
& =\frac{x_{0}}{\sqrt{2}}\langle z| z+z^{*}|z\rangle \\
& =\frac{x_{0}}{\sqrt{2}}\left(z+z^{*}\right)  \tag{2.442}\\
& =\sqrt{\frac{2 \hbar}{m \omega}} \frac{z+z^{*}}{2},
\end{align*}
$$

$$
\begin{align*}
\langle p\rangle & =\frac{i \hbar}{x_{0} \sqrt{2}}\langle z| a^{\dagger}-a|z\rangle \\
& =\frac{-i \hbar}{x_{0} \sqrt{2}}\langle z| z-z^{*}|z\rangle  \tag{2.443}\\
& =\sqrt{2 m \hbar \omega} \frac{z-z^{*}}{2 i},
\end{align*}
$$

$$
\left\langle x^{2}\right\rangle=\frac{x_{0}^{2}}{2}\langle z|\left(a+a^{\dagger}\right)^{2}|z\rangle
$$

$$
=\frac{x_{0}^{2}}{2}\langle z|\left(a^{2}+\left(a^{\dagger}\right)^{2}+a a^{\dagger}+a^{\dagger} a\right)|z\rangle
$$

$$
\begin{equation*}
=\frac{x_{0}^{2}}{2}\langle z|\left(a^{2}+\left(a^{\dagger}\right)^{2}+\left[a, a^{\dagger}\right]+2 a^{\dagger} a\right)|z\rangle \tag{2.444}
\end{equation*}
$$

$$
=\frac{x_{0}^{2}}{2}\left(z^{2}+\left(z^{*}\right)^{2}+1+2 z^{*} z\right)
$$

$$
=\frac{x_{0}^{2}}{2}\left(\left(z+z^{*}\right)^{2}+1\right)
$$

$$
=\frac{\hbar}{2 m \omega}\left(\left(z+z^{*}\right)^{2}+1\right),
$$

and

$$
\begin{align*}
\left\langle p^{2}\right\rangle & =\frac{-\hbar^{2}}{2 x_{0}^{2}}\langle z|\left(a^{\dagger}-a\right)^{2}|z\rangle \\
& =\frac{-\hbar^{2}}{2 x_{0}^{2}}\langle z|\left(\left(a^{\dagger}\right)^{2}+a^{2}-a a^{\dagger}-a^{\dagger} a\right)|z\rangle \\
& =\frac{-\hbar^{2}}{2 x_{0}^{2}}\langle z|\left(\left(a^{\dagger}\right)^{2}+a^{2}-\left[a, a^{\dagger}\right]-2 a^{\dagger} a\right)|z\rangle  \tag{2.445}\\
& =\frac{-\hbar^{2}}{2 x_{0}^{2}}\left(\left(z^{*}\right)^{2}+z^{2}-1-2 z^{*} z\right) \\
& =\frac{m \hbar \omega}{2}\left(1-\left(z-z^{*}\right)^{2}\right)
\end{align*}
$$

As a check against what was stated in class, observe that the minimum uncertainty are satisfied

$$
\begin{align*}
\left\langle x^{2}\right\rangle-\langle x\rangle^{2} & =\frac{x_{0}^{2}}{2}\left(\left(z+z^{*}\right)^{2}+1-\left(z+z^{*}\right)^{2}\right)  \tag{2.446}\\
& =\frac{x_{0}^{2}}{2}
\end{align*}
$$

and

$$
\begin{align*}
\left\langle p^{2}\right\rangle-\langle p\rangle^{2} & =\frac{\hbar^{2}}{2 x_{0}^{2}}\left(1-\left(z+z^{*}\right)^{2}--\left(z-z^{*}\right)^{2}\right)  \tag{2.447}\\
& =\frac{\hbar^{2}}{2 x_{0}^{2}}
\end{align*}
$$

so we have

$$
\begin{equation*}
\Delta x \Delta p=\frac{\hbar}{2} \tag{2.448}
\end{equation*}
$$

Part d. Suppose that $\{|\psi\rangle\}$ is a basis for the observable $A$. In the Heisenberg picture the matrix element for that operator is

$$
\begin{align*}
\langle\psi| A(t)\left|\psi^{\prime}\right\rangle & =\langle\psi| e^{i H t / \hbar} A e^{-i H t / \hbar}\left|\psi^{\prime}\right\rangle \\
& =\sum_{\psi^{\prime \prime}, \psi^{\prime \prime \prime}}\langle\psi| e^{i H t / \hbar}\left|\psi^{\prime \prime}\right\rangle\left\langle\psi^{\prime \prime}\right| A\left|\psi^{\prime \prime \prime}\right\rangle\left\langle\psi^{\prime \prime \prime}\right| e^{-i H t / \hbar}\left|\psi^{\prime}\right\rangle \tag{2.449}
\end{align*}
$$

This product of three matrix elements has the structure of a similarity transformation $\tilde{U}^{\dagger} \tilde{A} \tilde{U}$, where $\tilde{U}$ is the matrix element of the time evolution operator and $\tilde{A}$ is the matrix element of the observable $A$.

Compare this to the Schrödinger picture matrix element with respect to time evolved states

$$
\begin{align*}
\langle\psi(t)| A\left|\psi^{\prime}(t)\right\rangle & =\left(\langle\psi| e^{i H t / \hbar}\right) A\left(e^{-i H t / \hbar}\left|\psi^{\prime}\right\rangle\right) \\
& =\sum_{\psi^{\prime \prime}, \psi^{\prime \prime \prime}}\langle\psi| e^{i H t / \hbar}\left|\psi^{\prime \prime}\right\rangle\left\langle\psi^{\prime \prime}\right| A\left|\psi^{\prime \prime \prime}\right\rangle\left\langle\psi^{\prime \prime \prime}\right| e^{-i H t / \hbar}\left|\psi^{\prime}\right\rangle . \tag{2.450}
\end{align*}
$$

This has exactly the same structure as in the Heisenberg picture. Since average quantities are matrix elements with respect to the same pair of states, this shows that measurements are independent of whether the Heisenberg or Schrödinger picture is used to describe those measurements.

Part e. With time evolution in the mix using the Heisenberg representation of the annihilation operator $a(t)=a e^{-i \omega t}$, the $x$ expectation is

$$
\begin{align*}
\langle x(t)\rangle & =\frac{x_{0}}{\sqrt{2}}\langle z|\left(a e^{-i \omega t}+a^{\dagger} e^{i \omega t}\right)|z\rangle \\
& =\frac{x_{0}}{\sqrt{2}}\left(z e^{-i \omega t}+z^{*} e^{i \omega t}\right) . \tag{2.451}
\end{align*}
$$

It's clear how to generalize the stationary state calculations in part $c$, and can do so by inspection

$$
\begin{align*}
\langle x\rangle & =\sqrt{\frac{\hbar}{2 m \omega}}\left(z e^{-i \omega t}+z^{*} e^{i \omega t}\right) \\
\langle x p\rangle & =\frac{i \hbar}{2}\left(z e^{-i \omega t}+z^{*} e^{i \omega t}\right)\left(z e^{-i \omega t}-z^{*} e^{i \omega t}\right) \\
\langle p\rangle & =-i \sqrt{\frac{m \hbar \omega}{2}}\left(z e^{-i \omega t}-z^{*} e^{i \omega t}\right)  \tag{2.452}\\
\left\langle x^{2}\right\rangle & =\frac{\hbar}{2 m \omega}\left(\left(z e^{-i \omega t}+z^{*} e^{i \omega t}\right)^{2}+1\right)=\langle x\rangle^{2}+\frac{\hbar}{2 m \omega} \\
\left\langle p^{2}\right\rangle & =\frac{m \hbar \omega}{2}\left(1-\left(z e^{-i \omega t}-z^{*} e^{i \omega t}\right)^{2}\right)=\langle p\rangle^{2}+\frac{m \hbar \omega}{2} .
\end{align*}
$$

In class, symmetric and antisymmetric conjugate sums of $z$ were identified as position and momentum, with

$$
\begin{align*}
& x_{0} \equiv \sqrt{\frac{2 \hbar}{m \omega}} \operatorname{Re} z=\sqrt{\frac{\hbar}{2 m \omega}}\left(z+z^{*}\right)  \tag{2.453}\\
& p_{0} \equiv \sqrt{2 m \hbar \omega} \operatorname{Im} z=-i \sqrt{\frac{m \hbar \omega}{2}}\left(z-z^{*}\right)
\end{align*}
$$

With that identification the expectation values above are

$$
\begin{align*}
& \langle x\rangle=x_{0} \cos (\omega t)+\frac{p_{0}}{m \omega} \sin (\omega t)  \tag{2.454}\\
& \langle p\rangle=p_{0} \cos (\omega t)-m \omega x_{0} \sin (\omega t)
\end{align*}
$$

These expectations are analogous to the phase space trajectories of classical particles.

Partf. The Poisson distribution has the form

$$
\begin{equation*}
P(n)=\frac{\mu^{n} e^{-\mu}}{n!} \tag{2.455}
\end{equation*}
$$

Here $\mu$ is the mean of the distribution

$$
\begin{align*}
\langle n\rangle & =\sum_{n=0}^{\infty} n P(n) \\
& =\sum_{n=1}^{\infty} n \frac{\mu^{n} e^{-\mu}}{n!} \\
& =\mu e^{-\mu} \sum_{n=1}^{\infty} \frac{\mu^{n-1}}{(n-1)!}  \tag{2.456}\\
& =\mu e^{-\mu} e^{\mu} \\
& =\mu
\end{align*}
$$

We found that the coherent state had the form

$$
\begin{equation*}
|z\rangle=c_{0} \sum_{n=0} \frac{z^{n}}{\sqrt{n!}}|n\rangle \tag{2.457}
\end{equation*}
$$

so the probability coefficients for $|n\rangle$ are

$$
\begin{align*}
P(n) & =c_{0}^{2} \frac{\left|z^{n}\right|^{2}}{n!}  \tag{2.458}\\
& =e^{-|z|^{2}} \frac{\left|z^{n}\right|^{2}}{n!}
\end{align*}
$$

This has the structure of the Poisson distribution with mean $\mu=|z|^{2}$. The most probable value of $n$ is that for which $|f(n)|^{2}$ is the largest. This is, in general, hard to compute, since we have a maximization problem in the integer domain that falls outside the normal toolbox. If we assume that $n$ is large, so that Stirling's approximation can be used to approximate the factorial, and also seek a non-integer value that maximizes the distribution, the most probable value will be the closest integer to that, and this can be computed. Let

$$
\begin{align*}
g(n) & =|f(n)|^{2} \\
& =\frac{e^{-\mu} \mu^{n}}{n!} \\
& =\frac{e^{-\mu} \mu^{n}}{e^{\ln n!}}  \tag{2.459}\\
& \approx e^{-\mu-n \ln n+n} \mu^{n} \\
& =e^{-\mu-n \ln n+n+n \ln \mu} .
\end{align*}
$$

This is maximized when

$$
\begin{align*}
0 & =\frac{d g}{d n}  \tag{2.460}\\
& =(-\ln n-1+1+\ln \mu) g(n),
\end{align*}
$$

which is maximized at $n=\mu$. One of the integers $n=\lfloor\mu\rfloor$ or $n=\lceil\mu\rceil$ that brackets this value $\mu=|z|^{2}$ is the most probable. So, if an energy measurement is made of a coherent state $|z\rangle$, the most probable value will be one of

$$
\begin{equation*}
E=\hbar\left(\left\lfloor|z|^{2}\right\rfloor+\frac{1}{2}\right), \tag{2.461}
\end{equation*}
$$

or

$$
\begin{equation*}
E=\hbar\left(\left[|z|^{2}\right\rceil+\frac{1}{2}\right) . \tag{2.462}
\end{equation*}
$$

Exercise 2.25 Aharonov Bohm effect. (2015 ps3.1)
a. Consider Young's double slit experiment with electrons, having a monoenergetic source of electrons hitting a double slit with slit spacing $d$, with the electrons then landing on a screen at a distance $D$ away from the double slit. For electrons with energy E, find
the de Broglie wavelength $\lambda$, and hence the spacing between the fringes on the screen. You can ignore the drop in intensity as the electron beam 'spreads' when it travels from the slits to the screen (recall that the slits act as effective point sources), so just take phase changes into account along the travel path.
b. Next, imagine a thin solenoidal flux $\Phi$ being placed between the two slits, so that electron paths which encircle the flux once will pick up an Aharonov Bohm phase $e \Phi / \hbar c$. Compute the resulting shift in the interference pattern on the screen. Show that when the flux $\Phi$ is increased from $0 \rightarrow h c / e$, the interference pattern shifts by exactly one fringe, so the new pattern appears the same as the old. This is the same flux periodicity we saw in class for the energy levels versus flux for a particle on a ring.

Answer for Exercise 2.25

Part a. In general, the superposition of two equal amplitude wave packets with the same wavelength can be factored into a phase and amplitude

$$
\begin{align*}
e^{i\left(\omega t-k L_{1}\right)}+e^{i\left(\omega t-k L_{1}\right)} & =e^{i\left(\omega t-k L_{1} / 2-k L_{2} / 2\right)}\left(e^{-i k L_{1} / 2+i k L_{2} / 2}+e^{i k L_{1} / 2-i k L_{2} / 2}\right) \\
& =2 e^{i\left(\omega t-k L_{1} / 2-k L_{2} / 2\right)} \cos \left(k\left(L_{1}-L_{2}\right) / 2\right) \tag{2.463}
\end{align*}
$$

Now consider the geometry of this screen configuration as sketched in fig. 2.13. The upper and lower path lengths are


Figure 2.13: Double slit interference.

$$
\begin{align*}
L_{1,2} & =\sqrt{D^{2}+(y \mp d / 2)^{2}} \\
& =D \sqrt{1+\frac{(y \mp d / 2)^{2}}{D^{2}}} \\
& \approx D\left(1+\frac{1}{2} \frac{(y \mp d / 2)^{2}}{D^{2}}\right)  \tag{2.464}\\
& =D+\frac{1}{2 D}(y \mp d / 2)^{2} .
\end{align*}
$$

To first order the length difference is

$$
\begin{align*}
L_{1}-L_{2} & =\frac{1}{2 D}(y-d / 2)^{2}-\frac{1}{2 D}(y+d / 2)^{2} \\
& =-\frac{y d}{D} \tag{2.465}
\end{align*}
$$

The amplitude of the interference pattern, at height $y$ on the screen is gated by the cosine

$$
\begin{equation*}
\cos \left(\frac{k y d}{2 D}\right) \tag{2.466}
\end{equation*}
$$

This has peaks and zeros separated by

$$
\begin{equation*}
\frac{k \Delta y d}{2 D}=\pi \tag{2.467}
\end{equation*}
$$

or

$$
\begin{equation*}
\Delta y=\frac{2 \pi D}{k d} \tag{2.468}
\end{equation*}
$$

The electron wave number $(k=2 \pi / \lambda)$ is

$$
\begin{equation*}
k=\frac{\sqrt{2 m E}}{\hbar} \tag{2.469}
\end{equation*}
$$

so the peak separation is

$$
\begin{equation*}
\Delta y=\frac{D h}{d \sqrt{2 m E}} \tag{2.470}
\end{equation*}
$$

Part b. Suppose the upper electron path has a positive orientation with respect to the vector potential direction, while the lower electron path has a negative orientation. The sum of the wave packets will have the form

$$
\begin{aligned}
& e^{i \omega t-k L_{1}-e \Phi / \hbar c}+e^{i \omega t-k L_{2}+e \Phi / \hbar c} \\
&=2 e^{i\left(\omega t-k L_{1} / 2-k L_{2} / 2\right)}\left(e^{-i k L_{1} / 2+i k L_{2} / 2-e \Phi / \hbar c}+e^{i k L_{1} / 2-i k L_{2} / 2+e \Phi / \hbar c}\right) \\
&=2 e^{i\left(\omega t-k L_{1} / 2-k L_{2} / 2\right)} \cos \left(k\left(L_{1}-L_{2}\right) / 2+\frac{e \Phi}{\hbar c}\right)
\end{aligned}
$$

As $\Phi \rightarrow c h / e$ the additional phase term approaches

$$
\begin{equation*}
\frac{h}{\hbar}=2 \pi \tag{2.472}
\end{equation*}
$$

so the entire interference pattern is shifted exactly one full cycle.

## Exercise $2.26 \quad$ Landau Levels - Symmetric gauge. (2015 ps3.2)

Consider a charged particle moving in two dimensions (xy-plane) in a uniform magnetic field $B_{0} \hat{\mathbf{z}}$ perpendicular to the plane. Let us work in a different gauge from the Landau gauge we discussed in class, namely, let us set

$$
\begin{equation*}
\mathbf{A}=\frac{B_{0}}{2}(x \hat{\mathbf{y}}-y \hat{\mathbf{x}}), \tag{2.473}
\end{equation*}
$$

where $(x, y)$ denotes the particle position. This is called the 'symmetric gauge'.

In this gauge,
a. work out the energy spectrum, and the eigenfunctions, and
b. provide a crude counting of the number of states per energy level (i.e., the degeneracy) for an electron on a disk of radius $R$.

Answer for Exercise 2.26

Part a. Using the approach suggested by our practise problem [15] pr. 2.39, the Hamiltonian for magnetic field driven motion constrained to a plane, can be factored into raising and lowering style operators

$$
\begin{align*}
H & =\frac{1}{2 m}\left(\mathbf{p}-\frac{e}{c} \mathbf{A}\right)^{2} \\
& =\frac{1}{2 m}\left(\Pi_{x}^{2}+\Pi_{y}^{2}\right)^{2}  \tag{2.474}\\
& =\frac{1}{2 m}\left(\left(\Pi_{x}-i \Pi_{y}\right)\left(\Pi_{x}+i \Pi_{y}\right)-i\left(\Pi_{x} \Pi_{y}-\Pi_{y} \Pi_{x}\right)\right),
\end{align*}
$$

That commutator term is proportional to the magnetic field strength

$$
\begin{align*}
\Pi_{x} \Pi_{y}-\Pi_{y} \Pi_{x} & =\left[\Pi_{x}, \Pi_{y}\right] \\
& =\left[p_{x}-\frac{e}{c} A_{x}, p_{y}-\frac{e}{c} A_{y}\right] \\
& =\left[p_{x}, p_{y}\right\rceil-\frac{e}{c}\left(\left[A_{x}, p_{y}\right]+\left[p_{x}, A_{y}\right]\right)+\left(\frac{e}{c}\right)^{2}\left[A_{x}, A_{y}\right] \\
& =-\frac{e}{c}(-i \hbar)\left(\frac{\partial A_{y}}{\partial x}-\frac{\partial A_{x}}{\partial y}\right) \\
& =i \frac{e \hbar}{c} B_{z} \\
& =i \frac{e \hbar B_{0}}{c} \tag{2.475}
\end{align*}
$$

so

$$
\begin{equation*}
H=\frac{1}{2 m}\left(\Pi_{x}-i \Pi_{y}\right)\left(\Pi_{x}+i \Pi_{y}\right)+\frac{e \hbar B_{0}}{2 m c} \tag{2.476}
\end{equation*}
$$

Writing

$$
\begin{equation*}
\omega=\frac{e B_{0}}{m c} \tag{2.477}
\end{equation*}
$$

this appears to have the structure of the 1D Harmonic oscillator

$$
\begin{equation*}
H=\frac{1}{2 m}\left(\Pi_{x}-i \Pi_{y}\right)\left(\Pi_{x}+i \Pi_{y}\right)+\frac{\hbar \omega}{2} \tag{2.478}
\end{equation*}
$$

Observe that

$$
\begin{align*}
{\left[\Pi_{x}+i \Pi_{y}, \Pi_{x}-i \Pi_{y}\right] } & =i\left[\Pi_{y}, \Pi_{x}\right]-i\left[\Pi_{x}, \Pi_{y}\right] \\
& =-2 i\left[\Pi_{x}, \Pi_{y}\right] \\
& =\frac{2 e \hbar B_{0}}{c}  \tag{2.479}\\
& =2 m \omega
\end{align*}
$$

With

$$
\begin{equation*}
b=\frac{1}{\sqrt{2 m \omega \hbar}}\left(\Pi_{x}+i \Pi_{y}\right) \tag{2.480}
\end{equation*}
$$

the Hamiltonian has the form

$$
\begin{equation*}
H=\hbar \omega\left(b^{\dagger} b+\frac{1}{2}\right) \tag{2.481}
\end{equation*}
$$

where

$$
\begin{equation*}
\left[b, b^{\dagger}\right]=1, \tag{2.482}
\end{equation*}
$$

just like the 1D SHO. The energy levels are therefore

$$
\begin{equation*}
E_{n}=\hbar \omega\left(n+\frac{1}{2}\right) . \tag{2.483}
\end{equation*}
$$

For the symmetric gauge where $A_{x}=-B_{0} y / 2, A_{y}=B_{0} x / 2$, the lowering operator has the form

$$
\begin{align*}
b & =\frac{1}{\sqrt{2 m \omega \hbar}}\left(p_{x}+i p_{y}-\frac{e}{c} \frac{B_{0}}{2}(-y+i x)\right) \\
& =\frac{-i \hbar}{\sqrt{2 m \omega \hbar}}\left(\partial_{x}+i \partial_{y}+\frac{e B_{0}}{2 \hbar c}(i y+x)\right)  \tag{2.484}\\
& =\frac{-i \hbar}{\sqrt{2 m \omega \hbar}}\left(\partial_{x}+i \partial_{y}+\frac{m \omega}{2 \hbar}(x+i y)\right) .
\end{align*}
$$

With

$$
\begin{equation*}
\alpha=\frac{m \omega}{2 \hbar}=\frac{e B_{0}}{2 \hbar c}, \tag{2.485}
\end{equation*}
$$

The first state is defined by

$$
\begin{align*}
0 & =\langle x, y| b|0\rangle \\
& \propto\left(\partial_{x}+i \partial_{y}+\alpha(x+i y)\right) u(x, y) . \tag{2.486}
\end{align*}
$$

For integer $n$, this has solutions of the form

$$
\begin{equation*}
u_{n}(x, y)=c_{n}(x+i y)^{n} e^{-\alpha\left(x^{2}+y^{2}\right) / 2} \tag{2.487}
\end{equation*}
$$

which can be verified directly

$$
\begin{align*}
& \left(\partial_{x}+i \partial_{y}+\alpha(x+i y)\right) u_{n}(x, y) \\
& =\left(\partial_{x}+i \partial_{y}+\alpha(x+i y)\right) c_{n}(x+i y)^{n} e^{-\alpha\left(x^{2}+y^{2}\right) / 2} \\
& =c_{n} e^{-\alpha\left(x^{2}+y^{2}\right) / 2}\left(n(x+i y)^{n-1}\left(1+i^{2}\right) \times\right.  \tag{2.488}\\
& \left.\quad-\alpha(x+i y)^{n}(2 x+2 y i) / 2+\alpha(x+i y)^{n+1}\right) \\
& \quad=0 .
\end{align*}
$$

The normalization is given by

$$
\begin{align*}
1 & =\int\left|u_{n}(x, y)\right|^{2} d x d y \\
& =\left|c_{n}\right|^{2} \int|x+i y|^{2 n} e^{-\alpha\left(x^{2}+y^{2}\right)} d x d y  \tag{2.489}\\
& =\left|c_{n}\right|^{2} \int_{0}^{\infty} 2 \pi r r^{2 n} e^{-\alpha r^{2}} d r
\end{align*}
$$

Let $\alpha r^{2}=t$, with $2 r d r=d t / \alpha$, for

$$
\begin{align*}
1 & =\left|c_{n}\right|^{2} \frac{\pi}{\alpha} \int_{0}^{\infty}\left(\frac{t}{\alpha}\right)^{n} e^{-t} d t \\
& =\left|c_{n}\right|^{2} \frac{\pi}{\alpha^{n+1}} \int_{0}^{\infty} t^{(n+1)-1} e^{-t} d t  \tag{2.490}\\
& =\left|c_{n}\right|^{2} \frac{\pi}{\alpha^{n+1}} \Gamma(n+1) \\
& =\left|c_{n}\right|^{2} \frac{\pi}{\alpha^{n+1}} n!
\end{align*}
$$

so

$$
\begin{equation*}
c_{n}=\sqrt{\frac{\alpha^{n+1}}{n!\pi}} \tag{2.491}
\end{equation*}
$$

and

$$
\begin{equation*}
u_{n}(x, y)=\sqrt{\frac{\alpha^{n+1}}{n!\pi}}(x+i y)^{n} e^{-\alpha\left(x^{2}+y^{2}\right) / 2} \tag{2.492}
\end{equation*}
$$

Part $b$. The wave functions decay exponentially, and beyond a certain threshold (dependent on $n$ ), not much of the wave function will contribute to the probability density. To get a feel for this, the probability density $2 \pi r\left|u_{n}(r)\right|^{2}$ is plotted for $n=2,4,6$ with $\alpha=1$ in fig. 2.14.


Figure 2.14: Some representative probability density plots.

What is the value of $r$ for which the probability density is maximized for a given value of $n$ ? That is

$$
\begin{align*}
0 & =\frac{d}{d r}\left(2 \pi r\left|u_{n}(r)\right|^{2}\right) \\
& =2 \pi \frac{\alpha^{n+1}}{\pi n!} \frac{d}{d r}\left(r r^{2 n} e^{-\alpha r^{2}}\right)  \tag{2.493}\\
& =2 \pi \frac{\alpha^{n+1}}{\pi n!}\left((2 n+1) r^{2 n} r^{2 n+1}(-2 \alpha r)\right) e^{-\alpha r^{2}} \\
& =2 \pi \frac{\alpha^{n+1}}{\pi n!}\left((2 n+1)-2 \alpha r^{2}\right) r^{2 n} e^{-\alpha r^{2}},
\end{align*}
$$

which occurs at

$$
\begin{align*}
r^{2} & =\frac{(2 n+1)}{2 \alpha} \\
& =\left(n+\frac{1}{2}\right) \frac{2 \hbar c}{e B_{0}} \tag{2.494}
\end{align*}
$$

If this is less than $R^{2}$, and assuming $n$ large with respect to $1 / 2$, and $r^{2} \leq$ $R^{2}$, this is

$$
\begin{equation*}
n \frac{2 \hbar c}{e B_{0}} \leq R^{2} \tag{2.495}
\end{equation*}
$$

or

$$
\begin{align*}
n & \leq \frac{e B_{0} R^{2}}{2 \hbar c} \\
& =\frac{e B_{0} \pi R^{2}}{h c}  \tag{2.496}\\
& =\frac{e \Phi}{h c} .
\end{align*}
$$

The approximate number of ground states if the particle is confined to a disk of radius $R$ is proportional to the magnetic flux $\Phi=\int \mathbf{B} \cdot d \mathbf{S}=B_{0} \pi R^{2}$ through that disk.

## Exercise 2.27 Aharonov Bohm effect. ([15] pr. 2.28, 2015 ps3.3)

Consider an electron confined to the interior of a finite hollow cylinder with its axis being $\hat{\mathbf{z}}$. Let the inner and outer walls of the cylinder be at radial coordinates $\rho_{a}$ and $\rho_{b}>\rho_{a}$ respectively. Let the cylinder have its top and bottom ends at $z=0, L$.
a. Find the eigenstates for a particle confined to this cylinder (ignore normalization), and show that its energies are given by

$$
\begin{equation*}
E_{l m n}=\frac{\hbar^{2}}{2 m}\left(k_{m n}^{2}+\left(\frac{\pi l}{L}\right)^{2}\right) \quad(l=1,2,3, \cdots ; m=0,1,2, \cdots) \tag{2.497}
\end{equation*}
$$

where $k_{m n}$ is the $n$th root of the equation

$$
\begin{equation*}
J_{m}\left(k_{m n} \rho_{b}\right) N_{m}\left(k_{m n} \rho_{a}\right)-N_{m}\left(k_{m n} \rho_{b}\right) J_{m}\left(k_{m n} \rho_{a}\right)=0 \tag{2.498}
\end{equation*}
$$

b. Repeat this problem with a uniform magnetic field $B \hat{\mathbf{z}}$ which is confined to the region $0<\rho<\rho_{a}$ (i.e., only in the hollow part of the cylinder).
c. Show that there is a periodicity of the energy levels with the field, with the period being such that $\pi \rho_{a}^{2} B=2 \pi N \hbar c / e$.

Answer for Exercise 2.27
We can model this geometrical constraints of these two configurations by

$$
\begin{equation*}
H=\frac{1}{2 m}\left(-i \hbar \nabla-\frac{e}{c} \mathbf{A}\right)+V \tag{2.499}
\end{equation*}
$$

where

$$
\begin{equation*}
\boldsymbol{\nabla}=\hat{\boldsymbol{\rho}} \partial_{\rho}+\frac{1}{\rho} \partial_{\phi}+\hat{\mathbf{z}} \partial_{z} \tag{2.500}
\end{equation*}
$$

and

$$
V= \begin{cases}0 & \text { if } \rho \in\left[\rho_{a}, \rho_{b}\right], z \in[0, L]  \tag{2.501}\\ \infty & \text { otherwise }\end{cases}
$$

The effects of the potential require that a wavefunction $\psi$ solution of this equation satisfies

$$
\begin{equation*}
\left.\psi(z)\right|_{z=0, L}=0 \tag{2.502}
\end{equation*}
$$

and

$$
\begin{equation*}
\left.\psi(\rho)\right|_{\rho=\rho_{a}, \rho_{b}}=0 \tag{2.503}
\end{equation*}
$$

leaving

$$
\begin{equation*}
H=\frac{1}{2 m}\left(-i \hbar \boldsymbol{\nabla}-\frac{e}{c} \mathbf{A}\right)^{2} \tag{2.504}
\end{equation*}
$$

in the interior region of the cylinder where the electron is free to move.

Part a. Without a magnetic field, in the interior of the cylinder, the Hamiltonian is

$$
\begin{align*}
H \psi & =\frac{-\hbar^{2}}{2 m} \nabla^{2} \psi \\
& =\frac{-\hbar^{2}}{2 m}\left(\frac{1}{\rho} \partial_{\rho}\left(\rho \partial_{\rho} \psi\right)+\frac{1}{\rho^{2}} \partial_{\phi \phi} \psi+\partial_{z z} \psi\right) \tag{2.505}
\end{align*}
$$

Assuming a solution is possible using separation of variables, let

$$
\begin{equation*}
\psi(\rho, \phi, z)=P(\rho) \Phi(\phi) Z(z) \tag{2.506}
\end{equation*}
$$

so that

$$
\begin{align*}
H \psi & =E \psi \\
& =E P \Phi Z  \tag{2.507}\\
& =\frac{-\hbar^{2}}{2 m}\left(\Phi Z \frac{1}{\rho} \partial_{\rho}\left(\rho \partial_{\rho} P\right)+P Z \frac{1}{\rho^{2}} \partial_{\phi \phi} \Phi+P \Phi \partial_{z z} Z\right)
\end{align*}
$$

or

$$
\begin{equation*}
E=\frac{-\hbar^{2}}{2 m}\left(\frac{1}{\rho P} \partial_{\rho}\left(\rho \partial_{\rho} P\right)+\frac{1}{\rho^{2} \Phi} \partial_{\phi \phi} \Phi+\frac{1}{Z} \partial_{z z} Z\right) \tag{2.508}
\end{equation*}
$$

Let $E=E^{\prime}+E_{z}$, where

$$
\begin{equation*}
E_{z}=\frac{-\hbar^{2}}{2 m} \frac{1}{Z} Z^{\prime \prime} \tag{2.509}
\end{equation*}
$$

This has solution

$$
\begin{equation*}
Z=e^{i k_{z} z} \tag{2.510}
\end{equation*}
$$

where $k_{z}=\sqrt{2 m E_{z}} / \hbar$. The $z=0, L$ boundary condition requires that

$$
\begin{equation*}
k_{z} L=\pi l, \quad l \in \mathbb{Z} \tag{2.511}
\end{equation*}
$$

or

$$
\begin{equation*}
k_{z}=\frac{\pi l}{L} \tag{2.512}
\end{equation*}
$$

That means that the total energy is of the form

$$
\begin{equation*}
E=E^{\prime}+\frac{\hbar^{2}}{2 m}\left(\frac{\pi l}{L}\right)^{2} \tag{2.513}
\end{equation*}
$$

Now we consider the remaining subset of the eigenvalue equation

$$
\begin{equation*}
E^{\prime} \rho^{2}=\frac{-\hbar^{2}}{2 m}\left(\frac{\rho}{P} \partial_{\rho}\left(\rho \partial_{\rho} P\right)+\frac{1}{\Phi} \partial_{\phi \phi} \Phi\right) \tag{2.514}
\end{equation*}
$$

We are free to let

$$
\begin{equation*}
\frac{-\hbar^{2}}{2 m} \frac{1}{\Phi} \partial_{\phi \phi} \Phi=E_{\phi} \tag{2.515}
\end{equation*}
$$

which has solution

$$
\begin{equation*}
\Phi=e^{i k_{\phi} \phi} \tag{2.516}
\end{equation*}
$$

where $k_{\phi}=\sqrt{2 m E_{\phi}} / \hbar$. Geometry requires $k_{\phi}(2 \pi)=2 \pi \nu, v \in \mathbb{Z} \geq 0$, or

$$
\begin{equation*}
E_{\phi}=\frac{\hbar^{2} v^{2}}{2 m} \tag{2.517}
\end{equation*}
$$

This leaves

$$
\begin{equation*}
E^{\prime} \rho^{2}=\frac{-\hbar^{2}}{2 m} \frac{\rho}{P} \partial_{\rho}\left(\rho \partial_{\rho} P\right)+\frac{\hbar^{2} v^{2}}{2 m} \tag{2.518}
\end{equation*}
$$

or

$$
\begin{equation*}
0=\rho^{2} \partial_{\rho \rho} P+\rho \partial_{\rho} P+\left(\frac{2 m}{\hbar^{2}} E^{\prime} \rho^{2}-v^{2}\right) P . \tag{2.519}
\end{equation*}
$$

With $\rho^{\prime}=\sqrt{\frac{2 m E^{\prime}}{\hbar^{2}}} \rho$, this is the standard form (eq. 10.2.1 [5]) for Bessel's equation as a function of $\rho^{\prime}$, with solutions $J_{ \pm v}\left(\rho^{\prime}\right), N_{v}\left(\rho^{\prime}\right)$. In particular, the linear combination

$$
\begin{equation*}
a J_{ \pm v}\left(\rho^{\prime}\right)+b N_{v}\left(\rho^{\prime}\right) \tag{2.520}
\end{equation*}
$$

is a solution. The geometry requires this vanish at $\rho_{a}, \rho_{b}$. With

$$
\begin{equation*}
k=\sqrt{\frac{2 m E^{\prime}}{\hbar^{2}}} \tag{2.521}
\end{equation*}
$$

those boundary value constraints can be written as

$$
0=\left[\begin{array}{ll}
J_{ \pm v}\left(k \rho_{a}\right) & N_{v}\left(k \rho_{b}\right)  \tag{2.522}\\
J_{ \pm v}\left(k \rho_{b}\right) & N_{v}\left(k \rho_{b}\right)
\end{array}\right]\left[\begin{array}{l}
a \\
b
\end{array}\right] .
$$

This is satisfied when the determinant is zero

$$
\begin{equation*}
0=J_{ \pm v}\left(k \rho_{a}\right) N_{v}\left(k \rho_{b}\right)-J_{ \pm v}\left(k \rho_{b}\right) N_{v}\left(k \rho_{a}\right) . \tag{2.523}
\end{equation*}
$$

The total energy eigenvalue is

$$
\begin{equation*}
E=\frac{1}{2 m}(\hbar k)^{2}+\frac{\hbar^{2}}{2 m}\left(\frac{\pi l}{L}\right)^{2} . \tag{2.524}
\end{equation*}
$$

With minor differences in indexing notation, this completes the demonstration of eq. (2.497) and eq. (2.498).

The complete wavefunction associated with this energy eigenvalue is

$$
\begin{equation*}
\psi=\left(a J_{ \pm v}(k \rho)+b N_{v}(k \rho)\right) e^{i v \phi} e^{i \pi l z / L} \tag{2.525}
\end{equation*}
$$

Part b. A magnetic field that is isolated to the hole of the cavity is described by the piecewise vector potential

$$
\mathbf{A}= \begin{cases}\frac{B}{2} \frac{\rho_{a}^{2}}{\rho} \hat{\boldsymbol{\phi}} & \text { if } \rho \geq \rho_{a}  \tag{2.526}\\ \frac{B}{2} \rho \hat{\boldsymbol{\phi}} & \text { if } \rho<\rho_{a}\end{cases}
$$

Checking this for $\rho \geq \rho_{a}$ we have

$$
\begin{align*}
\boldsymbol{\nabla} \times \mathbf{A} & =\frac{B}{2} \rho_{a}^{2}\left(\hat{\boldsymbol{\rho}} \partial_{\rho}+\frac{\hat{\boldsymbol{\phi}}}{\rho} \partial_{\phi}\right) \times \frac{\hat{\boldsymbol{\phi}}}{\rho} \\
& =\frac{B}{2} \rho_{a}^{2}\left(\hat{\boldsymbol{\rho}} \times \frac{\hat{\boldsymbol{\phi}}}{-\rho^{2}}+\frac{\hat{\boldsymbol{\phi}}}{\rho^{2}} \partial_{\phi} \hat{\boldsymbol{\phi}}\right)  \tag{2.527}\\
& =\frac{B \rho_{a}^{2}}{2 \rho^{2}}(\hat{\boldsymbol{\rho}} \times \hat{\boldsymbol{\phi}}+\hat{\boldsymbol{\phi}} \times(-\hat{\boldsymbol{\rho}})) \\
& =0,
\end{align*}
$$

and for $\rho<\rho_{a}$

$$
\begin{align*}
\boldsymbol{\nabla} \times \mathbf{A} & =\frac{B}{2}\left(\hat{\boldsymbol{\rho}} \partial_{\rho}+\frac{\hat{\boldsymbol{\phi}}}{\rho} \partial_{\phi}\right) \times(\rho \hat{\boldsymbol{\phi}}) \\
& =\frac{B}{2}\left(\hat{\boldsymbol{\rho}} \times \hat{\boldsymbol{\phi}}+\frac{\hat{\boldsymbol{\phi}}}{\rho} \times(-\rho \hat{\boldsymbol{\rho}})\right)  \tag{2.528}\\
& =B \hat{\mathbf{z}} .
\end{align*}
$$

In the conduction zone of the cylinder the Hamiltonian is now

$$
\begin{align*}
H & =\frac{1}{2 m}\left(\mathbf{p}-\frac{e}{c} \mathbf{A}\right)^{2} \\
& =\frac{1}{2 m}\left(-i \hbar \boldsymbol{\nabla}-\frac{e}{c} \frac{B}{2} \rho_{a}^{2} \frac{\hat{\boldsymbol{\phi}}}{\rho}\right)^{2}  \tag{2.529}\\
& =\frac{1}{2 m}\left(-\hbar^{2} \boldsymbol{\nabla}^{2}+i \hbar \frac{e}{c} \frac{B}{2} \rho_{a}^{2}\left(\boldsymbol{\nabla} \cdot \frac{\hat{\boldsymbol{\phi}}}{\rho}+\frac{\hat{\boldsymbol{\phi}}}{\rho} \cdot \boldsymbol{\nabla}\right)+\left(\frac{e B \rho_{a}^{2}}{2 c \rho}\right)^{2}\right)
\end{align*}
$$

Expanding the cross terms we have

$$
\begin{align*}
\boldsymbol{\nabla} \cdot\left(\frac{\hat{\boldsymbol{\phi}}}{\rho} \psi\right) & =\boldsymbol{\nabla} \psi \cdot \frac{\hat{\boldsymbol{\phi}}}{\rho}+\psi \boldsymbol{\nabla} \cdot \frac{\hat{\boldsymbol{\phi}}}{\rho} \\
& =\left(\hat{\boldsymbol{\rho}} \partial_{\rho} \psi+\frac{\hat{\boldsymbol{\phi}}}{\rho} \partial_{\phi} \psi+\partial_{z} \psi\right) \cdot \frac{\hat{\boldsymbol{\phi}}}{\rho}+\psi\left(\hat{\boldsymbol{\rho}} \cdot \frac{\hat{\boldsymbol{\phi}}}{-\rho^{2}}+\frac{\hat{\boldsymbol{\phi}}}{\rho} \cdot \frac{(-\hat{\boldsymbol{\rho}})}{\rho}\right)  \tag{2.530}\\
& =\frac{1}{\rho^{2}} \partial_{\phi} \psi,
\end{align*}
$$

and

$$
\begin{align*}
\frac{\hat{\boldsymbol{\phi}}}{\rho} \cdot \nabla \psi & =\frac{\hat{\boldsymbol{\phi}}}{\rho} \cdot \frac{\hat{\boldsymbol{\phi}}}{\rho} \partial_{\phi} \psi  \tag{2.531}\\
& =\frac{1}{\rho^{2}} \partial_{\phi} \psi
\end{align*}
$$

so we have

$$
\begin{equation*}
\boldsymbol{\nabla} \cdot \frac{\hat{\boldsymbol{\phi}}}{\rho}+\frac{\hat{\boldsymbol{\phi}}}{\rho} \cdot \boldsymbol{\nabla}=\frac{2}{\rho^{2}} \partial_{\phi} \tag{2.532}
\end{equation*}
$$

The complete Hamiltonian is
$H \psi=E \psi$

$$
\begin{equation*}
=\frac{-\hbar^{2}}{2 m}\left(\frac{1}{\rho} \partial_{\rho}\left(\rho \partial_{\rho} \psi\right)+\frac{1}{\rho^{2}} \partial_{\phi \phi} \psi+\partial_{z z} \psi\right)+\frac{i \hbar e B \rho_{a}^{2}}{2 m c \rho^{2}} \partial_{\phi} \psi+\frac{1}{2 m}\left(\frac{e B \rho_{a}^{2}}{2 c \rho}\right)^{2} \psi \tag{2.533}
\end{equation*}
$$

Proceeding the same way with separation of variables using $\psi=P \Phi Z$, we have as before

$$
\begin{align*}
Z & =e^{i \pi l / L} \\
E & =E^{\prime}+\frac{\hbar^{2}}{2 m}\left(\frac{\pi l}{L}\right)^{2} \tag{2.534}
\end{align*}
$$

and are left with
$E^{\prime} \rho^{2}=\frac{-\hbar^{2}}{2 m}\left(\frac{\rho}{P} \partial_{\rho}\left(\rho \partial_{\rho} P\right)+\frac{1}{\Phi} \partial_{\phi \phi} \Phi\right)+\frac{i \hbar}{2 m} \frac{e}{c} \frac{B}{2} \rho_{a}^{2} \frac{1}{\Phi} 2 \partial_{\phi} \Phi+\frac{1}{2 m}\left(\frac{e B \rho_{a}^{2}}{2 c}\right)^{2}$.

Now set

$$
\begin{equation*}
\frac{-\hbar^{2}}{2 m \Phi} \partial_{\phi \phi} \Phi+\frac{i \hbar e B \rho_{a}^{2}}{2 m c \Phi} \partial_{\phi} \Phi+\frac{1}{2 m}\left(\frac{e B \rho_{a}^{2}}{2 c}\right)^{2}=E_{\phi} \tag{2.536}
\end{equation*}
$$

With $\Phi=e^{i v \phi}$, that gives

$$
\begin{align*}
E_{\phi} & =\frac{-\hbar^{2}}{2 m}(-i v)^{2}+\frac{i \hbar e B \rho_{a}^{2}}{2 m c \Phi}(i v)+\frac{1}{2 m}\left(\frac{e B \rho_{a}^{2}}{2 c}\right)^{2} \\
& =\frac{\hbar^{2} v^{2}}{2 m}-\frac{\hbar e B \rho_{a}^{2}}{2 m c} v+\frac{1}{2 m}\left(\frac{e B \rho_{a}^{2}}{2 c}\right)^{2} \\
& =\frac{\hbar^{2}}{2 m}\left(v^{2}-\frac{e B \rho_{a}^{2}}{\hbar c} v\right)+\frac{1}{2 m}\left(\frac{e B \rho_{a}^{2}}{2 c}\right)^{2}  \tag{2.537}\\
& =\frac{\hbar^{2}}{2 m}\left(\left(v-\frac{e B \rho_{a}^{2}}{2 \hbar c}\right)^{2}-\left(\frac{e B \rho_{a}^{2}}{2 \hbar c}\right)^{2}\right)+\frac{1}{2 m}\left(\frac{e B \rho_{a}^{2}}{2 c}\right)^{2} \\
& =\frac{\hbar^{2}}{2 m}\left(v-\frac{e B \rho_{a}^{2}}{2 \hbar c}\right)^{2}
\end{align*}
$$

We are left with

$$
\begin{align*}
E^{\prime} \rho^{2} & =\frac{-\hbar^{2}}{2 m}\left(\frac{\rho}{P} \partial_{\rho}\left(\rho \partial_{\rho} P\right)\right)+E_{\phi} \\
& =\frac{-\hbar^{2}}{2 m}\left(\frac{\rho}{P} \partial_{\rho}\left(\rho \partial_{\rho} P\right)\right)+\frac{\hbar^{2}}{2 m}\left(v-\frac{e B \rho_{a}^{2}}{2 \hbar c}\right)^{2} \tag{2.538}
\end{align*}
$$

or

$$
\begin{equation*}
\rho^{2} P^{\prime \prime}+\rho P+\left(\frac{2 m E^{\prime}}{\hbar^{2}} \rho^{2}-\left(v-\frac{e B \rho_{a}^{2}}{2 \hbar c}\right)^{2}\right) P=0 \tag{2.539}
\end{equation*}
$$

With

$$
\begin{equation*}
v^{\prime}=v-\frac{e B \rho_{a}^{2}}{2 \hbar c} \tag{2.540}
\end{equation*}
$$

and

$$
\begin{equation*}
k=\frac{\sqrt{2 m E^{\prime}}}{\hbar} \tag{2.541}
\end{equation*}
$$

this is, once again, a Bessel equation with solution

$$
\begin{equation*}
P=a J_{ \pm v^{\prime}}(k \rho)+b N_{v^{\prime}}(k \rho) \tag{2.542}
\end{equation*}
$$

The full solution is

$$
\begin{equation*}
\psi=\left(a J_{ \pm v^{\prime}}(k \rho)+b N_{v^{\prime}}(k \rho)\right) e^{i v \phi} e^{i \pi l z / L} \tag{2.543}
\end{equation*}
$$

Part c. With $\theta=e B \rho_{a}^{2} / 2 \hbar c$, and $N=v$, eq. (2.537) takes the form

$$
\begin{equation*}
\frac{2 m E_{\phi}}{\hbar^{2}}=(N-\theta)^{2} \tag{2.544}
\end{equation*}
$$

which has zeros when $N=\theta$, or

$$
\begin{equation*}
N=\frac{e B \rho_{a}^{2}}{2 \hbar c}, \tag{2.545}
\end{equation*}
$$

Shuffling terms, this is

$$
\begin{equation*}
\frac{2 \pi N \hbar c}{e}=B \pi \rho_{a}^{2} \tag{2.546}
\end{equation*}
$$

which is the desired result.
Exercise $2.28 \quad$ Two spin time evolution. (midterm pr. 1.iii)
Compute the time evolution of a two particle state

$$
\begin{equation*}
\psi=\frac{1}{\sqrt{2}}(|\uparrow \downarrow\rangle-|\downarrow \uparrow\rangle) \tag{2.547}
\end{equation*}
$$

under the action of the Hamiltonian

$$
\begin{equation*}
H=-B S_{z, 1}+2 B S_{x, 2}=\frac{\hbar B}{2}\left(-\sigma_{z, 1}+2 \sigma_{x, 2}\right) \tag{2.548}
\end{equation*}
$$

Answer for Exercise 2.28
We have to know the action of the Hamiltonian on all the states

$$
\begin{align*}
& H|\uparrow \uparrow\rangle=\frac{B \hbar}{2}(-|\uparrow \uparrow\rangle+2|\uparrow \downarrow\rangle) \\
& H|\uparrow \downarrow\rangle=\frac{B \hbar}{2}(-|\uparrow \downarrow\rangle+2|\uparrow \uparrow\rangle)  \tag{2.549}\\
& H|\downarrow \uparrow\rangle=\frac{B \hbar}{2}(|\downarrow \uparrow\rangle+2|\downarrow \downarrow\rangle) \\
& H|\downarrow \downarrow\rangle=\frac{B \hbar}{2}(|\downarrow \downarrow\rangle+2|\downarrow \uparrow\rangle) .
\end{align*}
$$

With respect to the basis $\{|\uparrow \uparrow\rangle,|\uparrow \downarrow\rangle,|\downarrow \uparrow\rangle,|\downarrow \downarrow\rangle\}$, the matrix of the Hamiltonian is

$$
H=\frac{\hbar B}{2}\left[\begin{array}{cccc}
-1 & 2 & 0 & 0  \tag{2.550}\\
2 & -1 & 0 & 0 \\
0 & 0 & 1 & 2 \\
0 & 0 & 2 & 1
\end{array}\right]
$$

Utilizing the block diagonal form (and ignoring the $\hbar B / 2$ factor for now), the characteristic equation is

$$
\begin{align*}
0 & =\left|\begin{array}{cc}
-1-\lambda & 2 \\
2 & -1-\lambda
\end{array}\right|\left|\begin{array}{cc}
1-\lambda & 2 \\
2 & 1-\lambda
\end{array}\right|  \tag{2.551}\\
& =\left((1+\lambda)^{2}-4\right)\left((1-\lambda)^{2}-4\right) .
\end{align*}
$$

This has solutions

$$
\begin{equation*}
1 \pm \lambda= \pm 2 \tag{2.552}
\end{equation*}
$$

or, with the $\hbar B / 2$ factors put back in

$$
\begin{equation*}
\lambda= \pm \hbar B / 2, \pm 3 \hbar B / 2 \tag{2.553}
\end{equation*}
$$

I was thinking that we needed to compute the time evolution operator

$$
\begin{equation*}
U=e^{-i H t / \hbar} \tag{2.554}
\end{equation*}
$$

but we actually only need the eigenvectors, and the inverse relations. We can find the eigenvectors by inspection in each case from

$$
\begin{align*}
H-(1) \frac{\hbar B}{2} & =\frac{\hbar B}{2}\left[\begin{array}{cccc}
-2 & 2 & 0 & 0 \\
2 & -2 & 0 & 0 \\
0 & 0 & 0 & 2 \\
0 & 0 & 2 & 0
\end{array}\right] \\
H-(-1) \frac{\hbar B}{2} & =\frac{\hbar B}{2}\left[\begin{array}{llll}
0 & 2 & 0 & 0 \\
2 & 0 & 0 & 0 \\
0 & 0 & 2 & 2 \\
0 & 0 & 2 & 2
\end{array}\right]  \tag{2.555}\\
H-(3) \frac{\hbar B}{2} & =\frac{\hbar B}{2}\left[\begin{array}{cccc}
-4 & 2 & 0 & 0 \\
2 & -4 & 0 & 0 \\
0 & 0 & -2 & 2 \\
0 & 0 & 2 & -2
\end{array}\right] \\
H-(-3) \frac{\hbar B}{2} & =\frac{\hbar B}{2}\left[\begin{array}{llll}
2 & 2 & 0 & 0 \\
2 & 2 & 0 & 0 \\
0 & 0 & 4 & 2 \\
0 & 0 & 2 & 1
\end{array}\right] .
\end{align*}
$$

The eigenkets are

$$
\begin{array}{lc}
|1\rangle=\frac{1}{\sqrt{2}}\left[\begin{array}{l}
1 \\
1 \\
0 \\
0
\end{array}\right], \quad|-1\rangle=\frac{1}{\sqrt{2}}\left[\begin{array}{c}
0 \\
0 \\
1 \\
-1
\end{array}\right],  \tag{2.556}\\
|3\rangle=\frac{1}{\sqrt{2}}\left[\begin{array}{l}
0 \\
0 \\
1 \\
1
\end{array}\right], \quad|-3\rangle=\frac{1}{\sqrt{2}}\left[\begin{array}{c}
1 \\
-1 \\
0 \\
0
\end{array}\right],
\end{array}
$$

or

$$
\begin{align*}
\sqrt{2}|1\rangle & =|\uparrow \uparrow\rangle+|\uparrow \downarrow\rangle \\
\sqrt{2}|-1\rangle & =|\downarrow \uparrow\rangle-|\downarrow \downarrow\rangle  \tag{2.557}\\
\sqrt{2}|3\rangle & =|\downarrow \uparrow\rangle+|\downarrow \downarrow\rangle \\
\sqrt{2}|-3\rangle & =|\uparrow \uparrow\rangle-|\uparrow \downarrow\rangle .
\end{align*}
$$

We can invert these

$$
\begin{align*}
& |\uparrow \uparrow\rangle=\frac{1}{\sqrt{2}}(|1\rangle+|-3\rangle) \\
& |\uparrow \downarrow\rangle=\frac{1}{\sqrt{2}}(|1\rangle-|-3\rangle) \\
& |\downarrow \uparrow\rangle=\frac{1}{\sqrt{2}}(|3\rangle+|-1\rangle)  \tag{2.558}\\
& |\downarrow \downarrow\rangle=\frac{1}{\sqrt{2}}(|3\rangle-|-1\rangle) .
\end{align*}
$$

The original state of interest can now be expressed in terms of the eigenkets

$$
\begin{equation*}
\psi=\frac{1}{2}(|1\rangle-|-3\rangle-|3\rangle-|-1\rangle) \tag{2.559}
\end{equation*}
$$

The time evolution of this ket is

$$
\begin{align*}
& \psi(t)= \frac{1}{2}\left(e^{-i B t / 2}|1\rangle-e^{3 i B t / 2}|-3\rangle-e^{-3 i B t / 2}|3\rangle-e^{i B t / 2}|-1\rangle\right) \\
&= \frac{1}{2 \sqrt{2}}\left(e^{-i B t / 2}(|\uparrow \uparrow\rangle+|\uparrow \downarrow\rangle)-e^{3 i B t / 2}(|\uparrow \uparrow\rangle-|\uparrow \downarrow\rangle)\right. \\
&\left.-e^{-3 i B t / 2}(|\downarrow \uparrow\rangle+|\downarrow \downarrow\rangle)-e^{i B t / 2}(|\downarrow \uparrow\rangle-|\downarrow \downarrow\rangle)\right) \\
&= \frac{1}{2 \sqrt{2}}\left(\left(e^{-i B t / 2}-e^{3 i B t / 2}\right)|\uparrow \uparrow\rangle+\left(e^{-i B t / 2}+e^{3 i B t / 2}\right)|\uparrow \downarrow\rangle\right. \\
&=\frac{\left.-\left(e^{-3 i B t / 2}+e^{i B t / 2}\right)|\downarrow \uparrow\rangle+\left(e^{i B t / 2}-e^{-3 i B t / 2}\right)|\downarrow \downarrow\rangle\right)}{2 \sqrt{2}}\left(e^{i B t / 2}\left(e^{-2 i B t / 2}-e^{2 i B t / 2}\right)|\uparrow \uparrow\rangle+e^{i B t / 2}\left(e^{-2 i B t / 2}+e^{2 i B t / 2}\right)|\uparrow \downarrow\rangle\right. \\
&\left.\quad-e^{-i B t / 2}\left(e^{-2 i B t / 2}+e^{2 i B t / 2}\right)|\downarrow \uparrow\rangle+e^{-i B t / 2}\left(e^{2 i B t / 2}-e^{-2 i B t / 2}\right)|\downarrow \downarrow\rangle\right) \\
&=\frac{1}{\sqrt{2}}\left(i \sin (B t)\left(e^{-i B t / 2}|\downarrow \downarrow\rangle-e^{i B t / 2}|\uparrow \uparrow\rangle\right)\right. \\
&\left.\quad+\cos (B t)\left(e^{i B t / 2}|\uparrow \downarrow\rangle-e^{-i B t / 2}|\downarrow \uparrow\rangle\right)\right) .
\end{align*}
$$

Note that this returns to the original state when $t=\frac{2 \pi n}{B}, n \in \mathbb{Z}$. I think I've got it right this time (although I got a slightly different answer on paper before typing it up.)

## Exercise $2.29 \quad$ Particle in uniform fields. (2015 midterm p2)

Find the energy eigenvalues and states for a charged particle moving the in the $x, y$ plane in a uniform magnetic field $B \hat{\mathbf{z}}$ and a uniform electric field $E \hat{\mathbf{y}}$.
Answer for Exercise 2.29
The Hamiltonian for such a problem has the form

$$
\begin{equation*}
H=\frac{\left(p_{x}-q A_{x} / c\right)^{2}}{2 m}+\frac{\left(p_{y}-q A_{y} / c\right)^{2}}{2 m}+q \phi \tag{2.561}
\end{equation*}
$$

where $\mathbf{A}, \phi$ are the potentials for the electromagnetic field. Since we don't want a time dependent electric potential, our only choice is

$$
\begin{equation*}
E \hat{\mathbf{y}}=-\boldsymbol{\nabla} \phi=-\boldsymbol{\nabla}(-E y) . \tag{2.562}
\end{equation*}
$$

We have many choices for the magnetic field, but it will have to be of the form $\mathbf{A}=B(-a y, b x, 0)$, for example $a=b=1 / 2$. The Hamiltonian becomes

$$
\begin{equation*}
H=\frac{\left(p_{x}+q B a y / c\right)^{2}}{2 m}+\frac{\left(p_{y}-q B b x / c\right)^{2}}{2 m}-q E y . \tag{2.563}
\end{equation*}
$$

We seek a wave function that makes this separable. If we try $\psi=e^{i k y} \phi(x)$ we get

$$
\begin{equation*}
\frac{H \psi}{e^{i k y}}=\left(\frac{\left(p_{x}+q B a y / c\right)^{2}}{2 m}+\frac{(\hbar k-q B b x / c)^{2}}{2 m}-q E y\right) \phi(x), \tag{2.564}
\end{equation*}
$$

and if we try $\psi=e^{i k y} \phi(x)$ we get

$$
\begin{equation*}
\frac{H \psi}{e^{i k x}}=\left(\frac{(\hbar k+q B a y / c)^{2}}{2 m}+\frac{\left(p_{y}-q B b x / c\right)^{2}}{2 m}-q E y\right) \phi(y) . \tag{2.565}
\end{equation*}
$$

The latter is separable if we set $b=0$, which requires $a=1$, leaving an eigenvalue equation for $\phi(y)$

$$
\begin{align*}
H^{\prime}= & \frac{(\hbar k+q B y / c)^{2}}{2 m}+\frac{p_{y}^{2}}{2 m}-q E y \\
= & \frac{p_{y}^{2}}{2 m}-(q E-\hbar k q B / m c) y+\frac{1}{2 m}\left(\frac{q B y}{c}\right)^{2}+\frac{(\hbar k)^{2}}{2 m} \\
= & \frac{p_{y}^{2}}{2 m}+\frac{1}{2 m}\left(\frac{q B}{c}\right)^{2}\left(-\frac{2}{m}\left(\frac{m c}{q B}\right)^{2}(q E-\hbar k q B / m c) y+y^{2}\right)+\frac{(\hbar k)^{2}}{2 m} \\
= & \frac{p_{y}^{2}}{2 m}+\frac{1}{2} m\left(\frac{q B}{m c}\right)^{2}\left(y-\frac{1}{m}\left(\frac{m c}{q B}\right)^{2}(q E-\hbar k q B / m c)\right)^{2} \\
& \quad-\frac{1}{2 m}\left(\frac{m c}{q B}\right)^{2}(q E-\hbar k q B / m c)^{2}+\frac{(\hbar k)^{2}}{2 m} \tag{2.566}
\end{align*}
$$

Let

$$
\begin{align*}
\omega & =\frac{q B}{m c} \\
y_{0} & =\frac{1}{m \omega^{2}}(q E-\hbar k \omega)  \tag{2.567}\\
E_{0} & =\frac{(\hbar k)^{2}}{2 m}-\frac{1}{2} m \omega^{2} y_{0}^{2},
\end{align*}
$$

leaving

$$
\begin{equation*}
H^{\prime}=\frac{p_{y}^{2}}{2 m}+\frac{1}{2} m \omega^{2}\left(y-y_{0}\right)^{2}+E_{0} \tag{2.568}
\end{equation*}
$$

The energy eigenvalues are therefore

$$
\begin{equation*}
E=\hbar \omega\left(n+\frac{1}{2}\right)+E_{0} \tag{2.569}
\end{equation*}
$$

and the eigenfunctions are

$$
\begin{equation*}
\psi=e^{i k x} \phi_{n}\left(y-y_{0}\right) \tag{2.570}
\end{equation*}
$$

where $\phi_{n}(y)$ is the n-th Harmonic oscillator wavefunction.

### 3.1 CONSTRUCTION OF THE DIRAC EQUATION.

Schrödinger Derivation Recall that a "derivation" of the Schrödinger equation can be associated with the following equivalences

$$
\begin{align*}
& E \leftrightarrow \hbar \omega \leftrightarrow i \hbar \frac{\partial}{\partial t},  \tag{3.1}\\
& p \leftrightarrow \hbar k \leftrightarrow-i \hbar \frac{\partial}{\partial t}, \tag{3.2}
\end{align*}
$$

so that the classical energy relationship

$$
\begin{equation*}
E=\frac{p^{2}}{2 m} \tag{3.3}
\end{equation*}
$$

takes the form

$$
\begin{equation*}
i \hbar \frac{\partial}{\partial t}=-\frac{\hbar^{2}}{2 m} \tag{3.4}
\end{equation*}
$$

How do we do this in a relativistic context where the energy momentum relationship is

$$
\begin{equation*}
E=\sqrt{p^{2} c^{2}+m^{2} c^{4}} \approx m c^{2}+\frac{p^{2}}{2 m}+\cdots \tag{3.5}
\end{equation*}
$$

where $m$ is the rest mass and $c$ is the speed of light?

Attempt I

$$
\begin{equation*}
E=m c^{2}+\frac{p^{2}}{2 m}+(\ldots) p^{4}+(\ldots) p^{6}+\cdots \tag{3.6}
\end{equation*}
$$

First order in time, but infinite order in space $\partial / \partial x$. Useless.

## Attempt II

$$
\begin{equation*}
E^{2}=p^{2} c^{2}+m^{2} c^{4} \tag{3.7}
\end{equation*}
$$

This gives

$$
\begin{equation*}
-\hbar^{2} \frac{\partial^{2} \psi}{\partial t^{2}}=-\hbar^{2} c^{2} \frac{\partial^{2} \psi}{\partial x^{2}}+m^{2} c^{4} \psi \tag{3.8}
\end{equation*}
$$

This is the Klein-Gordon equation, which is second order in time.

Attempt III Suppose that we have the matrix

$$
\left[\begin{array}{cc}
p c & m c^{2}  \tag{3.9}\\
m c^{2} & -p c
\end{array}\right]
$$

or

$$
\left[\begin{array}{cc}
m c^{2} & i p c  \tag{3.10}\\
-i p c & -m c^{2}
\end{array}\right],
$$

These both happen to have eigenvalues $\lambda_{ \pm}= \pm \sqrt{p^{2} c^{2}}$. For those familiar with the Dirac matrices, this amounts to a choice for different representations of the gamma matrices.

Working with eq. (3.9), which has some nicer features than other possible representations, we seek a state

$$
\psi=\left[\begin{array}{l}
\psi_{1}(x, t)  \tag{3.11}\\
\psi_{2}(x, t)
\end{array}\right]
$$

where we aim to write down an equation for this composite state.

$$
\begin{equation*}
i \hbar \frac{\partial \psi}{\partial t}=\mathbf{H} \psi \tag{3.12}
\end{equation*}
$$

Assuming the matrix is the Hamiltonian, multiplying that with the composite state gives

$$
\begin{align*}
{\left[\begin{array}{l}
i \hbar \frac{\partial \psi_{1}}{\partial t} \\
i \hbar \frac{\partial \psi_{1}}{\partial t}
\end{array}\right] } & =\left[\begin{array}{cc}
\hat{p} c & m c^{2} \\
m c^{2} & -\hat{p} c
\end{array}\right]\left[\begin{array}{l}
\psi_{1}(x, t) \\
\psi_{2}(x, t)
\end{array}\right]  \tag{3.13}\\
& =\left[\begin{array}{l}
\hat{p} c \psi_{1}+m c^{2} \psi_{2} \\
m c^{2} \psi_{1}-\hat{p} c \psi_{2}
\end{array}\right]
\end{align*}
$$

What happens when we square this? We get

$$
\begin{align*}
\left(i \hbar \frac{\partial}{\partial t}\right)^{2} \psi & =\mathbf{H} \mathbf{H} \boldsymbol{\psi} \\
& =\left[\begin{array}{cc}
\hat{p} c & m c^{2} \\
m c^{2} & -\hat{p} c
\end{array}\right]\left[\begin{array}{cc}
\hat{p} c & m c^{2} \\
m c^{2} & -\hat{p} c
\end{array}\right] \psi  \tag{3.14}\\
& =\left[\begin{array}{cc}
\hat{p}^{2} c^{2}+m^{2} c^{4} & 0 \\
0 & \hat{p}^{2} c^{2}+m^{2} c^{4}
\end{array}\right] \psi
\end{align*}
$$

or

$$
\begin{equation*}
-\hbar^{2} \frac{\partial^{2}}{\partial t^{2}} \boldsymbol{\psi}=\left(\hat{p}^{2} c^{2}+m^{2} c^{4}\right) \mathbf{1} \psi \tag{3.15}
\end{equation*}
$$

The operator squaring has decoupled the equation, leaving a pair of scalar relationships

$$
\begin{equation*}
-\hbar^{2} \frac{\partial^{2}}{\partial t^{2}} \psi_{1,2}=\left(\hat{p}^{2} c^{2}+m^{2} c^{4}\right) \psi_{1,2} \tag{3.16}
\end{equation*}
$$

We have recovered a Klein Gordon equation for each of the wave functions $\psi_{1}, \psi_{2}$.

### 3.2 Plane wave solution.

Instead of squaring the operators, lets try to solve the first order equation. To do so we'll want to diagonalize $\mathbf{H}$.

Before doing that, let's write out the Hamiltonian in an alternate but useful form

$$
\begin{align*}
\mathbf{H} & =\hat{p} c\left[\begin{array}{cc}
1 & 0 \\
0 & -1
\end{array}\right]+m c^{2}\left[\begin{array}{ll}
0 & 1 \\
1 & 0
\end{array}\right]  \tag{3.17}\\
& =\hat{p} c \hat{\sigma}_{z}+m c^{2} \hat{\sigma}_{x} .
\end{align*}
$$

We have two types of operators in the mix here. We have matrix operators that act on the wave function matrices, as well as derivative operators that act on the components of those matrices.

The action of the Pauli matrices are

$$
\hat{\sigma}_{z}\left[\begin{array}{l}
\psi_{1}  \tag{3.18}\\
\psi_{2}
\end{array}\right]=\left[\begin{array}{c}
\psi_{1} \\
-\psi_{2}
\end{array}\right]
$$

and

$$
\hat{\sigma}_{x}\left[\begin{array}{l}
\psi_{1}  \tag{3.19}\\
\psi_{2}
\end{array}\right]=\left[\begin{array}{l}
\psi_{2} \\
\psi_{1}
\end{array}\right]
$$

Because the derivative actions of $\hat{p}$ and the matrix operators are independent, we see that these operators commute. For example

$$
\begin{align*}
\hat{\sigma}_{z} \hat{p}\left[\begin{array}{l}
\psi_{1} \\
\psi_{2}
\end{array}\right] & =\hat{\sigma}_{z}\left[\begin{array}{c}
-i \hbar \frac{\partial \psi_{1}}{\partial x} \\
-i \hbar \frac{\partial \psi_{2}}{\partial x}
\end{array}\right] \\
& =\left[\begin{array}{c}
-i \hbar \frac{\partial \psi_{1}}{\partial x} \\
i \hbar \frac{\partial \psi_{2}}{\partial x}
\end{array}\right]  \tag{3.20}\\
& =\hat{p} \hat{\sigma}_{z}\left[\begin{array}{l}
\psi_{1} \\
\psi_{2}
\end{array}\right] .
\end{align*}
$$

Diagonalizing it Suppose the wave function matrix has the structure

$$
\psi=\left[\begin{array}{l}
f_{+}  \tag{3.21}\\
f_{-}
\end{array}\right] e^{i k x}
$$

We'll plug this into the Schrödinger equation and see what we get.

Where we left off

$$
-i \hbar \frac{\partial}{\partial t}\left[\begin{array}{l}
\psi_{1}  \tag{3.22}\\
\psi_{2}
\end{array}\right]=\left[\begin{array}{cc}
-i \hbar c \frac{\partial}{\partial x} & m c^{2} \\
m c^{2} & i \hbar c \frac{\partial}{\partial x}
\end{array}\right]
$$

With a potential this would be

$$
-i \hbar \frac{\partial}{\partial t}\left[\begin{array}{l}
\psi_{1}  \tag{3.23}\\
\psi_{2}
\end{array}\right]=\left[\begin{array}{cc}
-i \hbar c \frac{\partial}{\partial x}+V(x) & m c^{2} \\
m c^{2} & i \hbar c \frac{\partial}{\partial x}+V(x)
\end{array}\right]
$$

This means that the potential is raising the energy eigenvalue of the system.

Free Particle Assuming a form

$$
\left[\begin{array}{l}
\psi_{1}(x, t)  \tag{3.24}\\
\psi_{2}(x, t)
\end{array}\right]=e^{i k x}\left[\begin{array}{l}
f_{1}(t) \\
f_{2}(t)
\end{array}\right]
$$

and plugging back into the Dirac equation we have

$$
-i \hbar \frac{\partial}{\partial t}\left[\begin{array}{l}
f_{1}  \tag{3.25}\\
f_{2}
\end{array}\right]=\left[\begin{array}{cc}
k \hbar c & m c^{2} \\
m c^{2} & -\hbar k c
\end{array}\right]\left[\begin{array}{l}
f_{1} \\
f_{2}
\end{array}\right] .
$$

We can use a diagonalizing rotation

$$
\left[\begin{array}{l}
f_{1}  \tag{3.26}\\
f_{2}
\end{array}\right]=\left[\begin{array}{cc}
\cos \theta_{k} & -\sin \theta_{k} \\
\sin \theta_{k} & \cos \theta_{k}
\end{array}\right]\left[\begin{array}{l}
f_{+} \\
f_{-}
\end{array}\right]
$$

Plugging this in reduces the system to the form

$$
-i \hbar \frac{\partial}{\partial t}\left[\begin{array}{l}
f_{+}  \tag{3.27}\\
f_{-}
\end{array}\right]=\left[\begin{array}{cc}
E_{k} & 0 \\
0 & -E_{k}
\end{array}\right]\left[\begin{array}{l}
f_{+} \\
f_{-}
\end{array}\right] .
$$

Where the rotation angle is found to be given by

$$
\begin{align*}
\sin \left(2 \theta_{k}\right) & =\frac{m c^{2}}{\sqrt{(\hbar k c)^{2}+m^{2} c^{4}}} \\
\cos \left(2 \theta_{k}\right) & =\frac{\hbar k c}{\sqrt{(\hbar k c)^{2}+m^{2} c^{4}}}  \tag{3.28}\\
E_{k} & =\sqrt{(\hbar k c)^{2}+m^{2} c^{4}}
\end{align*}
$$

## 3.3 dirac sea and pair creation.

See fig. 3.1 for a sketch of energy vs momentum. The asymptotes are the limiting cases when $m c^{2} \rightarrow 0$. The + branch is what we usually associate with particles. What about the other energy states. For Fermions Dirac argued that the lower energy states could be thought of as "filled up", using the Pauli principle to leave only the positive energy states available. This was called the "Dirac Sea". This isn't a good solution, and won't work for example for Bosons.

Another way to rationalize this is to employ ideas from solid state theory. For example consider a semiconductor with a valence and conduction band as sketched in fig. 3.2.

A photon can excite an electron from the valence band to the conduction band, leaving all the valence band states filled except for one (a hole). For an electron we can use almost the same picture, as sketched in fig. 3.3.

A photon with energy $E_{k}-\left(-E_{k}\right)$ can create a positron-electron pair from the vacuum, where the energy of the electron and positron pair is $E_{k}$. At high enough energies, we can see this pair creation occur.


Figure 3.1: Dirac equation solution space.


Figure 3.2: Solid state valence and conduction band transition.


Figure 3.3: Pair creation.

### 3.4 ZITTERBEWEGUNG.

If a particle is created at a non-eigenstate such as one on the asymptotes, then oscillations between the positive and negative branches are possible as sketched in fig. 3.4. Only "vertical" oscillations between the positive


Figure 3.4: Zitterbewegung oscillation.
and negative locations on these branches is possible since those are the points that match the particle momentum. Examining this will be the aim of one of the problem set problems.

### 3.5 PROBABILITY AND CURRENT DENSITY.

If we define a probability density

$$
\begin{equation*}
\rho(x, t)=\left|\psi_{1}\right|^{2}+\left|\psi_{2}\right|^{2} \tag{3.29}
\end{equation*}
$$

does this satisfy a probability conservation relation

$$
\begin{equation*}
\frac{\partial \rho}{\partial t}+\frac{\partial j}{\partial x}=0 \tag{3.30}
\end{equation*}
$$

where $j$ is the probability current. Plugging in the density, we have

$$
\begin{equation*}
\frac{\partial \rho}{\partial t}=\frac{\partial \psi_{1}^{*}}{\partial t} \psi_{1}+\psi_{1}^{*} \frac{\partial \psi_{1}}{\partial t}+\frac{\partial \psi_{2}^{*}}{\partial t} \psi_{2}+\psi_{2}^{*} \frac{\partial \psi_{2}}{\partial t} \tag{3.31}
\end{equation*}
$$

It turns out that the probability current has the form

$$
\begin{equation*}
j(x, t)=c\left(\psi_{1}^{*} \psi_{1}-\psi_{2}^{*} \psi_{2}\right) \tag{3.32}
\end{equation*}
$$

Here the speed of light $c$ is the slope of the line in the plots above. We can think of this current density as right movers minus the left movers. Any state that is given can be thought of as a combination of right moving and left moving states, neither of which are eigenstates of the free particle Hamiltonian.

### 3.6 Potential step.

The next logical thing to think about, as in non-relativistic quantum mechanics, is to think about what occurs when the particle hits a potential step, as in fig. 3.5. The approach is the same. We write down the wave


Figure 3.5: Reflection off a potential barrier.
functions for the $V=0$ region (I), and the higher potential region (II).
The eigenstates are found on the solid lines above the asymptotes on the right hand movers side as sketched in fig. 3.6. The right and left moving designations are based on the phase velocity $\partial E / \partial k$ (approaching $\pm c$ on the top-right and top-left quadrants respectively). For $k>0$, an eigenstate


Figure 3.6: Right movers and left movers.
for the incident wave is

$$
\psi_{\mathrm{inc}}(x)=\left[\begin{array}{c}
\cos \theta_{k}  \tag{3.33}\\
\sin \theta_{k}
\end{array}\right] e^{i k x}
$$

For the reflected wave function, we pick a function on the left moving side of the positive energy branch.

$$
\psi_{\mathrm{ref}}(x)=\left[\begin{array}{l}
?  \tag{3.34}\\
?
\end{array}\right] e^{-i k x}
$$

We'll go through this in more detail next time.

## 3.7 dirac scattering off a potential step.

For the non-relativistic case we have

$$
\begin{align*}
& E<V_{0} \Longrightarrow T=0, R=1  \tag{3.35}\\
& E>V_{0} \Longrightarrow T>0, R<1
\end{align*}
$$

What happens for a relativistic 1D particle? Referring to fig. 3.7. the re-


Figure 3.7: Potential step.
gion I Hamiltonian is

$$
H=\left[\begin{array}{cc}
\hat{p} c & m c^{2}  \tag{3.36}\\
m c^{2} & -\hat{p} c
\end{array}\right],
$$

for which the solution is

$$
\Phi=e^{i k_{1} x}\left[\begin{array}{c}
\cos \theta_{1}  \tag{3.37}\\
\sin \theta_{1}
\end{array}\right],
$$

where

$$
\begin{align*}
\cos 2 \theta_{1} & =\frac{\hbar c k_{1}}{E_{k_{1}}} \\
\sin 2 \theta_{1} & =\frac{m c^{2}}{E_{k_{1}}} \tag{3.38}
\end{align*}
$$

To consider the $k_{1}<0$ case, note that

$$
\begin{align*}
\cos ^{2} \theta_{1}-\sin ^{2} \theta_{1} & =\cos 2 \theta_{1} \\
2 \sin \theta_{1} \cos \theta_{1} & =\sin 2 \theta_{1}, \tag{3.39}
\end{align*}
$$

so after flipping the signs on all the $k_{1}$ terms we find for the reflected wave

$$
\Phi=e^{-i k_{1} x}\left[\begin{array}{c}
\sin \theta_{1}  \tag{3.40}\\
\cos \theta_{1}
\end{array}\right]
$$

FIXME: this reasoning doesn't entirely make sense to me. Make sense of this by trying this solution as was done for the form of the incident wave solution.

The region I wave has the form

$$
\Phi_{I}=A e^{i k_{1} x}\left[\begin{array}{c}
\cos \theta_{1}  \tag{3.41}\\
\sin \theta_{1}
\end{array}\right]+B e^{-i k_{1} x}\left[\begin{array}{c}
\sin \theta_{1} \\
\cos \theta_{1}
\end{array}\right] .
$$

By the time we are done we want to have computed the reflection coefficient

$$
\begin{equation*}
R=\frac{|B|^{2}}{|A|^{2}} \tag{3.42}
\end{equation*}
$$

The region I energy is

$$
\begin{equation*}
E=\sqrt{\left(m c^{2}\right)^{2}+\left(\hbar c k_{1}\right)^{2}} \tag{3.43}
\end{equation*}
$$

We must have

$$
\begin{equation*}
E=\sqrt{\left(m c^{2}\right)^{2}+\left(\hbar c k_{2}\right)^{2}}+V_{0}=\sqrt{\left(m c^{2}\right)^{2}+\left(\hbar c k_{1}\right)^{2}} \tag{3.44}
\end{equation*}
$$

SO

$$
\begin{align*}
\left(\hbar c k_{2}\right)^{2} & =\left(E-V_{0}\right)^{2}-\left(m c^{2}\right)^{2} \\
& =\underbrace{\left(E-V_{0}+m c^{2}\right)}_{r_{1}} \underbrace{\left(E-V_{0}-m c^{2}\right)}_{r_{2}} . \tag{3.45}
\end{align*}
$$

The $r_{1}$ and $r_{2}$ branches are sketched in fig. 3.8.
For low energies, we have a set of potentials for which we will have propagation, despite having a potential barrier. For still higher values of the potential barrier the product $r_{1} r_{2}$ will be negative, so the solutions will be decaying. Finally, for even higher energies, there will again be propagation.

The non-relativistic case is sketched in fig. 3.9. For the relativistic case we must consider three different cases, sketched in fig. 3.10, fig. 3.11, and fig. 3.12 respectively. For the low potential energy, a particle with positive group velocity (what we've called right moving) can be matched to an equal energy portion of the potential shifted parabola in region II.


Figure 3.8: Energy signs.


Figure 3.9: Effects of increasing potential for non-relativistic case.

This is a case where we have transmission, but no antiparticle creation. There will be an energy region where the region II wave function has only a dissipative term, since there is no region of either of the region II parabolic branches available at the incident energy. When the potential is shifted still higher so that $V_{0}>E+m c^{2}$, a positive group velocity in region I with a given energy can be matched to an antiparticle branch in the region II parabolic energy curve.


Figure 3.10: Low potential energy.

Boundary value conditions We want to ensure that the current across the barrier is conserved (no particles are lost), as sketched in fig. 3.13. Recall that given a wave function

$$
\Psi=\left[\begin{array}{l}
\psi_{1}  \tag{3.46}\\
\psi_{2}
\end{array}\right]
$$



Figure 3.11: High enough potential energy for no propagation.


Figure 3.12: High potential energy.
the density and currents are respectively

$$
\begin{align*}
\rho & =\psi_{1}^{*} \psi_{1}+\psi_{2}^{*} \psi_{2}  \tag{3.47}\\
j & =\psi_{1}^{*} \psi_{1}-\psi_{2}^{*} \psi_{2}
\end{align*}
$$

Matching boundary value conditions requires

1. For both the relativistic and non-relativistic cases we must have

$$
\begin{equation*}
\Psi_{\mathrm{L}}=\Psi_{\mathrm{R}}, \quad \text { at } x=0 \tag{3.48}
\end{equation*}
$$

2. For the non-relativistic case we want

$$
\begin{align*}
& \int_{-\epsilon}^{\epsilon}-\frac{\hbar^{2}}{2 m} \frac{\partial^{2} \Psi}{\partial x^{2}}=\int_{-\epsilon}^{\epsilon}(E-V(x)) \Psi(x)  \tag{3.49}\\
& -\frac{\hbar^{2}}{2 m}\left(\left.\frac{\partial \Psi}{\partial x}\right|_{\mathrm{R}}-\left.\frac{\partial \Psi}{\partial x}\right|_{\mathrm{L}}\right)=0 \tag{3.50}
\end{align*}
$$

For the relativistic case

$$
\begin{equation*}
-i \hbar \sigma_{z} \int_{-\epsilon}^{\epsilon} \frac{\partial \Psi}{\partial x}+m c^{2} \sigma_{x} \int_{-\epsilon}^{\epsilon} \psi=\int_{-\epsilon}^{\epsilon}(E-\sqrt{0}) \psi \tag{3.51}
\end{equation*}
$$

$$
\begin{equation*}
-i \hbar c \sigma_{z}(\psi(\epsilon)-\psi(-\epsilon))=-i \hbar c \sigma_{z}\left(\psi_{\mathrm{R}}-\psi_{\mathrm{L}}\right) \tag{3.52}
\end{equation*}
$$



Figure 3.13: Transmitted, reflected and incident components.
so we must match

$$
\begin{equation*}
\sigma_{z} \psi_{\mathrm{R}}=\sigma_{z} \psi_{\mathrm{L}} \tag{3.53}
\end{equation*}
$$

It appears that things are simpler, because we only have to match the wave function values at the boundary, and don't have to match the derivatives too. However, we have a two component wave function, so there are still two tasks.

Solving the system Let's look for a solution for the $E+m c^{2}>V_{0}$ case on the right branch, as sketched in fig. 3.14. While the right branch in this


Figure 3.14: High potential region. Anti-particle transmission.
case is left going, this might work out since that is an antiparticle. We could try both. First

$$
\Psi_{I I}=D e^{i k_{2} x}\left[\begin{array}{c}
-\sin \theta_{2}  \tag{3.54}\\
\cos \theta_{2}
\end{array}\right]
$$

This is justified by

$$
+E \rightarrow\left[\begin{array}{c}
\cos \theta  \tag{3.55}\\
\sin \theta
\end{array}\right]
$$

so

$$
-E \rightarrow\left[\begin{array}{c}
-\sin \theta  \tag{3.56}\\
\cos \theta
\end{array}\right]
$$

At $x=0$ the exponentials vanish, so equating the waves at that point means

$$
\left[\begin{array}{c}
\cos \theta_{1}  \tag{3.57}\\
\sin \theta_{1}
\end{array}\right]+\frac{B}{A}\left[\begin{array}{c}
\sin \theta_{1} \\
\cos \theta_{1}
\end{array}\right]=\frac{D}{A}\left[\begin{array}{c}
-\sin \theta_{2} \\
\cos \theta_{2}
\end{array}\right] .
$$

Solving this yields

$$
\begin{equation*}
\frac{B}{A}=-\frac{\cos \left(\theta_{1}-\theta_{2}\right)}{\sin \left(\theta_{1}+\theta_{2}\right)} \tag{3.58}
\end{equation*}
$$

This yields

$$
\begin{equation*}
R=\frac{1+\cos \left(2 \theta_{1}-2 \theta_{2}\right)}{1-\cos \left(2 \theta_{1}-2 \theta_{2}\right)} \tag{3.59}
\end{equation*}
$$

As $V_{0} \rightarrow \infty$ this simplifies to

$$
\begin{equation*}
R=\frac{E-\sqrt{E^{2}-\left(m c^{2}\right)^{2}}}{E+\sqrt{E^{2}-\left(m c^{2}\right)^{2}}} \tag{3.60}
\end{equation*}
$$

Filling in the details for these results is left to problem set 4.

### 3.8 PROBLEMS.

Exercise 3.1 Calculate the right going diagonalization.
a. Prove eq. (3.28).

Answer for Exercise 3.1

Part a. To determine the relations for $\theta_{k}$ we have to solve

$$
\left[\begin{array}{cc}
E_{k} & 0  \tag{3.61}\\
0 & -E_{k}
\end{array}\right]=R^{-1} H R
$$

Working with $\hbar=c=1$ temporarily, and $C=\cos \theta_{k}, S=\sin \theta_{k}$, that is

$$
\begin{align*}
{\left[\begin{array}{cc}
E_{k} & 0 \\
0 & -E_{k}
\end{array}\right] } & =\left[\begin{array}{cc}
C & S \\
-S & C
\end{array}\right]\left[\begin{array}{cc}
k & m \\
m & -k
\end{array}\right]\left[\begin{array}{cc}
C & -S \\
S & C
\end{array}\right] \\
& =\left[\begin{array}{cc}
C & S \\
-S & C
\end{array}\right]\left[\begin{array}{cc}
k C+m S & -k S+m C \\
m C-k S & -m S-k C
\end{array}\right] \\
& =\left[\begin{array}{cc}
k C^{2}+m S C+m C S-k S^{2} & -k S C+m C^{2}-m S^{2}-k C S \\
-k C S-m S^{2}+m C^{2}-k S C & k S^{2}-m C S-m S C-k C^{2}
\end{array}\right] \\
& =\left[\begin{array}{ll}
k \cos \left(2 \theta_{k}\right)+m \sin \left(2 \theta_{k}\right) & m \cos \left(2 \theta_{k}\right)-k \sin \left(2 \theta_{k}\right) \\
m \cos \left(2 \theta_{k}\right)-k \sin \left(2 \theta_{k}\right) & -k \cos \left(2 \theta_{k}\right)-m \sin \left(2 \theta_{k}\right)
\end{array}\right] . \tag{3.62}
\end{align*}
$$

This gives

$$
\begin{align*}
E_{k}\left[\begin{array}{l}
1 \\
0
\end{array}\right] & =\left[\begin{array}{l}
k \cos \left(2 \theta_{k}\right)+m \sin \left(2 \theta_{k}\right) \\
m \cos \left(2 \theta_{k}\right)-k \sin \left(2 \theta_{k}\right)
\end{array}\right]  \tag{3.63}\\
& =\left[\begin{array}{cc}
k & m \\
m & -k
\end{array}\right]\left[\begin{array}{l}
\cos \left(2 \theta_{k}\right) \\
\sin \left(2 \theta_{k}\right)
\end{array}\right] .
\end{align*}
$$

Adding back in the $\hbar$ 's and $c$ 's this is

$$
\begin{align*}
{\left[\begin{array}{c}
\cos \left(2 \theta_{k}\right) \\
\sin \left(2 \theta_{k}\right)
\end{array}\right] } & =\frac{E_{k}}{-(\hbar k c)^{2}-\left(m c^{2}\right)^{2}}\left[\begin{array}{cc}
-\hbar k c & -m c^{2} \\
-m c^{2} & \hbar k c
\end{array}\right]\left[\begin{array}{l}
1 \\
0
\end{array}\right] \\
& =\frac{1}{E_{k}}\left[\begin{array}{c}
\hbar k c \\
m c^{2}
\end{array}\right] . \tag{3.64}
\end{align*}
$$

Exercise $3.2 \quad$ Verify the plane wave eigenstate.
a. Verify eq. (3.33).
b. Find the form of the reflected wave.

Answer for Exercise 3.2

Parta. With

$$
\begin{align*}
H_{k} & =\left[\begin{array}{cc}
\hbar k c & m c^{2} \\
m c^{2} & -\hbar k c
\end{array}\right] \\
& =R\left[\begin{array}{cc}
E_{k} & 0 \\
0 & -E_{k}
\end{array}\right] R^{-1} \tag{3.65}
\end{align*}
$$

, We wish to show that

$$
H_{k}\left[\begin{array}{c}
\cos \theta_{k}  \tag{3.66}\\
\sin \theta_{k}
\end{array}\right] e^{i k x}=E_{k}\left[\begin{array}{c}
\cos \theta_{k} \\
\sin \theta_{k}
\end{array}\right] e^{i k x} .
$$

The LHS side expands to

$$
\begin{align*}
{\left[\begin{array}{cc}
C & -S \\
S & C
\end{array}\right]\left[\begin{array}{cc}
E_{k} & 0 \\
0 & -E_{k}
\end{array}\right]\left[\begin{array}{cc}
C & S \\
-S & C
\end{array}\right]\left[\begin{array}{c}
C \\
S
\end{array}\right] e^{i k x} } & =\left[\begin{array}{cc}
C & -S \\
S & C
\end{array}\right]\left[\begin{array}{cc}
E_{k} & 0 \\
0 & -E_{k}
\end{array}\right]\left[\begin{array}{c}
C^{2}+S^{2} \\
0
\end{array}\right] e^{i k x} \\
& =\left[\begin{array}{cc}
C & -S \\
S & C
\end{array}\right]\left[\begin{array}{c}
E_{k} \\
0
\end{array}\right] e^{i k x} \\
& =E_{k}\left[\begin{array}{l}
C \\
S
\end{array}\right] e^{i k x} . \tag{3.67}
\end{align*}
$$

Part b. For the reflected wave, let's assume that the reflected wave has the form

$$
\Psi=\left[\begin{array}{c}
\sin \theta_{k}  \tag{3.68}\\
-\cos \theta_{k}
\end{array}\right] e^{-i k x} .
$$

Let's verify this

$$
\begin{align*}
& {\left[\begin{array}{cc}
C & -S \\
S & C
\end{array}\right]\left[\begin{array}{cc}
E_{k} & 0 \\
0 & -E_{k}
\end{array}\right]\left[\begin{array}{cc}
C & S \\
-S & C
\end{array}\right]\left[\begin{array}{c}
S \\
-C
\end{array}\right] e^{-i k x} } \\
&=\left[\begin{array}{cc}
C & -S \\
S & C
\end{array}\right]\left[\begin{array}{cc}
E_{k} & 0 \\
0 & -E_{k}
\end{array}\right]\left[\begin{array}{c}
C S-S C \\
-S^{2}-C^{2}
\end{array}\right] e^{-i k x} \\
&=\left[\begin{array}{cc}
C & -S \\
S & C
\end{array}\right]\left[\begin{array}{c}
0 \\
E_{k}
\end{array}\right] e^{-i k x}  \tag{3.69}\\
&=E_{k}\left[\begin{array}{c}
-S \\
C
\end{array}\right] e^{-i k x} \\
&=-E_{k}\left[\begin{array}{c}
S \\
-C
\end{array}\right] e^{-i k x}
\end{align*}
$$

However, note that we have a different rotation angle $\theta_{k}$ for the forward going and reverse going waves.

For the incident wave, $k>0$, we have

$$
\begin{equation*}
\tan 2 \theta_{k}=\frac{\hbar k c}{m c^{2}} \tag{3.70}
\end{equation*}
$$

and for the reflected wave, $k<0$, we have

$$
\begin{equation*}
\tan 2 \theta_{k}=\frac{-\hbar k c}{m c^{2}} \tag{3.71}
\end{equation*}
$$

The rotation angle for both cases can therefore be expressed as

$$
\begin{equation*}
\theta_{k}=\frac{1}{2} \operatorname{atan}\left(\left|\frac{\hbar k}{m c}\right|\right) \tag{3.72}
\end{equation*}
$$

Exercise $3.3 \quad$ Verify the Dirac current relationship.
Prove eq. (3.32).
Answer for Exercise 3.3
The components of the Schrödinger equation are

$$
\begin{align*}
& -i \hbar \frac{\partial \psi_{1}}{\partial t}=-i \hbar c \frac{\partial \psi_{1}}{\partial x}+m c^{2} \psi_{2}  \tag{3.73}\\
& -i \hbar \frac{\partial \psi_{2}}{\partial t}=m c^{2} \psi_{1}+i \hbar c \frac{\partial \psi_{2}}{\partial x}
\end{align*}
$$

The conjugates of these are

$$
\begin{align*}
& i \hbar \frac{\partial \psi_{1}^{*}}{\partial t}=i \hbar c \frac{\partial \psi_{1}^{*}}{\partial x}+m c^{2} \psi_{2}^{*}  \tag{3.74}\\
& i \hbar \frac{\partial \psi_{2}^{*}}{\partial t}=m c^{2} \psi_{1}^{*}-i \hbar c \frac{\partial \psi_{2}^{*}}{\partial x}
\end{align*}
$$

This gives

$$
\begin{align*}
i \hbar \frac{\partial \rho}{\partial t} & =\left(i \hbar c \frac{\partial \psi_{1}^{*}}{\partial x}+m c^{2} \psi_{2}^{*}\right) \psi_{1} \\
& +\psi_{1}^{*}\left(i \hbar c \frac{\partial \psi_{1}}{\partial x}-m c^{2} \psi_{2}\right) \\
& +\left(m c^{2} \psi_{1}^{*}-i \hbar c \frac{\partial \psi_{2}^{*}}{\partial x}\right) \psi_{2}  \tag{3.75}\\
& +\psi_{2}^{*}\left(-m c^{2} \psi_{1}-i \hbar c \frac{\partial \psi_{2}}{\partial x}\right)
\end{align*}
$$

All the non-derivative terms cancel leaving

$$
\begin{align*}
\frac{1}{c} \frac{\partial \rho}{\partial t} & =\frac{\partial \psi_{1}^{*}}{\partial x} \psi_{1}+\psi_{1}^{*} \frac{\partial \psi_{1}}{\partial x}-\frac{\partial \psi_{2}^{*}}{\partial x} \psi_{2}-\psi_{2}^{*} \frac{\partial \psi_{2}}{\partial x}  \tag{3.76}\\
& =\frac{\partial}{\partial x}\left(\psi_{1}^{*} \psi_{1}-\psi_{2}^{*} \psi_{2}\right)
\end{align*}
$$

## Exercise $3.4 \quad$ Zitterbewegung in one dimension. (2015 ps4 pl)

Consider the Dirac Hamiltonian $H=c \hat{p} \sigma_{z}+m c^{2} \sigma_{x}$. Using the Heisenberg equations of motion, derive a second order equation of motion (eom) for the velocity operator $\hat{v}=d \hat{x} / d t$. For a state

$$
\Psi=\left[\begin{array}{l}
1  \tag{3.77}\\
0
\end{array}\right] e^{i k x}
$$

with $k=0$, average this eom in state $\Psi$ to get a homogeneous second order differential equation for $\langle\hat{v}\rangle$. Using this equation of motion and the initial conditions on the velocity and its time-derivative, obtain $\langle\hat{v}\rangle(t)$ and $\langle\hat{x}(t)\rangle$. Show that these oscillate with a rapid frequency $2 m c^{2} / h$, with the oscillation of the position having an amplitude $\hbar / m c$ which is the Compton wavelength. This 'trembling' motion is called zitterbewegung.

## Answer for Exercise 3.4

Using the hint from class that zitterbewegung is associated with oscillation of the particle between ordinary-particle and anti-particle states, let's look for a combination of such $k>0$ states to represent the state of this problem

$$
\left[\begin{array}{l}
1  \tag{3.78}\\
0
\end{array}\right] e^{i k x}=\left.\left(a\left[\begin{array}{c}
\cos \theta \\
\sin \theta
\end{array}\right] e^{i k x-i E t / \hbar}+b\left[\begin{array}{c}
-\sin \theta \\
\cos \theta
\end{array}\right] e^{i k x+i E t / \hbar}\right)\right|_{t=0}
$$

Observe that $\left[\begin{array}{c}-\sin \theta \\ \cos \theta\end{array}\right]$ and $\left[\begin{array}{c}\cos \theta \\ \sin \theta\end{array}\right]$ are orthogonal, so

$$
\begin{align*}
& a=\left[\begin{array}{ll}
1 & 0
\end{array}\right]\left[\begin{array}{c}
\cos \theta \\
\sin \theta
\end{array}\right]=\cos \theta .  \tag{3.79}\\
& b=\left[\begin{array}{ll}
1 & 0
\end{array}\right]\left[\begin{array}{c}
-\sin \theta \\
\cos \theta
\end{array}\right]=-\sin \theta . \tag{3.80}
\end{align*}
$$

With

$$
\begin{align*}
& |a\rangle=\left[\begin{array}{c}
\cos \theta \\
\sin \theta
\end{array}\right] e^{i k x-i E t / \hbar}  \tag{3.81}\\
& |b\rangle=\left[\begin{array}{c}
-\sin \theta \\
\cos \theta
\end{array}\right] e^{i k x+i E t / \hbar}
\end{align*}
$$

The wave function of eq. (3.77) can be represented as

$$
\begin{equation*}
\Psi=\cos \theta|a\rangle-\sin \theta|b\rangle . \tag{3.82}
\end{equation*}
$$

The action of the Hamiltonian on this wave function is

$$
\begin{align*}
H \psi & =i \hbar\left(-i \frac{E}{\hbar} C|a\rangle-i \frac{E}{\hbar} S|b\rangle\right)  \tag{3.83}\\
& =E(C|a\rangle+S|b\rangle),
\end{align*}
$$

Writing $\epsilon=\sqrt{\left(m c^{2}\right)^{2}+(\hbar k c)^{2}}$, and noting that $|a\rangle$, and $|b\rangle$ are positive and negative eigenstates respectively, the spatial action of the Hamiltonian on this wave function is
$H \Psi$

$$
\begin{align*}
& =(+\epsilon) C|a\rangle-(-\epsilon) S|b\rangle  \tag{3.84}\\
& =\epsilon(C|a\rangle+S|b\rangle) .
\end{align*}
$$

This provides a relationship between the energy $E$ and the eigenvalues

$$
\begin{equation*}
E=\epsilon=\sqrt{\left(m c^{2}\right)^{2}+(\hbar k c)^{2}} \tag{3.85}
\end{equation*}
$$

In particular, when $k=0$, this is $E=m c^{2}$. Using the Heisenberg equations of motion, the velocity operator is

$$
\begin{align*}
\hat{v} I & =\frac{d \hat{x}}{d t} I \\
& =\frac{1}{i \hbar}\left[\hat{x} I, \hat{p} c \sigma_{z}+m c^{2} \sigma_{x}\right] \\
& =\frac{1}{i \hbar}\left(\hat{x} \hat{p} c \sigma_{z}-\hat{p} c \sigma_{z} \hat{x}+\hat{x} m c^{2} \sigma_{x}-m c^{2} \sigma_{x} \hat{x}\right)  \tag{3.86}\\
& =\frac{1}{i \hbar} c \sigma_{z}[\hat{x}, \hat{p}] \\
& =c \sigma_{z}
\end{align*}
$$

As we've seen with the Harmonic oscillator, an expectation with respect to a state that is not a single eigenstate, will have non-zero time dependence. With $C=\cos \theta, S=\sin \theta$ that expectation value with respect to the state as expressed in eq. (3.82) is

$$
\begin{align*}
&\langle\hat{v}\rangle=\langle\Psi| \hat{v}|\Psi\rangle \\
&=c(C\langle a|-S\langle b|) \sigma_{z}(C|a\rangle-S|b\rangle) \\
&=c(C\langle a|-S\langle b|)\left[\begin{array}{cc}
1 & 0 \\
0 & -1
\end{array}\right]\left(\left[\begin{array}{l}
C \\
S
\end{array}\right] e^{i k x-i E t / \hbar}-S\left[\begin{array}{c}
-S \\
C
\end{array}\right] e^{i k x+i E t / \hbar}\right) \\
&=c\left(C\left[\begin{array}{ll}
C & S
\end{array}\right] e^{-i k x+i E t / \hbar}-S\left[\begin{array}{ll}
-S & C
\end{array}\right] e^{-i k x-i E t / \hbar}\right) \times \\
&\left(C\left[\begin{array}{c}
C \\
-S
\end{array}\right] e^{i k x-i E t / \hbar}+S\left[\begin{array}{l}
S \\
C
\end{array}\right] e^{i k x+i E t / \hbar}\right) \\
&=c\left(C^{2} \cos (2 \theta)-S^{2} \cos (2 \theta)+S C \sin (2 \theta) e^{-2 i E t / \hbar}+S C \sin (2 \theta) e^{2 i E t / \hbar}\right) \\
&=c\left(\cos ^{2}(2 \theta)+\sin ^{2}(2 \theta) \cos (2 E t / \hbar)\right) \\
&\left.=\frac{(\hbar c k)^{2}}{\left(m c^{2}\right)^{2}+(\hbar k c)^{2}}+\frac{\left(m c^{2}\right)^{2}}{\left(m c^{2}\right)^{2}+(\hbar k c)^{2}} \cos \left(2 \pi \frac{2 E}{h} t\right)\right) \\
&\left(m c^{2}\right)^{2}+(\hbar k c)^{2} \tag{3.87}
\end{align*}\left((\hbar c k)^{2}+\left(m c^{2}\right)^{2} \cos \left(2 \pi \frac{2 E}{h} t\right)\right) .
$$

For $k=0$, this is

$$
\begin{equation*}
\langle\hat{v}\rangle=c \cos \left(2 \pi \frac{2 m c^{2}}{h} t\right) \tag{3.88}
\end{equation*}
$$

The frequency of this oscillation is $2 m c^{2} / h$ as the problem states. For the position expectation with respect to this state, we have

$$
\begin{align*}
\langle\hat{x}\rangle & =\langle\hat{x}\rangle(0)+\frac{c \hbar}{2 m c^{2}} \sin \left(\frac{2 m c^{2}}{\hbar} t\right)  \tag{3.89}\\
& =\langle\hat{x}\rangle(0)+\frac{\hbar}{2 m c} \sin \left(\frac{2 m c^{2}}{\hbar} t\right) .
\end{align*}
$$

The amplitude of this oscillation is

$$
\begin{equation*}
2 \times \frac{\hbar}{2 m c}=\frac{\hbar}{m c}, \tag{3.90}
\end{equation*}
$$

as expected.

## Exercise 3.5 Jackiw-Rebbi problem. (2015 ps4 p2)

Recall that the energy of a relativistic particle is $E(p)=\sqrt{p^{2} c^{2}+m^{2} c^{4}}$, which is independent of the sign of $m$. Thus $m>0$ and $m<0$ lead to the same dispersion relation. Set aside for now, the physical meaning of $m<0$. Assume $V(x)=0$ but let us assume the mass $m$ is a function of position $m(x)$. This leads to

$$
H=\left[\begin{array}{cc}
c \hat{p} & m(x) c^{2}  \tag{3.91}\\
m(x) c^{2} & -c \hat{p}
\end{array}\right] .
$$

Let $m(x)$ be such that $m(-x)=-m(x)$, i.e., an odd function of position which changes sign at $x=0$.
a. Show that the operator $\mathscr{P} \widehat{\mathcal{P}}_{\text {Dirac }}=\sigma_{y} \mathscr{P} \widehat{\mathscr{P}}$ commutes with the Hamiltonian, where

$$
\sigma_{y}=\left[\begin{array}{cc}
0 & -i  \tag{3.92}\\
i & 0
\end{array}\right],
$$

is the y-Pauli matrix, and $\mathscr{P} \widehat{\mathscr{P}}$ is the parity operator which sends $x \rightarrow-x$.
b. Consider the wavefunction

$$
\Phi(x)=\left[\begin{array}{c}
f(x)  \tag{3.93}\\
-i f(x)
\end{array}\right]
$$

where $f(-x)=f(x)$ is an even function. Show $\Phi(x)$ is an eigenstate of $\mathscr{P} \widehat{\mathscr{P}}_{\text {Dirac }}$ with eigenvalue -1 .
c. Next, assuming $m(x>0)=m_{0}$ and $m(x<0)=-m_{0}$, where $m_{0}>0$, find $f(x)$ such that $\Phi(x)$ is an eigenstate of H with zero energy.
d. Normalize the wavefunction $\Phi(x)$.

## Answer for Exercise 3.5

Part $a$. Let $\pi$ represent a parity operator that acts on a scalar (operator), so the Dirac parity operator $\mathscr{P} \widehat{\mathscr{P}}_{\text {Dirac }}$ takes the form

$$
\mathscr{P} \widehat{\mathscr{D}}_{\text {Dirac }}=\left[\begin{array}{cc}
0 & -i \pi  \tag{3.94}\\
i \pi & 0
\end{array}\right] .
$$

The commutator with the Hamiltonian is

$$
\begin{align*}
{\left[H, \mathscr{P} \widehat{\mathscr{P}}_{\text {Dirac }}\right] } & =H \mathscr{P} \widehat{\mathscr{P}}_{\text {Dirac }}-\mathscr{P} \widehat{\mathscr{P}}_{\text {Dirac }} H \\
& =\left[\begin{array}{cc}
c \hat{p} & m(x) c^{2} \\
m(x) c^{2} & -c \hat{p}
\end{array}\right]\left[\begin{array}{cc}
0 & -i \pi \\
i \pi & 0
\end{array}\right]-\left[\begin{array}{cc}
0 & -i \pi \\
i \pi & 0
\end{array}\right]\left[\begin{array}{cc}
c \hat{p} & m(x) c^{2} \\
m(x) c^{2} & -c \hat{p}
\end{array}\right] \\
& =\left[\begin{array}{cc}
i c^{2} m(x) \pi & -i c \hat{p} \pi \\
-i c \hat{p} \pi & -i c^{2} m(x) \pi
\end{array}\right]-\left[\begin{array}{cc}
-i c^{2} \pi m(x) & i c \pi \hat{p} \\
i c \pi \hat{p} & i c^{2} \pi m(x)
\end{array}\right] \\
& =\left[\begin{array}{cc}
i c^{2}\{m(x), \pi\} & -i c\{\hat{p}, \pi\} \\
-i c\{\hat{p}, \pi\} & -i c^{2}\{m(x), \pi\}
\end{array}\right] . \tag{3.95}
\end{align*}
$$

Now consider the matrix element of this commutator with respect to a position basis wavefunction $\langle x|\left[H, \mathscr{P} \widehat{\mathscr{P}}_{\text {Dirac }}\right]|\Psi\rangle$. To compute this we must
first understand the behavior of the anticommutators of scalar operators $m, \hat{p}$ with $\pi$. That is

$$
\begin{align*}
\langle x|\{m(x), \pi\}|\psi\rangle & =\{m(x), \pi\} \psi(x) \\
& =m(x) \pi \psi(x)+\pi(m(x) \psi(x)) \\
& =m(x) \psi(-x)+m(-x) \psi(-x)  \tag{3.96}\\
& =m(x) \psi(-x)-m(x) \psi(-x) \\
& =0,
\end{align*}
$$

and

$$
\begin{align*}
\langle x|\{\hat{p}, \pi\}|\psi\rangle & =\left\{-i \hbar \frac{\partial}{\partial x}, \pi\right\} \psi(x) \\
& =-i \hbar \frac{\partial}{\partial x} \pi \psi(x)+\pi\left(-i \hbar \frac{\partial}{\partial x} \psi(x)\right) \\
& =-i \hbar \frac{\partial}{\partial x} \psi(-x)+(-i \hbar) \frac{\partial}{\partial(-x)} \psi(-x)  \tag{3.97}\\
& =i \hbar \frac{\partial}{\partial(-x)} \psi(-x)-i \hbar \frac{\partial}{\partial(-x)} \psi(-x) \\
& =0 .
\end{align*}
$$

This shows that $\langle x|\left[H, \mathscr{P} \widehat{\mathscr{P}}_{\text {Dirac }}\right]|\Psi\rangle=0$. Since this is true for any $|\Psi\rangle$, we've shown that the Hamiltonian commutes with the Dirac parity operator

$$
\begin{equation*}
H \mathscr{P} \widehat{\mathscr{D}}_{\text {Dirac }}=\mathscr{P} \widehat{\mathscr{D}}_{\text {Dirac }} H \tag{3.98}
\end{equation*}
$$

Part b. This follows with direct substitution

$$
\begin{align*}
\mathscr{P} \widehat{\mathscr{P}}_{\text {Dirac }} \Phi(x) & =\left[\begin{array}{cc}
0 & -i \pi \\
i \pi & 0
\end{array}\right]\left[\begin{array}{c}
f(x) \\
-i f(x)
\end{array}\right] \\
& =\left[\begin{array}{c}
-\pi f(x) \\
i \pi f(x)
\end{array}\right] \\
& =\left[\begin{array}{c}
-f(-x) \\
i f(-x)
\end{array}\right]  \tag{3.99}\\
& =-\left[\begin{array}{c}
f(x) \\
-i f(x)
\end{array}\right]
\end{align*}
$$

Part c. The eigenvalue equation has the form

$$
\begin{align*}
\lambda \Phi & =H \Phi \\
& =\left[\begin{array}{cc}
c \hat{p} & m(x) c^{2} \\
m(x) c^{2} & -c \hat{p}
\end{array}\right]\left[\begin{array}{c}
f \\
-i f
\end{array}\right]  \tag{3.100}\\
& =\left[\begin{array}{l}
c \hat{p} f-i m(x) c^{2} f \\
m(x) c^{2} f+i c \hat{p} f
\end{array}\right],
\end{align*}
$$

or

$$
\begin{align*}
& c(-i \hbar) \frac{\partial f}{\partial x}-i m(x) c^{2} f=\lambda f \\
& i c(-i \hbar) \frac{\partial f}{\partial x}+m(x) c^{2} f=-i \lambda f . \tag{3.101}
\end{align*}
$$

This special selection of $\Phi$ leads to two identical equations, which should simplify things nicely. For the zero energy eigenstate, we must solve

$$
\begin{equation*}
\frac{\partial f}{\partial x}=-\frac{1}{c \hbar} m(x) c^{2} f \tag{3.102}
\end{equation*}
$$

which integrates to

$$
\begin{equation*}
\ln f=\ln f(0)-\int_{0}^{x} \frac{1}{c \hbar} m(x) c^{2} d x \tag{3.103}
\end{equation*}
$$

or

$$
\begin{equation*}
f(x)=f(0) \exp \left(-\int_{0}^{x} \frac{c}{\hbar} m(x) d x\right) \tag{3.104}
\end{equation*}
$$

For $x>0$ this is

$$
\begin{align*}
f(x) & =f(0) \exp \left(-\int_{0}^{x} \frac{c}{\hbar} m_{0} d x\right)  \tag{3.105}\\
& =f(0) \exp \left(-\frac{m_{0} c x}{\hbar}\right),
\end{align*}
$$

and for $x<0$ this is

$$
\begin{align*}
f(x) & =f(0) \exp \left(\int_{0}^{x} \frac{c}{\hbar} m_{0} d x\right)  \tag{3.106}\\
& =f(0) \exp \left(\frac{m_{0} c x}{\hbar}\right) .
\end{align*}
$$

These can be combined as

$$
\begin{equation*}
f(x)=f(0) \exp \left(-\frac{m_{0} c|x|}{\hbar}\right) \tag{3.107}
\end{equation*}
$$

Part d. The normalization is given by

$$
\begin{align*}
1 & =\int_{-\infty}^{\infty} \Phi^{\dagger} \Phi d x \\
& =\int_{-\infty}^{\infty} 2|f|^{2} d x \\
& =2 f^{2}(0) \int_{-\infty}^{\infty} e^{-2 m_{0} c|x| / \hbar} d x \\
& =4 f^{2}(0) \int_{0}^{\infty} e^{-2 m_{0} c x / \hbar} d x  \tag{3.108}\\
& =\left.4 f^{2}(0) \frac{e^{-2 m_{0} c x / \hbar}}{-2 m_{0} c / \hbar}\right|_{0} ^{\infty} \\
& =\frac{2 \hbar}{m_{0} c} f^{2}(0),
\end{align*}
$$

or

$$
\begin{equation*}
f(0)=\sqrt{\frac{m_{0} c}{2 \hbar}} . \tag{3.109}
\end{equation*}
$$

The fully normalized wave function is therefore

$$
\Phi(x)=\sqrt{\frac{m_{0} c}{2 \hbar}} e^{-m_{0} c|x| / \hbar}\left[\begin{array}{c}
1  \tag{3.110}\\
-i
\end{array}\right]
$$

Exercise $3.6 \quad$ Scattering off a potential step. (2015 ps4 p3)
Consider the 1D Dirac Hamiltonian as

$$
H=\left[\begin{array}{cc}
c \hat{p}+V(x) & m c^{2}  \tag{3.111}\\
m c^{2} & -c \hat{p}+V(x)
\end{array}\right],
$$

where the operator $\hat{p}=-i \hbar \partial / \partial x$ is the rest mass, and $c$ is the speed of light, and with 2 -component wavefunctions

$$
\Psi(x, t) \equiv\left[\begin{array}{l}
\psi_{1}(x, t)  \tag{3.112}\\
\psi_{2}(x, t)
\end{array}\right],
$$

such that $i \hbar \partial \Psi(x, t) / \partial t=H \Psi(x, t)$. Assuming a potential step, where $V(x<0)=0$ and $V(x>0)=V_{0}$, with $V_{0}>0$ as in class, complete the details of the scattering onto the step which was done in class. Discuss
the incident current, reflected current, and transmitted current for the case where the incident energy is such that $E>0$ and $V_{0}>2 m c^{2}$, and $E>0$ and $V_{0}<2 m c^{2}$. Draw pictures to illustrate the parabola and the location in momentum of the incident and reflected particles.
Answer for Exercise 3.6

Mostly background. We talked about diagonalization of the Dirac Hamiltonian by introducing a rotation, and then figuring out the rotation angle required.

To understand the form of the eigenkets for particles and antiparticles in both forward and backwards moving configurations, lets do this diagonalization explicitly for both forwards and backwards solutions.

For the forward solution, given $\Psi=\Psi_{0} e^{i(k x-E t / \hbar)}$, the Dirac equation is

$$
i \hbar(-i E / \hbar) \Psi=\left[\begin{array}{cc}
-i \hbar(i k)+V_{0} & m c^{2}  \tag{3.113}\\
m c^{2} & i \hbar(i k)
\end{array}\right] \Psi,
$$

or

$$
\left[\begin{array}{cc}
E-V_{0} & 0  \tag{3.114}\\
0 & E-V_{0}
\end{array}\right] \Psi=\left[\begin{array}{cc}
\hbar k & m c^{2} \\
m c^{2} & -\hbar k
\end{array}\right] \Psi .
$$

Similarly, for the backwards moving wave $\Psi=e^{i(-k x-E t / \hbar)}$, we have

$$
\left[\begin{array}{cc}
E-V_{0} & 0  \tag{3.115}\\
0 & E-V_{0}
\end{array}\right] \Psi=\left[\begin{array}{cc}
-\hbar k & m c^{2} \\
m c^{2} & \hbar k
\end{array}\right] \Psi .
$$

Working with $\hbar=c=1$ temporarily, we want to compute the eigensolutions for the matrix

$$
H_{ \pm k}=\left[\begin{array}{cc} 
\pm k & m  \tag{3.116}\\
m & \mp k
\end{array}\right] .
$$

The eigenvalues $\epsilon$ of both are the same

$$
\begin{align*}
0 & =\left|H_{ \pm k}-\epsilon\right| \\
& =( \pm k-\epsilon)(\mp k-\epsilon)-m^{2} \\
& =(\mp k+\epsilon)( \pm k+\epsilon)-m^{2}  \tag{3.117}\\
& =\epsilon^{2}-k^{2}-m^{2},
\end{align*}
$$

or

$$
\begin{equation*}
\epsilon= \pm \sqrt{k^{2}+m^{2}} \tag{3.118}
\end{equation*}
$$

Eigenkets for $H_{k} \quad$ For the positive(negative) energy eigenvalues, we have

$$
0=\left[\begin{array}{cc}
k \mp \epsilon & m  \tag{3.119}\\
m & -k \mp \epsilon
\end{array}\right]\left[\begin{array}{l}
a \\
b
\end{array}\right]
$$

for some $a, b$. That is

$$
\begin{equation*}
(k \mp \epsilon) a+m b=0, \tag{3.120}
\end{equation*}
$$

or

$$
\left[\begin{array}{l}
a  \tag{3.121}\\
b
\end{array}\right] \propto\left[\begin{array}{c}
-m \\
k \mp \epsilon
\end{array}\right]
$$

For the normalization note that

$$
\begin{align*}
m^{2}+(k \mp \epsilon)^{2} & =m^{2}+k^{2}+\epsilon^{2} \mp 2 k \epsilon \\
& =2 \epsilon^{2} \mp 2 k \epsilon  \tag{3.122}\\
& =2 \epsilon(\epsilon \mp k),
\end{align*}
$$

so the normalized kets are

$$
|k ; \pm \epsilon\rangle=\frac{1}{\sqrt{2 \epsilon(\epsilon \mp k)}}\left[\begin{array}{c} 
\pm m  \tag{3.123}\\
\epsilon \mp k
\end{array}\right] .
$$

Eigenkets for $H_{-k} \quad$ This time, for the positive(negative) energy eigenvalues, we have

$$
0=\left[\begin{array}{cc}
-k \mp \epsilon & m  \tag{3.124}\\
m & k \mp \epsilon
\end{array}\right]\left[\begin{array}{l}
a \\
b
\end{array}\right],
$$

for some $a, b$. That is

$$
\begin{equation*}
(-k \mp \epsilon) a+m b=0 \tag{3.125}
\end{equation*}
$$

or

$$
\begin{align*}
{\left[\begin{array}{l}
a \\
b
\end{array}\right] } & \propto\left[\begin{array}{c}
-m \\
-k \mp \epsilon
\end{array}\right] \\
& \propto\left[\begin{array}{c}
m \\
k \pm \epsilon
\end{array}\right] . \tag{3.126}
\end{align*}
$$

For the normalization note that

$$
\begin{align*}
m^{2}+(k \pm \epsilon)^{2} & =m^{2}+k^{2}+\epsilon^{2} \pm 2 k \epsilon \\
& =2 \epsilon^{2} \pm 2 k \epsilon  \tag{3.127}\\
& =2 \epsilon(\epsilon \pm k)
\end{align*}
$$

so the normalized kets are

$$
|-k ; \pm \epsilon\rangle=\frac{1}{\sqrt{2 \epsilon(\epsilon \mp k)}}\left[\begin{array}{c} 
\pm m  \tag{3.128}\\
\epsilon \pm k
\end{array}\right]
$$

Simplification of the rotation matrices The eigenvalue equations have the form

$$
H\left[\begin{array}{cc}
|+\rangle & |-\rangle
\end{array}\right]=\left[\begin{array}{cc}
|+\rangle & |-\rangle
\end{array}\right]\left[\begin{array}{cc}
\epsilon & 0  \tag{3.129}\\
0 & -\epsilon
\end{array}\right]
$$

With $R=\left[\begin{array}{cc}|+\rangle & |-\rangle\end{array}\right]$, this has the form $H R=R \Omega$, or $H=R \Omega R^{-1}$. The rotation matrices have been found to be

$$
R_{+k}=\frac{1}{\sqrt{2 \epsilon}}\left[\begin{array}{cc}
\frac{m}{\sqrt{\epsilon-k}} & \frac{-m}{\sqrt{\epsilon+k}}  \tag{3.130}\\
\sqrt{\epsilon-k} & \sqrt{\epsilon+k}
\end{array}\right]
$$

and

$$
R_{-k}=\frac{1}{\sqrt{2 \epsilon}}\left[\begin{array}{cc}
\frac{m}{\sqrt{\epsilon+k}} & \frac{-m}{\sqrt{\epsilon-k}}  \tag{3.131}\\
\sqrt{\epsilon+k} & \sqrt{\epsilon-k}
\end{array}\right]
$$

These don't look very much like rotation matrices as is, but not all the terms are independent. Writing

$$
\begin{align*}
& a=\frac{m}{\sqrt{2 \epsilon(\epsilon-k)}} \\
& b=\frac{m}{\sqrt{2 \epsilon(\epsilon+k)}} \\
& \alpha=\sqrt{\frac{\epsilon+k}{2 \epsilon}}  \tag{3.132}\\
& \beta=\sqrt{\frac{\epsilon-k}{2 \epsilon}}
\end{align*}
$$

the rotation matrices are

$$
R_{+k}=\left[\begin{array}{cc}
a & -b  \tag{3.133}\\
\beta & \alpha
\end{array}\right]
$$

and

$$
R_{-k}=\left[\begin{array}{cc}
b & -a  \tag{3.134}\\
\alpha & \beta
\end{array}\right]
$$

Note that

$$
\begin{align*}
\frac{a}{\alpha} & =\frac{b}{\beta} \\
& =\frac{m}{\sqrt{\epsilon^{2}-k^{2}}}  \tag{3.135}\\
& =\frac{m}{\sqrt{m^{2}}} \\
& =1 .
\end{align*}
$$

So

$$
R_{+k}=\left[\begin{array}{cc}
a & -b  \tag{3.136}\\
b & a
\end{array}\right]
$$

and

$$
R_{-k}=\left[\begin{array}{cc}
b & -a  \tag{3.137}\\
a & b
\end{array}\right]
$$

These are expected to have a unit determinant, which is verified easily

$$
\begin{align*}
a^{2}+b^{2} & =\frac{m^{2}}{2 \epsilon}\left(\frac{1}{\epsilon+k}+\frac{1}{\epsilon-k}\right) \\
& =\frac{m^{2}}{2 \epsilon} \frac{2 \epsilon}{\epsilon_{2}-k^{2}}  \tag{3.138}\\
& =1
\end{align*}
$$

We see that both sets of matrices invert by transposition, so we are free to make a trigonometric identification

$$
\begin{equation*}
a=\cos \theta=\frac{m}{\sqrt{2 \epsilon(\epsilon-k)}} \tag{3.139}
\end{equation*}
$$

$$
\begin{equation*}
b=\sin \theta=\frac{m}{\sqrt{2 \epsilon(\epsilon+k)}} \tag{3.140}
\end{equation*}
$$

Using the double angle formulation used in class these are

$$
\begin{align*}
\cos (2 \theta) & =\frac{m^{2}}{2 \epsilon(\epsilon-k)}-\frac{m^{2}}{2 \epsilon(\epsilon+k)} \\
& =\frac{m^{2}}{2 \epsilon}\left(\frac{1}{\epsilon-k}-\frac{1}{\epsilon+k}\right)  \tag{3.141}\\
& =\frac{2 k m^{2}}{2 \epsilon\left(\epsilon^{2}-k^{2}\right)} \\
& =\frac{k}{\epsilon}
\end{align*}
$$

and

$$
\begin{align*}
\sin (2 \theta) & =2 \frac{m^{2}}{2 \epsilon} \frac{1}{\sqrt{\epsilon^{2}-k^{2}}}  \tag{3.142}\\
& =\frac{m}{\epsilon}
\end{align*}
$$

Putting back in the $\hbar$, and $c$ factors, the diagonalizing transformation is now fully specified

$$
\left.\begin{array}{rl}
H_{ \pm k} & =\left[\begin{array}{cc} 
\pm \hbar k c & m c^{2} \\
m c^{2} & \mp \hbar k c
\end{array}\right]=R_{ \pm k} \Omega R_{ \pm k}^{\mathrm{T}} \\
R_{+k} & =\left[\begin{array}{cc}
\cos \theta & -\sin \theta \\
\sin \theta & \cos \theta
\end{array}\right]=\left[\begin{array}{ll}
|+k ;+\epsilon\rangle & |+k ;-\epsilon\rangle
\end{array}\right] \\
R_{-k} & =\left[\begin{array}{cc}
\sin \theta & -\cos \theta \\
\cos \theta & \sin \theta
\end{array}\right]=[|-k ;+\epsilon\rangle  \tag{3.143}\\
|-k ;-\epsilon\rangle
\end{array}\right], \begin{array}{ll}
\tan (2 \theta) & =\frac{m c}{\hbar|k|} \\
\Omega & =\left[\begin{array}{cc}
\epsilon & 0 \\
0 & -\epsilon
\end{array}\right] \\
\epsilon & =\sqrt{(\hbar k c)^{2}+\left(m c^{2}\right)^{2}} .
\end{array}
$$

The functions $\Psi=| \pm k ; \pm \epsilon\rangle e^{ \pm i k x-i E t / \hbar}$ are eigenfunctions of the Hamiltonian. For example, for a non-antiparticle forward moving state

$$
\begin{align*}
\left(E-V_{0}\right)|k ;+\epsilon\rangle & =H_{k}|k ;+\epsilon\rangle \\
& =\left[\begin{array}{ll}
|k ;+\epsilon\rangle & |k ;-\epsilon\rangle
\end{array}\right]\left[\begin{array}{cc}
\epsilon & 0 \\
0 & -\epsilon
\end{array}\right]\left[\begin{array}{c}
\langle k ;+\epsilon| \\
\langle k ;-\epsilon|
\end{array}\right]|k ; \epsilon\rangle \\
& =\left[\begin{array}{ll}
|k ;+\epsilon\rangle & |k ;-\epsilon\rangle
\end{array}\right]\left[\begin{array}{cc}
\epsilon & 0 \\
0 & -\epsilon
\end{array}\right]\left[\begin{array}{cc}
\langle k ;+\epsilon \mid k ; \epsilon\rangle \\
\langle k ;-\epsilon \mid k ; \epsilon\rangle
\end{array}\right]  \tag{3.144}\\
& =\left[\begin{array}{ll}
|k ;+\epsilon\rangle & |k ;-\epsilon\rangle
\end{array}\right]\left[\begin{array}{cc}
\epsilon & 0 \\
0 & -\epsilon
\end{array}\right]\left[\begin{array}{l}
1 \\
0
\end{array}\right] \\
& =\left[\begin{array}{ll}
|k ;+\epsilon\rangle & |k ;-\epsilon\rangle
\end{array}\right]\left[\begin{array}{l}
\epsilon \\
0
\end{array}\right] \\
& =\epsilon|k ;+\epsilon\rangle .
\end{align*}
$$

Back to the core problem. The total energy for this particle is

$$
\begin{equation*}
E=V_{0} \pm \sqrt{(\hbar k c)^{2}+\left(m c^{2}\right)^{2}} \tag{3.145}
\end{equation*}
$$

This can be plotted nicely, as $\frac{E}{m c^{2}}$ vs. $\frac{\hbar k}{m c}$ if non-dimensionalized as

$$
\begin{equation*}
\frac{E}{m c^{2}}=\frac{V_{0}}{m c^{2}}+\sqrt{1+\frac{(\hbar k)^{2}}{(m c)^{2}}} \tag{3.146}
\end{equation*}
$$

For the step potential we have

$$
\begin{align*}
\frac{E}{m c^{2}} & =\sqrt{1+\frac{\hbar k_{1}^{2}}{m c}}  \tag{3.147}\\
& = \pm \sqrt{1+\frac{\hbar k_{2}^{2}}{m c}}+\frac{V_{0}}{m c^{2}}
\end{align*}
$$

or

$$
\begin{equation*}
\frac{\hbar k_{2}}{m c}=\left(\left( \pm \sqrt{1+\frac{\hbar k_{1}^{2}}{m c}}-\frac{V_{0}}{m c^{2}}\right)^{2}-1\right)^{1 / 2} \tag{3.148}
\end{equation*}
$$

When this is real valued, there is transmission in the barrier region. There are a few cases of interest, the $V_{0}<2 m c^{2}$ cases are plotted in fig. 3.15
showing the transmission $\left(V_{0}=0.7 m c^{2}\right)$ and decaying cases $\left(V_{0}=1.4 m c^{2}\right)$ respectively. Three $V_{0}>2 m c^{2}$ cases are plotted in fig. 3.16, showing the anti-particle transmission, decaying solution, and ordinary-particle transmission $\left(\left(V_{0} / m c^{2}, k \hbar / m c\right)=(3.4,1.7),(2.6,1.9),(2.7,6.3)\right)$. At low enough momentum there is antiparticle transmission, then reflection as momentum is increased, and finally at high enough momentum ordinary particle transmission.


Figure 3.15: $V_{0}<2 m c^{2}$.

(a)

(b)

(c)

Figure 3.16: $V_{0}>2 m c^{2}$.
We want to find the wave incident, reflected, and transmitted wave function. From the diagonalization above, these are

$$
\Psi_{\mathrm{inc}}=\left[\begin{array}{c}
\cos \theta_{k_{1}}  \tag{3.149}\\
\sin \theta_{k_{1}}
\end{array}\right] e^{i\left(k_{1} x-E t / \hbar\right)},
$$

$$
\Psi_{\mathrm{ref}}=\left[\begin{array}{l}
\sin \theta_{k_{1}}  \tag{3.150}\\
\cos \theta_{k_{1}}
\end{array}\right] e^{i\left(-k_{1} x-E t / \hbar\right)}
$$

The currents $j=c\left(\psi_{1}^{*} \psi_{1}-\psi_{2}^{*} \psi_{2}\right)$ for these wave functions are

$$
\begin{equation*}
j_{\mathrm{inc}}=c\left(\cos ^{2} \theta_{k_{1}}-\sin ^{2} \theta_{k_{1}}\right)=c \cos \left(2 \theta_{k_{1}}\right) \tag{3.151}
\end{equation*}
$$

and

$$
\begin{equation*}
j_{\mathrm{ref}}=c\left(\sin ^{2} \theta_{k_{1}}-\cos ^{2} \theta_{k_{1}}\right)=-c \cos \left(2 \theta_{k_{1}}\right) \tag{3.152}
\end{equation*}
$$

When there is a transmitted particle (anti-particle) in region II, the transmitted wave function are respectively

$$
\begin{align*}
& \Psi_{\text {trans }}=\left[\begin{array}{c}
\cos \theta_{k_{2}} \\
\sin \theta_{k_{2}}
\end{array}\right] e^{i\left(k_{2} x-E t / \hbar\right)},  \tag{3.153}\\
& \Psi_{\text {trans }}=\left[\begin{array}{c}
-\sin \theta_{k_{2}} \\
\cos \theta_{k_{2}}
\end{array}\right] e^{i\left(k_{2} x-E t / \hbar\right)} . \tag{3.154}
\end{align*}
$$

The currents for these are respectively

$$
\begin{equation*}
j=c\left(\cos ^{2} \theta_{k_{2}}-\sin ^{2} \theta_{k_{2}}\right)=c \cos \left(2 \theta_{k_{2}}\right) \tag{3.155}
\end{equation*}
$$

and

$$
\begin{equation*}
j=c\left(\sin ^{2} \theta_{k_{2}}-\cos ^{2} \theta_{k_{2}}\right)=-c \cos \left(2 \theta_{k_{2}}\right) \tag{3.156}
\end{equation*}
$$

To determine the weighting of the wave functions, we require matching at the boundary. There are a few cases to consider
(i) Total reflection. Are there any special values of momentum that allow for total reflection? If so, at the boundary both components must be zero

$$
\left[\begin{array}{c}
\cos \theta_{k_{1}}  \tag{3.157}\\
\sin \theta_{k_{1}}
\end{array}\right]+\frac{B}{A}\left[\begin{array}{c}
\sin \theta_{k_{1}} \\
\cos \theta_{k_{1}}
\end{array}\right]=\left[\begin{array}{l}
0 \\
0
\end{array}\right] .
$$

This is possible if $-B / A=\cot \theta_{k_{1}}=\tan \theta_{k_{1}}$, which requires

$$
\begin{equation*}
\theta_{k_{1}}=\frac{\pi}{4}(1+2 n), \quad n \in \mathbb{Z} \tag{3.158}
\end{equation*}
$$

However, we must also have

$$
\begin{equation*}
\tan 2 \theta_{1}=\frac{m c}{\hbar k} \tag{3.159}
\end{equation*}
$$

so

$$
\begin{equation*}
\theta_{1}=\frac{1}{2} \operatorname{atan}\left(\frac{m c}{\hbar k}\right) \tag{3.160}
\end{equation*}
$$

Simultaneous solutions of eq. (3.158), eq. (3.160) only occur at $k= \pm 0$, where $\tan ((1+2 n) \pi / 2)= \pm \infty$. Since a particle at rest is not at interest in a reflection scenario, this shows that a decaying solution in region II must be introduced to match the boundary value constraints.
(ii) Ordinary matter transmission. For this case at $x=0$, we have

$$
A\left[\begin{array}{c}
\cos \theta_{k_{1}}  \tag{3.161}\\
\sin \theta_{k_{1}}
\end{array}\right]+B\left[\begin{array}{c}
\sin \theta_{k_{1}} \\
\cos \theta_{k_{1}}
\end{array}\right]=D\left[\begin{array}{c}
\cos \theta_{2} \\
\sin \theta_{2}
\end{array}\right]
$$

With $a=B / A$ and $b=D / A, S_{1,2}=\sin \theta_{k_{1,2}}, C_{1,2}=\cos \theta_{k_{1,2}}$, this is

$$
\left[\begin{array}{l}
C_{1}  \tag{3.162}\\
S_{1}
\end{array}\right]=\left[\begin{array}{ll}
-S_{1} & C_{2} \\
-C_{1} & S_{2}
\end{array}\right]\left[\begin{array}{l}
a \\
b
\end{array}\right]
$$

or

$$
\begin{align*}
{\left[\begin{array}{l}
a \\
b
\end{array}\right] } & =\frac{1}{-S_{1} S_{2}+C_{1} C_{2}}\left[\begin{array}{ll}
S_{2} & -C_{2} \\
C_{1} & -S_{1}
\end{array}\right]\left[\begin{array}{l}
C_{1} \\
S_{1}
\end{array}\right] \\
& =\frac{1}{\cos \left(\theta_{k_{1}}+\theta_{k_{2}}\right)}\left[\begin{array}{c}
C_{1} S_{2}-S_{1} C_{2} \\
C_{1}^{2}-S_{1}^{2}
\end{array}\right], \tag{3.163}
\end{align*}
$$

which is

$$
\begin{align*}
\frac{B}{A} & =\frac{\cos \left(\theta_{k_{2}}-\theta_{k_{1}}\right)}{\cos \left(\theta_{k_{1}}+\theta_{k_{2}}\right)}  \tag{3.164}\\
\frac{D}{A} & =\frac{\cos \left(2 \theta_{k_{1}}\right)}{\cos \left(\theta_{k_{1}}+\theta_{k_{2}}\right)}
\end{align*}
$$

Let's verify that the currents in both regions match. With $A=1$, the region I current sum is

$$
\begin{align*}
j_{\text {inc }}+j_{\text {ref }} & =c \cos \left(2 \theta_{k_{1}}\right)-B^{2} c \cos \left(2 \theta_{k_{1}}\right) \\
& =c \cos \left(2 \theta_{k_{1}}\right)\left(1-\frac{\sin ^{2}\left(\theta_{k_{2}}-\theta_{k_{1}}\right)}{\cos ^{2}\left(\theta_{k_{1}}+\theta_{k_{2}}\right)}\right) \\
& =c \cos \left(2 \theta_{k_{1}}\right) \frac{\cos ^{2}\left(\theta_{k_{1}}+\theta_{k_{2}}\right)-\sin ^{2}\left(\theta_{k_{2}}-\theta_{k_{1}}\right)}{\cos ^{2}\left(\theta_{k_{1}}+\theta_{k_{2}}\right)}  \tag{3.165}\\
& =c \frac{\cos \left(2 \theta_{k_{1}}\right) \cos \left(2 \theta_{k_{1}}\right) \cos \left(2 \theta_{k_{2}}\right)}{\cos ^{2}\left(\theta_{k_{1}}+\theta_{k_{2}}\right)} .
\end{align*}
$$

Whereas, the transmitted (region II) current is

$$
\begin{align*}
j_{\text {trans }} & =c D^{2} \cos \left(2 \theta_{k_{2}}\right) \\
& =c \cos \left(2 \theta_{k_{2}}\right) \frac{\cos ^{2}\left(2 \theta_{k_{1}}\right)}{\cos ^{2}\left(\theta_{k_{1}}+\theta_{k_{2}}\right)} \tag{3.166}
\end{align*}
$$

so we see $j_{\text {inc }}+j_{\text {ref }}=j_{\text {trans }}$, as expected.
(iii) Anti-matter transmission.

For this case at $x=0$, we have

$$
A\left[\begin{array}{c}
\cos \theta_{k_{1}}  \tag{3.167}\\
\sin \theta_{k_{1}}
\end{array}\right]+B\left[\begin{array}{c}
\sin \theta_{k_{1}} \\
\cos \theta_{k_{1}}
\end{array}\right]=D\left[\begin{array}{c}
-\sin \theta_{2} \\
\cos \theta_{2}
\end{array}\right]
$$

With the same substitutions as above, this is

$$
\left[\begin{array}{l}
C_{1}  \tag{3.168}\\
S_{1}
\end{array}\right]=\left[\begin{array}{cc}
-S_{1} & -S_{2} \\
-C_{1} & C_{2}
\end{array}\right]\left[\begin{array}{l}
a \\
b
\end{array}\right]
$$

or

$$
\begin{align*}
{\left[\begin{array}{l}
a \\
b
\end{array}\right] } & =\frac{1}{-\left(S_{1} C_{2}+S_{2} C_{1}\right)}\left[\begin{array}{ll}
C_{2} & -S_{2} \\
C_{1} & -S_{1}
\end{array}\right]\left[\begin{array}{l}
C_{1} \\
S_{1}
\end{array}\right] \\
& =\frac{1}{-\sin \left(\theta_{k_{1}}+\theta_{k_{2}}\right)}\left[\begin{array}{c}
C_{1} C_{2}+S_{1} S_{2} \\
C_{1}^{2}-S_{1}^{2}
\end{array}\right], \tag{3.169}
\end{align*}
$$

which is

$$
\begin{align*}
\frac{B}{A} & =\frac{\cos \left(\theta_{k_{1}}-\theta_{k_{2}}\right)}{\sin \left(\theta_{k_{1}}+\theta_{k_{2}}\right)} \\
\frac{D}{A} & =-\frac{\cos \left(2 \theta_{k_{1}}\right)}{\sin \left(\theta_{k_{1}}+\theta_{k_{2}}\right)} \tag{3.170}
\end{align*}
$$

For the anti-particle transmission, the region I current is

$$
\begin{align*}
j_{\text {inc }}+j_{\text {ref }} & =c \cos \left(2 \theta_{k_{1}}\right)-B^{2} c \cos \left(2 \theta_{k_{1}}\right) \\
& =c \cos \left(2 \theta_{k_{1}}\right)\left(1-\frac{\cos ^{2}\left(\theta_{k_{1}}-\theta_{k_{2}}\right)}{\sin ^{2}\left(\theta_{k_{1}}+\theta_{k_{2}}\right)}\right) \\
& =c \cos \left(2 \theta_{k_{1}}\right) \frac{\sin ^{2}\left(\theta_{k_{1}}+\theta_{k_{2}}\right)-\cos ^{2}\left(\theta_{k_{1}}-\theta_{k_{2}}\right)}{\cos ^{2}\left(\theta_{k_{1}}+\theta_{k_{2}}\right)}  \tag{3.171}\\
& =-c \frac{\cos \left(2 \theta_{k_{1}}\right) \cos \left(2 \theta_{k_{1}}\right) \cos \left(2 \theta_{k_{2}}\right)}{\sin ^{2}\left(\theta_{k_{1}}+\theta_{k_{2}}\right)}
\end{align*}
$$

Whereas, the transmitted (region II) current is

$$
\begin{align*}
j_{\text {trans }} & =-c D^{2} \cos \left(2 \theta_{k_{2}}\right) \\
& =-c \cos \left(2 \theta_{k_{2}}\right) \frac{\cos ^{2}\left(2 \theta_{k_{1}}\right)}{\sin ^{2}\left(\theta_{k_{1}}+\theta_{k_{2}}\right)} \tag{3.172}
\end{align*}
$$

and again we see $j_{\text {inc }}+j_{\text {ref }}=j_{\text {trans }}$, as expected.
(iv) Decaying transmission. In units where $\hbar=c=1$, the eigenkets found for the Dirac Hamiltonian, before normalization, were found to be
where $\epsilon_{1}^{2}=m^{2}+k^{2}$. Letting $k \rightarrow i k$, this provides the form of the non-oscillatory wavefunctions in region II when there is no ordinary nor anti-particle transmission.
where

$$
\begin{equation*}
\epsilon_{2}^{2}=m^{2}-k^{2} \tag{3.175}
\end{equation*}
$$

Observe that the $\psi_{2}$ component of this wave function sits on a circle in the complex plane

$$
\begin{align*}
\left|\epsilon_{2} \pm i k\right|^{2} & =\epsilon^{2}+k^{2} \\
& =m^{2}-k^{2}+k^{2}  \tag{3.176}\\
& =m^{2} .
\end{align*}
$$

Putting back in the factors of $\hbar, c$, this allows the identification

$$
\begin{equation*}
\epsilon_{2} \pm i \hbar k c=m c^{2} e^{i \phi} . \tag{3.177}
\end{equation*}
$$

So, like the trigonometric representation of the oscillatory wave function, the normalized wave functions for exponential decay(increase) can be written
where the respective eigenvalues are $\epsilon_{2}= \pm \sqrt{\left(m c^{2}\right)^{2}-(\hbar k c)^{2}}$, and $\hbar k c<m c^{2}$.

Observe that the currents for the exponential wave functions are both light-like

$$
\begin{equation*}
j_{ \pm}=c\left(\left| \pm e^{ \pm i \phi / 2}\right|^{2}-\left|e^{\mp i \phi / 2}\right|^{2}\right)=0 \tag{3.179}
\end{equation*}
$$

which makes some sense since the particle is not able to propagate freely in the barrier region.
At the interface, we wish to solve

$$
A\left[\begin{array}{l}
C_{1}  \tag{3.180}\\
S_{1}
\end{array}\right]+B\left[\begin{array}{l}
S_{1} \\
C_{1}
\end{array}\right]=\frac{D}{\sqrt{2}}\left[\begin{array}{c}
e^{i \phi / 2} \\
e^{-i \phi / 2}
\end{array}\right] .
$$

With $a=A / D$, and $b=B / D$, this has solution

$$
\begin{align*}
{\left[\begin{array}{l}
a \\
b
\end{array}\right] } & =\frac{1}{\sqrt{2}}\left[\begin{array}{ll}
C_{1} & S_{1} \\
S_{1} & C_{1}
\end{array}\right]^{-1}\left[\begin{array}{c}
e^{i \phi / 2} \\
e^{-i \phi / 2}
\end{array}\right] \\
& =\frac{1}{\sqrt{2} \cos \left(2 \theta_{k_{1}}\right)}\left[\begin{array}{cc}
C_{1} & -S_{1} \\
-S_{1} & C_{1}
\end{array}\right]\left[\begin{array}{c}
e^{i \phi / 2} \\
e^{-i \phi / 2}
\end{array}\right]  \tag{3.181}\\
& =\frac{1}{\sqrt{2} \cos \left(2 \theta_{k_{1}}\right)}\left[\begin{array}{c}
C_{1} e^{i \phi / 2}-S_{1} e^{-i \phi / 2} \\
-S_{1} e^{i \phi / 2}+C_{1} e^{-i \phi / 2}
\end{array}\right] .
\end{align*}
$$

The reflection coefficient is unity

$$
\begin{aligned}
R & =\left|\frac{B}{A}\right|^{2} \\
& =\left|\frac{B / D}{A / D}\right|^{2} \\
& =\frac{\left|-S_{1} e^{i \phi / 2}+C_{1} e^{-i \phi / 2}\right|^{2}}{\left|C_{1} e^{i \phi / 2}-S_{1} e^{-i \phi / 2}\right|^{2}} \\
& =\frac{C_{1}^{2}+S_{1}^{2}-2 S_{1} C_{1} \operatorname{Re} e^{-i \phi}}{C_{1}^{2}+S_{1}^{2}-2 S_{1} C_{1} \operatorname{Re} e^{i \phi}} \\
& =\frac{1-\sin \left(2 \theta_{k_{1}}\right) \cos \phi}{1-\sin \left(2 \theta_{k_{1}}\right) \cos \phi} \\
& =1 .
\end{aligned}
$$

## Exercise 3.7 Dirac equation representations. (2015 midterm pr. 2)

Given an alternate representation of the Dirac equation

$$
H=\left[\begin{array}{cc}
m c^{2}+V_{0} & c \hat{p}  \tag{3.183}\\
c \hat{p} & -m c^{2}+V_{0}
\end{array}\right],
$$

calculate
a. the constant momentum plane wave solutions,
b. the constant momentum hyperbolic solutions,
c. the Heisenberg velocity operator $\hat{v}$, and
d. find the form of the probability density current.

## Answer for Exercise 3.7

Part a. The action of the Hamiltonian on

$$
\psi=e^{i k x-i E t / \hbar}\left[\begin{array}{l}
\psi_{1}  \tag{3.184}\\
\psi_{2}
\end{array}\right],
$$

is

$$
\begin{align*}
H \psi & =\left[\begin{array}{cc}
m c^{2}+V_{0} & c(-i \hbar) i k \\
c(-i \hbar) i k & -m c^{2}+V_{0}
\end{array}\right]\left[\begin{array}{l}
\psi_{1} \\
\psi_{2}
\end{array}\right] e^{i k x-i E t / \hbar} \\
& =\left[\begin{array}{cc}
m c^{2}+V_{0} & c \hbar k \\
c \hbar k & -m c^{2}+V_{0}
\end{array}\right] \psi \tag{3.185}
\end{align*}
$$

Writing

$$
H_{k}=\left[\begin{array}{cc}
m c^{2}+V_{0} & c \hbar k  \tag{3.186}\\
c \hbar k & -m c^{2}+V_{0}
\end{array}\right]
$$

the characteristic equation is

$$
\begin{align*}
0 & =\left(m c^{2}+V_{0}-\lambda\right)\left(-m c^{2}+V_{0}-\lambda\right)-(c \hbar k)^{2} \\
& =\left(\left(\lambda-V_{0}\right)^{2}-\left(m c^{2}\right)^{2}\right)-(c \hbar k)^{2}, \tag{3.187}
\end{align*}
$$

so

$$
\begin{equation*}
\lambda=V_{0} \pm \epsilon \tag{3.188}
\end{equation*}
$$

where

$$
\begin{equation*}
\epsilon^{2}=\left(m c^{2}\right)^{2}+(c \hbar k)^{2} \tag{3.189}
\end{equation*}
$$

We've got

$$
\begin{align*}
& H-\left(V_{0}+\epsilon\right)=\left[\begin{array}{cc}
m c^{2}-\epsilon & c \hbar k \\
c \hbar k & -m c^{2}-\epsilon
\end{array}\right] \\
& H-\left(V_{0}-\epsilon\right)=\left[\begin{array}{cc}
m c^{2}+\epsilon & c \hbar k \\
c \hbar k & -m c^{2}+\epsilon
\end{array}\right] \tag{3.190}
\end{align*}
$$

so the eigenkets are

$$
\begin{align*}
& \left|V_{0}+\epsilon\right\rangle \propto\left[\begin{array}{c}
-c \hbar k \\
m c^{2}-\epsilon
\end{array}\right] \\
& \left|V_{0}-\epsilon\right\rangle \propto\left[\begin{array}{c}
-c \hbar k \\
m c^{2}+\epsilon
\end{array}\right] . \tag{3.191}
\end{align*}
$$

Up to an arbitrary phase for each, these are

$$
\begin{align*}
& \left|V_{0}+\epsilon\right\rangle=\frac{1}{\sqrt{2 \epsilon\left(\epsilon-m c^{2}\right)}}\left[\begin{array}{c}
c \hbar k \\
\epsilon-m c^{2}
\end{array}\right] \\
& \left|V_{0}-\epsilon\right\rangle=\frac{1}{\sqrt{2 \epsilon\left(\epsilon+m c^{2}\right)}}\left[\begin{array}{c}
-c \hbar k \\
\epsilon+m c^{2}
\end{array}\right] . \tag{3.192}
\end{align*}
$$

We can now write

$$
H_{k}=E\left[\begin{array}{cc}
V_{0}+\epsilon & 0  \tag{3.193}\\
0 & V_{0}-\epsilon
\end{array}\right] E^{-1}
$$

where

$$
\begin{align*}
& E=\frac{1}{\sqrt{2 \epsilon}}\left[\begin{array}{cc}
\frac{c \hbar k}{\sqrt{\epsilon-m c^{2}}} & -\frac{c \hbar k}{\sqrt{\epsilon+m c^{2}}} \\
\sqrt{\epsilon-m c^{2}} & \sqrt{\epsilon+m c^{2}}
\end{array}\right],  \tag{3.194}\\
& E=\frac{1}{\sqrt{2 \epsilon}}\left[\begin{array}{cc}
-\frac{c \hbar k}{\sqrt{\epsilon-m c^{2}}} & -\frac{c \hbar k}{\sqrt{\epsilon+m c^{2}}} \\
-\sqrt{\epsilon-m c^{2}} & \sqrt{\epsilon+m c^{2}}
\end{array}\right], \quad k<0 .
\end{align*}
$$

Here the signs have been adjusted to ensure the transformation matrix has a unit determinant.

Observe that there's redundancy in this matrix since $c \hbar|k| / \sqrt{\epsilon-m c^{2}}=$ $\sqrt{\epsilon+m c^{2}}$, and $c \hbar|k| / \sqrt{\epsilon+m c^{2}}=\sqrt{\epsilon-m c^{2}}$, which allows the transformation matrix to be written in the form of a rotation matrix

$$
\begin{array}{ll}
E=\frac{1}{\sqrt{2 \epsilon}}\left[\begin{array}{cc}
\frac{c \hbar k}{\sqrt{\epsilon-m c^{2}}} & -\frac{c \hbar k}{\sqrt{\epsilon+m c^{2}}} \\
\frac{c \hbar k}{\sqrt{\epsilon+m c^{2}}} & \frac{c \hbar k}{\sqrt{\epsilon-m c^{2}}}
\end{array}\right], & k>0 \\
E=\frac{1}{\sqrt{2 \epsilon}}\left[\begin{array}{cc}
-\frac{c \hbar k}{\sqrt{\epsilon-m c^{2}}} & -\frac{c \hbar k}{\sqrt{\epsilon+m c^{2}}} \\
\frac{c \hbar k}{\sqrt{\epsilon+m c^{2}}} & -\frac{c \hbar k}{\sqrt{\epsilon-m c^{2}}}
\end{array}\right], & k<0 . \tag{3.195}
\end{array}
$$

With

$$
\begin{align*}
& \cos \theta=\frac{c \hbar|k|}{\sqrt{2 \epsilon\left(\epsilon-m c^{2}\right)}}=\frac{\sqrt{\epsilon+m c^{2}}}{\sqrt{2 \epsilon}} \\
& \sin \theta=\frac{c \hbar k}{\sqrt{2 \epsilon\left(\epsilon+m c^{2}\right)}}=\frac{\operatorname{sgn}(k) \sqrt{\epsilon-m c^{2}}}{\sqrt{2 \epsilon}} \tag{3.196}
\end{align*}
$$

the transformation matrix (and eigenkets) is

$$
E=\left[\begin{array}{ll}
\left|V_{0}+\epsilon\right\rangle & \left|V_{0}-\epsilon\right\rangle
\end{array}\right]=\left[\begin{array}{cc}
\cos \theta & -\sin \theta  \tag{3.197}\\
\sin \theta & \cos \theta
\end{array}\right]
$$

Observe that eq. (3.196) can be simplified by using double angle formulas

$$
\begin{align*}
\cos (2 \theta) & =\frac{\left(\epsilon+m c^{2}\right)}{2 \epsilon}-\frac{\left(\epsilon-m c^{2}\right)}{2 \epsilon} \\
& =\frac{1}{2 \epsilon}\left(\epsilon+m c^{2}-\epsilon+m c^{2}\right)  \tag{3.198}\\
& =\frac{m c^{2}}{\epsilon}
\end{align*}
$$

and

$$
\begin{align*}
\sin (2 \theta) & =2 \frac{1}{2 \epsilon} \operatorname{sgn}(k) \sqrt{\epsilon^{2}-\left(m c^{2}\right)^{2}} \\
& =\frac{\hbar k c}{\epsilon} \tag{3.199}
\end{align*}
$$

This allows all the $\theta$ dependence on $\hbar k c$ and $m c^{2}$ to be expressed as a ratio of momenta

$$
\begin{equation*}
\tan (2 \theta)=\frac{\hbar k}{m c} \tag{3.200}
\end{equation*}
$$

Part b. For a wave function of the form

$$
\psi=e^{k x-i E t / \hbar}\left[\begin{array}{l}
\psi_{1}  \tag{3.201}\\
\psi_{2}
\end{array}\right]
$$

some of the work above can be recycled if we substitute $k \rightarrow-i k$, which yields unnormalized eigenfunctions

$$
\begin{align*}
& \left|V_{0}+\epsilon\right\rangle \propto\left[\begin{array}{c}
i c \hbar k \\
m c^{2}-\epsilon
\end{array}\right] \\
& \left|V_{0}-\epsilon\right\rangle \propto\left[\begin{array}{c}
i c \hbar k \\
m c^{2}+\epsilon
\end{array}\right], \tag{3.202}
\end{align*}
$$

where

$$
\begin{equation*}
\epsilon^{2}=\left(m c^{2}\right)^{2}-(c \hbar k)^{2} \tag{3.203}
\end{equation*}
$$

The squared magnitude of these wavefunctions are

$$
\begin{aligned}
(c \hbar k)^{2}+\left(m c^{2} \mp \epsilon\right)^{2} & =(c \hbar k)^{2}+\left(m c^{2}\right)^{2}+\epsilon^{2} \mp 2 m c^{2} \epsilon \\
& =(c \hbar k)^{2}+\left(m c^{2}\right)^{2}+\left(m c^{2}\right)^{2} \mp(c \hbar k)^{2}-2 m c^{2} \epsilon \\
& =2\left(m c^{2}\right)^{2} \mp 2 m c^{2} \epsilon \\
& =2 m c^{2}\left(m c^{2} \mp \epsilon\right),
\end{aligned}
$$

so, up to a constant phase for each, the normalized kets are

$$
\begin{align*}
& \left|V_{0}+\epsilon\right\rangle=\frac{1}{\sqrt{2 m c^{2}\left(m c^{2}-\epsilon\right)}}\left[\begin{array}{c}
i c \hbar k \\
m c^{2}-\epsilon
\end{array}\right]  \tag{3.205}\\
& \left|V_{0}-\epsilon\right\rangle=\frac{1}{\sqrt{2 m c^{2}\left(m c^{2}+\epsilon\right)}}\left[\begin{array}{c}
i c \hbar k \\
m c^{2}+\epsilon
\end{array}\right] .
\end{align*}
$$

After the $k \rightarrow-i k$ substitution, $H_{k}$ is not Hermitian, so these kets aren't expected to be orthonormal, which is readily verified

$$
\begin{align*}
\left\langle V_{0}+\epsilon \mid V_{0}-\epsilon\right\rangle & =\frac{1}{\sqrt{2 m c^{2}\left(m c^{2}-\epsilon\right)}} \frac{1}{\sqrt{2 m c^{2}\left(m c^{2}+\epsilon\right)}}, \times \\
& {\left[-i c \hbar k \quad m c^{2}-\epsilon\right]\left[\begin{array}{c}
i c \hbar k \\
m c^{2}+\epsilon
\end{array}\right] }  \tag{3.206}\\
& =\frac{2(c \hbar k)^{2}}{2 m c^{2} \sqrt{(\hbar k c)^{2}}} \\
& =\operatorname{sgn}(k) \frac{\hbar k}{m c}
\end{align*}
$$

Part c.

$$
\begin{align*}
\hat{v} & =\frac{1}{i \hbar}[\hat{x}, H] \\
& =\frac{1}{i \hbar}\left[\hat{x}, m c^{2} \sigma_{z}+V_{0}+c \hat{p} \sigma_{x}\right]  \tag{3.207}\\
& =\frac{c \sigma_{x}}{i \hbar}[\hat{x}, \hat{p}] \\
& =c \sigma_{x} .
\end{align*}
$$

Part d. Acting against a completely general wavefunction the Hamiltonian action $H \psi$ is

$$
\begin{align*}
i \hbar \frac{\partial \psi}{\partial t} & =m c^{2} \sigma_{z} \psi+V_{0} \psi+c \hat{p} \sigma_{x} \psi \\
& =m c^{2} \sigma_{z} \psi+V_{0} \psi-i \hbar c \sigma_{x} \frac{\partial \psi}{\partial x} \tag{3.208}
\end{align*}
$$

Conversely, the conjugate $(H \psi)^{\dagger}$ is

$$
\begin{equation*}
-i \hbar \frac{\partial \psi^{\dagger}}{\partial t}=m c^{2} \psi^{\dagger} \sigma_{z}+V_{0} \psi^{\dagger}+i \hbar c \frac{\partial \psi^{\dagger}}{\partial x} \sigma_{x} \tag{3.209}
\end{equation*}
$$

These give

$$
\begin{align*}
i \hbar \psi^{\dagger} \frac{\partial \psi}{\partial t} & =m c^{2} \psi^{\dagger} \sigma_{z} \psi+V_{0} \psi^{\dagger} \psi-i \hbar c \psi^{\dagger} \sigma_{x} \frac{\partial \psi}{\partial x} \\
-i \hbar \frac{\partial \psi^{\dagger}}{\partial t} \psi & =m c^{2} \psi^{\dagger} \sigma_{z} \psi+V_{0} \psi^{\dagger} \psi+i \hbar c \frac{\partial \psi^{\dagger}}{\partial x} \sigma_{x} \psi \tag{3.210}
\end{align*}
$$

Taking differences

$$
\begin{equation*}
\psi^{\dagger} \frac{\partial \psi}{\partial t}+\frac{\partial \psi^{\dagger}}{\partial t} \psi=-c \psi^{\dagger} \sigma_{x} \frac{\partial \psi}{\partial x}-c \frac{\partial \psi^{\dagger}}{\partial x} \sigma_{x} \psi \tag{3.211}
\end{equation*}
$$

or

$$
\begin{equation*}
0=\frac{\partial}{\partial t}\left(\psi^{\dagger} \psi\right)+\frac{\partial}{\partial x}\left(c \psi^{\dagger} \sigma_{x} \psi\right) \tag{3.212}
\end{equation*}
$$

The probability current still has the usual form $\rho=\psi^{\dagger} \psi=\psi_{1}^{*} \psi_{1}+\psi_{2}^{*} \psi_{2}$, but the probability current with this representation of the Dirac Hamiltonian is

$$
j=c \psi^{\dagger} \sigma_{x} \psi=c\left[\begin{array}{ll}
\psi_{1}^{*} & \psi_{2}^{*}
\end{array}\right]\left[\begin{array}{l}
\psi_{2}  \tag{3.213}\\
\psi_{1}
\end{array}\right]=c\left(\psi_{1}^{*} \psi_{2}+\psi_{2}^{*} \psi_{1}\right)
$$

## 4.1 symmetry in classical mechanics.

In a classical context considering a Hamiltonian

$$
\begin{equation*}
H\left(q_{i}, p_{i}\right), \tag{4.1}
\end{equation*}
$$

a symmetry means that certain $q_{i}$ don't appear. In that case the rate of change of one of the generalized momenta is zero

$$
\begin{equation*}
\frac{d p_{k}}{d t}=-\frac{\partial H}{\partial q_{k}}=0, \tag{4.2}
\end{equation*}
$$

so $p_{k}$ is a constant of motion. This simplifies the problem by reducing the number of degrees of freedom. Another aspect of such a symmetry is that it relates trajectories. For example, assuming a rotational symmetry as in fig. 4.1. the trajectory of a particle after rotation is related by rotation to


Figure 4.1: Trajectory under rotational symmetry.
the trajectory of the unrotated particle.

## 4.2 symmetry in quantum mechanics.

Suppose that we have a symmetry operation that takes states from

$$
\begin{equation*}
|\psi\rangle \rightarrow|U \psi\rangle \tag{4.3}
\end{equation*}
$$

$$
\begin{equation*}
|\phi\rangle \rightarrow|U \phi\rangle, \tag{4.4}
\end{equation*}
$$

we expect that

$$
\begin{equation*}
|\langle\psi \mid \phi\rangle|^{2}=|\langle U \psi \mid U \phi\rangle|^{2} . \tag{4.5}
\end{equation*}
$$

This won't hold true for a general operator. Two cases where this does hold true is when

- $\langle\psi \mid \phi\rangle=\langle U \psi \mid U \phi\rangle$. Here $U$ is unitary, and the equivalence follows from

$$
\begin{equation*}
\langle U \psi \mid U \phi\rangle=\langle\psi| U^{\dagger} U \phi=\langle\psi| 1 \phi=\langle\psi \mid \phi\rangle . \tag{4.6}
\end{equation*}
$$

- $\langle\psi \mid \phi\rangle=\langle U \psi \mid U \phi\rangle^{*}$. Here $U$ is anti-unitary.

Unitary case If an "observable" is not changed by a unitary operation representing a symmetry we must have

$$
\begin{equation*}
\langle\psi| \hat{A}|\psi\rangle \rightarrow\langle U \psi| \hat{A}|U \psi\rangle=\langle\psi| U^{\dagger} \hat{A} U|\psi\rangle, \tag{4.7}
\end{equation*}
$$

so

$$
\begin{equation*}
U^{\dagger} \hat{A} U=\hat{A}, \tag{4.8}
\end{equation*}
$$

or

$$
\begin{equation*}
\hat{A} U=U \hat{A} . \tag{4.9}
\end{equation*}
$$

An observable that is unchanged by a unitary symmetry commutes $[\hat{A}, U]$ with the operator $U$ for that transformation.

Symmetries of the Hamiltonian Given

$$
\begin{equation*}
[H, U]=0 \tag{4.10}
\end{equation*}
$$

$H$ is invariant. Given

$$
\begin{equation*}
H\left|\phi_{n}\right\rangle=\epsilon_{n}\left|\phi_{n}\right\rangle, \tag{4.11}
\end{equation*}
$$

$$
\begin{align*}
U H\left|\phi_{n}\right\rangle & =H U\left|\phi_{n}\right\rangle  \tag{4.12}\\
& =\epsilon_{n} U\left|\phi_{n}\right\rangle
\end{align*}
$$

Such a state

$$
\begin{equation*}
\left|\psi_{n}\right\rangle=U\left|\phi_{n}\right\rangle, \tag{4.13}
\end{equation*}
$$

is also an eigenstate with the same energy. Suppose this process is repeated, finding other states

$$
\begin{equation*}
U\left|\psi_{n}\right\rangle=\left|\chi_{n}\right\rangle \tag{4.14}
\end{equation*}
$$

$$
\begin{equation*}
U\left|\chi_{n}\right\rangle=\left|\alpha_{n}\right\rangle \tag{4.15}
\end{equation*}
$$

Because such a transformation only generates states with the initial energy, this process cannot continue forever. At some point this process will enumerate a fixed size set of states. These states can be orthonormalized.

We can say that symmetry operations are generators of a group. For a set of symmetry operations we can

- form products that lie in a closed set

$$
\begin{equation*}
U_{1} U_{2}=U_{3} \tag{4.16}
\end{equation*}
$$

- and can define an inverse operation

$$
\begin{equation*}
U \leftrightarrow U^{-1} . \tag{4.17}
\end{equation*}
$$

- Such operators obey associative rules for multiplication

$$
\begin{equation*}
U_{1}\left(U_{2} U_{3}\right)=\left(U_{1} U_{2}\right) U_{3} . \tag{4.18}
\end{equation*}
$$

- and have an identity operation.

When $H$ has a symmetry, then degenerate eigenstates form irreducible representations (which cannot be further block diagonalized).

Parity symmetry

## Example 4.1: Inversion.

Consider a state and a parity operation $\hat{\Pi}$, with the transformation

$$
\begin{equation*}
|\psi\rangle \rightarrow \hat{\Pi}|\psi\rangle \tag{4.19}
\end{equation*}
$$

In one dimension, the parity operation is just inversion. In two dimensions, this is a set of flipping operations on two axes fig. 4.2.


Figure 4.2: 2D parity operation.
The operational effects of this operator are

$$
\begin{gather*}
\hat{x} \rightarrow-\hat{x} \\
\hat{p} \rightarrow-\hat{p} . \tag{4.20}
\end{gather*}
$$

Acting again with the parity operator produces the original value, so it is its own inverse, and $\hat{\Pi}^{\dagger}=\hat{\Pi}=\hat{\Pi}^{-1}$. In an expectation value

$$
\begin{equation*}
\langle\hat{\Pi} \psi| \hat{x}|\hat{\Pi} \psi\rangle=-\langle\psi| \hat{x}|\psi\rangle \tag{4.21}
\end{equation*}
$$

This means that

$$
\begin{equation*}
\hat{\Pi}^{\dagger} \hat{x} \hat{\Pi}=-\hat{x} \tag{4.22}
\end{equation*}
$$

or

$$
\begin{equation*}
\hat{x} \hat{\Pi}=-\hat{\Pi} \hat{x} \tag{4.23}
\end{equation*}
$$

$$
\begin{align*}
\hat{x} \hat{\Pi}\left|x_{0}\right\rangle & =-\hat{\Pi} \hat{x}\left|x_{0}\right\rangle \\
& =-\hat{\Pi} x_{0}\left|x_{0}\right\rangle  \tag{4.24}\\
& =-x_{0} \hat{\Pi}\left|x_{0}\right\rangle
\end{align*}
$$

so

$$
\begin{equation*}
\hat{\Pi}\left|x_{0}\right\rangle=\left|-x_{0}\right\rangle \tag{4.25}
\end{equation*}
$$

Acting on a wave function

$$
\begin{align*}
\langle x| \hat{\Pi}|\psi\rangle & =\langle-x \mid \psi\rangle  \tag{4.26}\\
& =\psi(-x)
\end{align*}
$$

What does this mean for eigenfunctions. Eigenfunctions are supposed to form irreducible representations of the group. The group has just two elements

$$
\begin{equation*}
\{1, \hat{\Pi}\} \tag{4.27}
\end{equation*}
$$

where $\hat{\Pi}^{2}=1$.
Suppose we have a Hamiltonian

$$
\begin{equation*}
H=\frac{\hat{p}^{2}}{2 m}+V(\hat{x}) \tag{4.28}
\end{equation*}
$$

where $V(\hat{x})$ is even, or $[V(\hat{x}), \hat{\Pi}]=0$. The squared momentum commutes with the parity operator

$$
\begin{align*}
{\left[\hat{p}^{2}, \hat{\Pi}\right] } & =\hat{p}^{2} \hat{\Pi}-\hat{\Pi} \hat{p}^{2} \\
& =\hat{p}^{2} \hat{\Pi}-(\hat{\Pi} \hat{p}) \hat{p} \\
& =\hat{p}^{2} \hat{\Pi}-(-\hat{p} \hat{\Pi}) \hat{p}  \tag{4.29}\\
& =\hat{p}^{2} \hat{\Pi}+\hat{p}(-\hat{p} \hat{\Pi}) \\
& =0 .
\end{align*}
$$

Only two functions are possible in the symmetry set $\{\Psi(x), \hat{\Pi} \Psi(x)\}$, since

$$
\begin{align*}
\hat{\Pi}^{2} \Psi(x) & =\hat{\Pi} \Psi(-x) \\
& =\Psi(x) \tag{4.30}
\end{align*}
$$

This symmetry severely restricts the possible solutions, making it so there can be only one dimensional forms of this problem with solutions that are either even or odd respectively

$$
\begin{align*}
& \phi_{e}(x)=\psi(x)+\psi(-x)  \tag{4.31}\\
& \phi_{o}(x)=\psi(x)-\psi(-x) .
\end{align*}
$$

Parity (review) The action of the parity operators on the position and momentum operators are

$$
\begin{equation*}
\hat{\Pi} \hat{x} \hat{\Pi}=-\hat{x}, \tag{4.32}
\end{equation*}
$$

and

$$
\begin{equation*}
\hat{\Pi} \hat{p} \hat{\Pi}=-\hat{p} \tag{4.33}
\end{equation*}
$$

These are polar vectors, in contrast to an axial vector such as $\mathbf{L}=\mathbf{r} \times \mathbf{p}$.

$$
\begin{equation*}
\hat{\Pi}^{2}=1 \tag{4.34}
\end{equation*}
$$

$$
\begin{equation*}
\Psi(x) \rightarrow \Psi(-x) \tag{4.35}
\end{equation*}
$$

If $[\hat{\Pi}, \hat{H}]=0$ then all the eigenstates are either

- even: $\hat{\Pi}$ eigenvalue is +1 ,
- odd: $\hat{\Pi}$ eigenvalue is -1 .

Note on parity in multiple dimensions A Hamiltonian can be constructed with parity symmetries in one or more directions. For example, given a potential

$$
\begin{equation*}
V(x, y)=a x+b x^{2}+c y^{2} . \tag{4.36}
\end{equation*}
$$

We don't have parity symmetry for $\mathbf{x}=(x, y)$, but do have parity symmetry in the $y$ direction. Assuming a separated variables form for the wave function, say $\psi(x, y)=X(x) Y(y)$, we can't say much about $X$ on the grounds of symmetry considerations only, but know that $Y$ has to be either an even or odd function.

### 4.3 Translations.

Define a (continuous) translation operator

$$
\begin{equation*}
\hat{T}_{\epsilon}|x\rangle=|x+\epsilon\rangle \tag{4.37}
\end{equation*}
$$

The action of this operator is sketched in fig. 4.3. This is a unitary operator


Figure 4.3: Translation operation.

$$
\begin{equation*}
\hat{T}_{-\epsilon}=\hat{T}_{\epsilon}^{\dagger}=\hat{T}_{\epsilon}^{-1} \tag{4.38}
\end{equation*}
$$

In a position basis, the action of this operator is

$$
\begin{equation*}
\langle x| \hat{T}_{\epsilon}|\psi\rangle=\langle x-\epsilon \mid \psi\rangle=\psi(x-\epsilon) \tag{4.39}
\end{equation*}
$$

$$
\begin{equation*}
\Psi(x-\epsilon) \approx \Psi(x)-\epsilon \frac{\partial \Psi}{\partial x} \tag{4.40}
\end{equation*}
$$

$$
\begin{equation*}
\langle x| \hat{T}_{\epsilon}|\Psi\rangle=\langle x \mid \Psi\rangle-\frac{\epsilon}{\hbar}\langle x| i \hat{p}|\Psi\rangle \tag{4.41}
\end{equation*}
$$

$$
\begin{equation*}
\hat{T}_{\epsilon} \approx\left(1-i \frac{\epsilon}{\hbar} \hat{p}\right) \tag{4.42}
\end{equation*}
$$

A non-infinitesimal translation can be composed of many small translations, as sketched in fig. 4.4. For $\epsilon \rightarrow 0, N \rightarrow \infty, N \epsilon=a$, the total


Figure 4.4: Composition of small translations.
translation operator is

$$
\begin{align*}
\hat{T}_{a} & =\hat{T}_{\epsilon}^{N} \\
& =\lim _{\epsilon \rightarrow 0, N \rightarrow \infty, N \epsilon=a}\left(1-\frac{\epsilon}{\hbar} \hat{p}\right)^{N}  \tag{4.43}\\
& =e^{-i a \hat{p} / \hbar} .
\end{align*}
$$

The momentum $\hat{p}$ is called a "Generator" of translations. If a Hamiltonian $H$ is translationally invariant, then

$$
\begin{equation*}
\left[\hat{T}_{a}, H\right]=0, \quad \forall a . \tag{4.44}
\end{equation*}
$$

This means that momentum will be a good quantum number

$$
\begin{equation*}
[\hat{p}, H]=0 . \tag{4.45}
\end{equation*}
$$

## 4.4 rotations.

Rotations form a non-Abelian group, since the order of rotations $\hat{R}_{1} \hat{R}_{2} \neq$ $\hat{R}_{2} \hat{R}_{1}$. Given a rotation acting on a bet

$$
\begin{equation*}
\hat{R}|\mathbf{r}\rangle=|R \mathbf{r}\rangle, \tag{4.46}
\end{equation*}
$$

observe that the action of the rotation operator on a wave function is invented

$$
\begin{align*}
\langle\mathbf{r}| \hat{R}|\Psi\rangle & =\left\langle R^{-1} \mathbf{r}\right||\Psi\rangle  \tag{4.47}\\
& =\Psi\left(R^{-1} \mathbf{r}\right) .
\end{align*}
$$

## Example 4.2: Z axis normal rotation

Consider an infinitesimal rotation about the z-axis as sketched in fig. 4.5.

(a)

(b)

Figure 4.5: Rotation about z-axis.

$$
\begin{align*}
& x^{\prime}=x-\epsilon y \\
& y^{\prime}=y+\epsilon y  \tag{4.48}\\
& z^{\prime}=z .
\end{align*}
$$

The rotated wave function is

$$
\begin{align*}
& \tilde{\Psi}(x, y, z)=\Psi(x+\epsilon y, y-\epsilon x, z) \\
&=\Psi(x, y, z)+\epsilon y \frac{\partial \Psi}{\partial x}-\epsilon x \frac{\partial \Psi}{\partial y} .  \tag{4.49}\\
& i \hat{p}_{x} / \hbar \quad i \hat{p}_{y} / \hbar
\end{align*}
$$

The state must then transform as

$$
\begin{equation*}
|\tilde{\Psi}\rangle=\left(1+i \frac{\epsilon}{\hbar} \hat{y} \hat{p}_{x}-i \frac{\epsilon}{\hbar} \hat{x} \hat{p}_{y}\right)|\Psi\rangle . \tag{4.50}
\end{equation*}
$$

Observe that the combination $\hat{x} \hat{p}_{y}-\hat{y} \hat{p}_{x}$ is the $\hat{L}_{z}$ component of angular momentum $\hat{\mathbf{L}}=\hat{\mathbf{r}} \times \hat{\mathbf{p}}$, so the infinitesimal rotation can be written

$$
\begin{equation*}
\hat{R}_{z}(\epsilon)|\Psi\rangle=\left(1-i \frac{\epsilon}{\hbar} \hat{L}_{z}\right)|\Psi\rangle \tag{4.51}
\end{equation*}
$$

For a finite rotation $\epsilon \rightarrow 0, N \rightarrow \infty, \phi=\epsilon N$, the total rotation is

$$
\begin{equation*}
\hat{R}_{z}(\phi)=\left(1-\frac{i \epsilon}{\hbar} \hat{L}_{z}\right)^{N} \tag{4.52}
\end{equation*}
$$

or

$$
\begin{equation*}
\hat{R}_{z}(\phi)=e^{-i \frac{\phi}{\hbar} \hat{L}_{z}} \tag{4.53}
\end{equation*}
$$

Note that $\left[\hat{L}_{x}, \hat{L}_{y}\right] \neq 0$.
By construction using Euler angles or any other method, a general rotation will include contributions from components of all the angular momentum operator, and will have the structure

$$
\begin{equation*}
\hat{R}_{\hat{\mathbf{n}}}(\phi)=e^{-i \frac{\phi}{\hbar}(\hat{\mathbf{L}} \cdot \hat{\mathbf{n}})} \tag{4.54}
\end{equation*}
$$

Rotationally invariant $\hat{H}$. Given a rotationally invariant Hamiltonian

$$
\begin{equation*}
\left[\hat{R}_{\hat{\mathbf{n}}}(\phi), \hat{H}\right]=0 \quad \forall \hat{\mathbf{n}}, \phi \tag{4.55}
\end{equation*}
$$

then every

$$
\begin{equation*}
[\mathbf{L} \cdot \hat{\mathbf{n}}, \hat{H}]=0 \tag{4.56}
\end{equation*}
$$

or

$$
\begin{equation*}
\left[L_{i}, \hat{H}\right]=0 \tag{4.57}
\end{equation*}
$$

Non-Abelian implies degeneracies in the spectrum.

### 4.5 TIME-REVERSAL.

Imagine that we have something moving along a curve at time $t=0$, and ending up at the final position at time $t=t_{f}$, as sketched in fig. 4.6. Now imagine that we flip the direction of motion (i.e. flipping the velocity) and run time backwards so the final-time state becomes the initial state. If the time reversal operator is designated $\hat{\Theta}$, with operation

$$
\begin{equation*}
\hat{\Theta}|\Psi\rangle=|\tilde{\Psi}\rangle \tag{4.58}
\end{equation*}
$$

so that

$$
\begin{equation*}
\hat{\Theta}^{-1} e^{-i \hat{H} t / \hbar} \hat{\Theta}|\Psi(t)\rangle=|\Psi(0)\rangle \tag{4.59}
\end{equation*}
$$



Figure 4.6: Time reversal trajectory.
or

$$
\begin{equation*}
\hat{\Theta}^{-1} e^{-i \hat{H} t / \hbar} \hat{\Theta}|\Psi(0)\rangle=|\Psi(-t)\rangle . \tag{4.60}
\end{equation*}
$$

Time reversal (cont.) Given a time reversed state

$$
\begin{equation*}
|\tilde{\Psi}(t)\rangle=\Theta|\Psi(0)\rangle \tag{4.61}
\end{equation*}
$$

which can alternately be written

$$
\begin{equation*}
\Theta^{-1}|\tilde{\Psi}(t)\rangle=|\Psi(-t)\rangle=e^{i \hat{H} t / \hbar}|\Psi(0)\rangle . \tag{4.62}
\end{equation*}
$$

The left hand side can be expanded as the evolution of the state as found at time - $t$

$$
\begin{align*}
\Theta^{-1}|\tilde{\Psi}(t)\rangle & =\Theta^{-1} e^{-i \hat{H} t / \hbar}|\tilde{\Psi}(-t)\rangle \\
& =\Theta^{-1} e^{-i \hat{H} t / \hbar} \Theta|\Psi(0)\rangle \tag{4.63}
\end{align*}
$$

To first order for a small time increment $\delta t$, we have

$$
\begin{equation*}
\left(1+i \frac{\hat{H}}{\hbar} \delta t\right)|\Psi(0)\rangle=\Theta^{-1}\left(1-i \frac{\hat{H}}{\hbar} \delta t\right) \Theta|\Psi(0)\rangle \tag{4.64}
\end{equation*}
$$

or

$$
\begin{equation*}
i \frac{\hat{H}}{\hbar} \delta t|\Psi(0)\rangle=\Theta^{-1}(-i) \frac{\hat{H}}{\hbar} \delta t \Theta|\Psi(0)\rangle \tag{4.65}
\end{equation*}
$$

Since this holds for any state $|\Psi(0)\rangle$, the time reversal operator satisfies

$$
\begin{equation*}
i \hat{H}=\Theta^{-1}(-i) \hat{H} \Theta \tag{4.66}
\end{equation*}
$$

Note that the factors of $i$ have not been cancelled on purpose, since we are allowing for the time reversal operator to not necessarily commute with imaginary numbers.

There are two possible solutions

- If $\Theta$ is unitary where $\Theta i=i \Theta$, then

$$
\begin{equation*}
\hat{H}=-\Theta^{-1} \hat{H} \Theta \tag{4.67}
\end{equation*}
$$

or

$$
\begin{equation*}
\Theta \hat{H}=-\hat{H} \Theta \tag{4.68}
\end{equation*}
$$

Consider the implications of this on energy eigenstates

$$
\begin{equation*}
\hat{H}\left|\Psi_{n}\right\rangle=E_{n}\left|\Psi_{n}\right\rangle \tag{4.69}
\end{equation*}
$$

$$
\begin{equation*}
\Theta \hat{H}\left|\Psi_{n}\right\rangle=E_{n} \Theta\left|\Psi_{n}\right\rangle \tag{4.70}
\end{equation*}
$$

but

$$
\begin{equation*}
-\hat{H} \Theta\left|\Psi_{n}\right\rangle=E_{n} \Theta\left|\Psi_{n}\right\rangle \tag{4.71}
\end{equation*}
$$

or

$$
\begin{equation*}
\hat{H}\left(\Theta\left|\Psi_{n}\right\rangle\right)=-E_{n}\left(\Theta\left|\Psi_{n}\right\rangle\right) \tag{4.72}
\end{equation*}
$$

This would mean that $\left(\Theta\left|\Psi_{n}\right\rangle\right)$ is an eigenket of $\hat{H}$, but with a negative energy eigenvalue.

- $\Theta$ is antiunitary, where $\Theta i=-i \Theta$.

This time

$$
\begin{equation*}
i \hat{H}=i \Theta^{-1} \hat{H} \Theta \tag{4.73}
\end{equation*}
$$

so

$$
\begin{equation*}
\Theta \hat{H}=\hat{H} \Theta \tag{4.74}
\end{equation*}
$$

Acting on an energy eigenket, we've got

$$
\begin{equation*}
\Theta \hat{H}\left|\Psi_{n}\right\rangle=E_{n}\left(\Theta\left|\Psi_{n}\right\rangle\right) \tag{4.75}
\end{equation*}
$$

and

$$
\begin{equation*}
(\hat{H} \Theta)\left|\Psi_{n}\right\rangle=\hat{H}\left(\Theta\left|\Psi_{n}\right\rangle\right) \tag{4.76}
\end{equation*}
$$

so $\Theta\left|\Psi_{n}\right\rangle$ is an eigenstate with energy $E_{n}$.

What properties do we expect from $\Theta$ ? We expect

$$
\begin{align*}
\hat{x} & \rightarrow \hat{x} \\
\hat{p} & \rightarrow-\hat{p}  \tag{4.77}\\
\hat{\mathbf{L}} & \rightarrow-\hat{\mathbf{L}}
\end{align*}
$$

where we have a sign flip in the time dependent momentum operator (and therefore angular momentum), but not for position. If we have

$$
\begin{equation*}
\Theta^{-1} \hat{x} \Theta=\hat{x}, \tag{4.78}
\end{equation*}
$$

then how about the momentum operator in the position basis

$$
\begin{align*}
\Theta^{-1} \hat{p} \Theta & =\Theta^{-1}\left(-i \hbar \frac{\partial}{\partial x}\right) \Theta \\
& =\Theta^{-1}(-i \hbar) \Theta \frac{\partial}{\partial x}=i \hbar \Theta^{-1} \Theta \frac{\partial}{\partial x}  \tag{4.79}\\
& =-\hat{p}
\end{align*}
$$

How about the $x, p$ commutator? For that we have

$$
\begin{align*}
\Theta^{-1}[\hat{x}, \hat{p}] \Theta & =\Theta^{-1}(i \hbar) \Theta \\
& =-i \hbar \Theta^{-1} \Theta  \tag{4.80}\\
& =-[\hat{x}, \hat{p}]
\end{align*}
$$

For the angular momentum operators

$$
\begin{equation*}
\hat{L}_{i}=\epsilon_{i j k} \hat{r}_{j} \hat{p}_{k}, \tag{4.81}
\end{equation*}
$$

the time reversal operator should flip the sign due to its action on $\hat{p}_{k}$.

Time reversal acting on spin 1/2 (Fermions). Attempt I. Consider two spin states $|\uparrow\rangle,|\downarrow\rangle$. What should the action of the time reversal operator on such a state be? Let's (incorrectly) start by supposing that the time reversal operator effects are

$$
\begin{align*}
& \Theta|\uparrow\rangle \stackrel{?}{=}|\downarrow\rangle \\
& \Theta|\downarrow\rangle \stackrel{?}{=}|\uparrow\rangle . \tag{4.82}
\end{align*}
$$

Given a general state so that if

$$
\begin{equation*}
|\Psi\rangle=a|\uparrow\rangle+b|\downarrow\rangle, \tag{4.83}
\end{equation*}
$$

the action of the time reversal operator would be

$$
\begin{equation*}
\Theta|\Psi\rangle=a^{*}|\downarrow\rangle+b^{*}|\uparrow\rangle \tag{4.84}
\end{equation*}
$$

That action is:

$$
\begin{align*}
a & \rightarrow b^{*} \\
b & \rightarrow a^{*} \tag{4.85}
\end{align*}
$$

Let's consider whether or not such an action a spin operator with properties

$$
\begin{equation*}
\left[\hat{S}_{i}, \hat{S}_{j}\right]=i \epsilon_{i j k} \hat{S}_{k} \tag{4.86}
\end{equation*}
$$

will produce the desired inversion of sign

$$
\begin{equation*}
\Theta^{-1} \hat{S}_{i} \Theta=-\hat{S}_{i} \tag{4.87}
\end{equation*}
$$

The expectations of the spin operators (without any application of time reversal) are

$$
\begin{align*}
\langle\Psi| \hat{S}_{x}|\Psi\rangle & =\frac{\hbar}{2}\left(a^{*}\langle\uparrow|+b^{*}\langle\downarrow|\right) \sigma_{x}(a|\uparrow\rangle+b|\downarrow\rangle) \\
& =\frac{\hbar}{2}\left(a^{*}\langle\uparrow|+b^{*}\langle\downarrow|\right)(a|\downarrow\rangle+b|\uparrow\rangle)  \tag{4.88}\\
& =\frac{\hbar}{2}\left(a^{*} b+b^{*} a\right), \\
\langle\Psi| \hat{S}_{y}|\Psi\rangle & =\frac{\hbar}{2}\left(a^{*}\langle\uparrow|+b^{*}\langle\downarrow|\right) \sigma_{y}(a|\uparrow\rangle+b|\downarrow\rangle) \\
& =\frac{i \hbar}{2}\left(a^{*}\langle\uparrow|+b^{*}\langle\downarrow|\right)(a|\downarrow\rangle-b|\uparrow\rangle)  \tag{4.89}\\
& =\frac{\hbar}{2 i}\left(a^{*} b-b^{*} a\right), \\
\langle\Psi| \hat{S}_{z}|\Psi\rangle & =\frac{\hbar}{2}\left(a^{*}\langle\uparrow|+b^{*}\langle\downarrow|\right) \sigma_{z}(a|\uparrow\rangle-b|\downarrow\rangle)  \tag{4.90}\\
& =\frac{\hbar}{2}\left(|a|^{2}-|b|^{2}\right) .
\end{align*}
$$

The time reversed actions are

$$
\begin{align*}
\langle\Psi| \Theta^{-1} \hat{S}_{x} \Theta|\Psi\rangle & =\frac{\hbar}{2}\left(a^{*}\langle\downarrow|+b^{*}\langle\uparrow|\right) \sigma_{x}(a|\downarrow\rangle+b|\uparrow\rangle) \\
& =\frac{\hbar}{2}\left(a^{*}\langle\downarrow|+b^{*}\langle\uparrow|\right)(a|\uparrow\rangle+b|\downarrow\rangle)  \tag{4.91}\\
& =\frac{\hbar}{2}\left(a^{*} b+b^{*} a\right)
\end{align*}
$$

$$
\begin{align*}
\langle\Psi| \Theta^{-1} \hat{S}_{y} \Theta|\Psi\rangle & =\frac{\hbar}{2}\left(a^{*}\langle\downarrow|+b^{*}\langle\uparrow|\right) \sigma_{y}(a|\downarrow\rangle+b|\uparrow\rangle) \\
& =\frac{i \hbar}{2}\left(a^{*}\langle\downarrow|+b^{*}\langle\uparrow|\right)(-a|\uparrow\rangle+b|\downarrow\rangle)  \tag{4.92}\\
& =\frac{\hbar}{2 i}\left(-a^{*} b+b^{*} a\right) \\
\langle\Psi| \Theta^{-1} \hat{S}_{z} \Theta|\Psi\rangle & =\frac{\hbar}{2}\left(a^{*}\langle\downarrow|+b^{*}\langle\uparrow|\right) \sigma_{z}(a|\downarrow\rangle+b|\uparrow\rangle) \\
& =\frac{\hbar}{2}\left(a^{*}\langle\downarrow|+b^{*}\langle\uparrow|\right)(-a|\downarrow\rangle+b|\uparrow\rangle)  \tag{4.93}\\
& =\frac{\hbar}{2}\left(-|a|^{2}+|b|^{2}\right)
\end{align*}
$$

We see that this is not right, because the sign for the x component has not been flipped.

Spin 1/2 (Fermions). Attempt II. Again assuming

$$
\begin{equation*}
|\Psi\rangle=a|\uparrow\rangle+b|\downarrow\rangle, \tag{4.94}
\end{equation*}
$$

now try the action

$$
\begin{equation*}
\Theta|\Psi\rangle=a^{*}|\downarrow\rangle-b^{*}|\uparrow\rangle . \tag{4.95}
\end{equation*}
$$

This is the action:

$$
\begin{align*}
a & \rightarrow-b^{*} \\
b & \rightarrow a^{*} \tag{4.96}
\end{align*}
$$

The correct action of time reversal on the basis states (up to a phase choice) is

$$
\begin{align*}
& \Theta|\uparrow\rangle=|\downarrow\rangle \\
& \Theta|\downarrow\rangle=-|\uparrow\rangle \tag{4.97}
\end{align*}
$$

Note that acting the time reversal operator twice has the effects

$$
\begin{equation*}
\Theta^{2}|\uparrow\rangle=\Theta|\downarrow\rangle=-|\uparrow\rangle \tag{4.98}
\end{equation*}
$$

$$
\begin{equation*}
\Theta^{2}|\downarrow\rangle=\Theta(-|\uparrow\rangle)=-|\uparrow\rangle \tag{4.99}
\end{equation*}
$$

We end up with the same state we started with, but with the opposite sign. This means that as an operator

$$
\begin{equation*}
\Theta^{2}=-1 \tag{4.100}
\end{equation*}
$$

This is try for half integer particles (Fermions) $S=1 / 2,3 / 2,5 / 2, \cdots$, but for Bosons with integer spin $S$.

$$
\begin{equation*}
\Theta^{2}=1 \tag{4.101}
\end{equation*}
$$

Kramer's degeneracy for Spin 1/2 (Fermions) Suppose we imagine there is state for which the action of the time reversal operator produces the same state, just different in phase. Let

$$
\begin{equation*}
|\psi\rangle^{\prime}=\Theta|\psi\rangle=e^{i \delta}|\psi\rangle . \tag{4.102}
\end{equation*}
$$

For a Fermion we have

$$
\begin{equation*}
\Theta^{2}|\psi\rangle=-|\psi\rangle \tag{4.103}
\end{equation*}
$$

but if such the time reversal action posited above is possible we also have

$$
\begin{align*}
\Theta^{2}|\psi\rangle & =\Theta e^{i \delta}|\psi\rangle \\
& =e^{-i \delta} \Theta|\psi\rangle  \tag{4.104}\\
& =e^{-i \delta} e^{i \delta}|\psi\rangle \\
& =|\psi\rangle \neq-|\psi\rangle .
\end{align*}
$$

This is a contradiction, so we must have at least a two-fold degeneracy. This is called Kramer's degeneracy. In the homework we will show that this is not the case for integer spin particles.

Time reversal implications for wave functions For spin and angular momentum states, the implications of time reversal on the states is worked out above. If a spinless Hamiltonian has time reversal symmetry then the implication is really just the fact that the wave functions can be real valued.
4.6 problems.

## Exercise $4.1 \quad$ No Kramers theorem for spin-1. (2015 ps5 pl)

Consider a spin-1 particle. Even with time-reversal invariance, there is no Kramers theorem, so each eigenvalue of a generic spin-1 Hamiltonian will be non-degenerate. Systematically construct a Hamiltonian which is time-reversal invariant and obviously also Hermitian to illustrate this point, making clear the logic of your construction (i.e., why you are including terms which you are including).
Answer for Exercise 4.1
One representation of the spin-one $J_{z}$ operator is

$$
J_{z}=\hbar\left[\begin{array}{ccc}
1 & 0 & 0  \tag{4.105}\\
0 & 0 & 0 \\
0 & 0 & -1
\end{array}\right]
$$

This operator changes sign under time reversal. The $L_{z}$ operator also changes sign under time reversal, and can be written

$$
\begin{equation*}
L_{z}=-i \hbar \frac{\partial}{\partial \phi} \tag{4.106}
\end{equation*}
$$

A product of these will be time reversal invariant, but not Hermitian. To make it Hermitian we can scale with an imaginary constant

$$
\begin{align*}
H & =-\frac{i}{2 m \rho^{2}} J_{z} L_{z} \\
& =-\frac{i(-i \hbar) \hbar}{2 m \rho^{2}}\left[\begin{array}{ccc}
1 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & -1
\end{array}\right] \frac{\partial}{\partial \phi}  \tag{4.107}\\
& =-\frac{\hbar^{2}}{2 m \rho^{2}}\left[\begin{array}{ccc}
\partial_{\phi} & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & -\partial_{\phi}
\end{array}\right]
\end{align*}
$$

A $1 / 2 m \rho^{2}$ factor has been included to ensure the dimensions of this spinone time-reversal invariant Hamiltonian also has the dimensions of energy.

This Hamiltonian can be solved directly using the trial function $\psi=$ $e^{-i E t / \hbar} \tilde{\psi}$, which provides an eigenvalue equation

$$
\begin{equation*}
E \tilde{\psi}=H \tilde{\psi} \tag{4.108}
\end{equation*}
$$

leading to solutions that have a hyperbolic form

$$
\psi \propto\left[\begin{array}{c}
C e^{-2 m \rho^{2} E \phi / \hbar^{2}}  \tag{4.109}\\
0 \\
C^{\prime} e^{2 m \rho^{2} E \phi / \hbar^{2}}
\end{array}\right] e^{-i E t / \hbar}
$$

Consistent with the no-Kramer's theorem result for a spin-one Hamiltonian, there is a continuum of energy solutions for this Hamiltonian, with nothing to constrain them.

## Exercise $4.2 \quad$ Boosts. (2015 ps5 p2)

The momentum operator $\hat{p}$ was shown, in class, to act as the generator of space translations. Show by following the exact same steps that the position operator $\hat{x}$ acts as the generator of momentum boosts (i.e., $\hat{x}$ is a generator of 'momentum translation').
Answer for Exercise 4.2
Borrowing the same notation as in class, for an infinitesimal change in momentum $\delta \tilde{p}$, define a momentum translation operator with the action on a momentum space state of

$$
\begin{equation*}
\hat{T}_{\delta \tilde{p}}|p\rangle=|p+\delta \tilde{p}\rangle \tag{4.110}
\end{equation*}
$$

A wave function matrix element for this momentum translation is

$$
\begin{align*}
\langle p| \hat{T}_{\delta \tilde{p}}|\psi\rangle & =\langle p-\delta \tilde{p} \mid \psi\rangle \\
& =\psi_{p}(p-\delta \tilde{p}) \approx \psi_{p}(p)-\delta \tilde{p} \frac{\partial \psi_{p}(p)}{\partial p} \tag{4.111}
\end{align*}
$$

Since the momentum space representation of the position operator is

$$
\begin{equation*}
x=i \hbar \frac{\partial}{\partial p} \tag{4.112}
\end{equation*}
$$

we have

$$
\begin{equation*}
\langle p| \hat{T}_{\delta \tilde{p}}|\psi\rangle \approx \psi_{p}(p)-\delta \tilde{p} \frac{x}{i \hbar} \psi_{p}(p)=\left(1+\frac{\delta \tilde{p} i x}{\hbar}\right) \psi_{p}(p) \tag{4.113}
\end{equation*}
$$

Given a finite change of momentum $\tilde{p}=N \delta \tilde{p}$, the matrix element has the limiting form

$$
\begin{align*}
\langle p| \hat{T}_{\tilde{p}}|\psi\rangle & =\lim _{N \rightarrow \infty, \delta \tilde{p} \rightarrow 0}\left(1+\frac{\tilde{p} i x}{N \hbar}\right)^{N} \psi_{p}(p)  \tag{4.114}\\
& =e^{i \tilde{p} x / \hbar} \psi_{p}(p),
\end{align*}
$$

showing that $x$ is the generator of the momentum translation operator

$$
\begin{equation*}
\hat{T}_{\tilde{p}}=e^{i \tilde{p} x / \hbar} \tag{4.115}
\end{equation*}
$$

## Exercise $4.3 \quad$ Simultaneous eigenstates. (2015 ps5 p3)

A quantum state $|\Psi\rangle$ is a simultaneous eigenstate of two anticommuting Hermitian operators $A, B$, with $A B+B A=0$. What can you say about the eigenvalues of $A, B$ for the state $|\Psi\rangle$ ? Illustrate your point using the parity operator and the momentum operator.
Answer for Exercise 4.3
For the eigenvalues of the respective states, let

$$
\begin{equation*}
A|\Psi\rangle=a|\Psi\rangle \tag{4.116}
\end{equation*}
$$

and

$$
\begin{equation*}
B|\Psi\rangle=b|\Psi\rangle \tag{4.117}
\end{equation*}
$$

The action of the anticommutator on this state is

$$
\begin{align*}
(A B+B A)|\Psi\rangle & =(a b+b a)|\Psi\rangle  \tag{4.118}\\
& =0
\end{align*}
$$

This means the product $2 a b$, is zero, and that at least one of $a$ or $b$ must be zero.

The concrete example of the parity operator $\hat{\Pi}$ and the momentum operator $\hat{p}$ demonstrates this. Because the eigenvalues of the parity operator can only be $\pm 1$, if such a simultaneous eigenstate exists in the momentum basis, it must have a $p=0$ eigenvalue. To see this explicitly, assume there is some value $p$ for which $|p\rangle$ is a simultaneous eigenstate of $\hat{\Pi}$ and $\hat{p}$. First consider

$$
\begin{align*}
\hat{\Pi} \hat{p}|p\rangle & =p \hat{\Pi}|p\rangle  \tag{4.119}\\
& =-p|p\rangle
\end{align*}
$$

Since these operators anticommute

$$
\begin{equation*}
\hat{\Pi} \hat{p}=-\hat{p} \hat{\Pi} \tag{4.120}
\end{equation*}
$$

we must also have

$$
\begin{align*}
\hat{\Pi} \hat{p}|p\rangle & =-\hat{p} \hat{\Pi}|p\rangle \\
& =\hat{p}|p\rangle  \tag{4.121}\\
& =p|p\rangle
\end{align*}
$$

The only way this can be satisfied is when the eigenvalue of this state is $p=0$.

## Exercise $4.4 \quad$ Angular momentum. (2015 ps5 p4)

a. What is the time-reversed version of $\mathscr{D}(R)|j, m\rangle$ ?
b. Prove that time-reversal implies $\Theta|j, m\rangle=(-1)^{m}|j,-m\rangle$.

## Answer for Exercise 4.4

Part $a$. Assuming the rotation is around the normal $\hat{\mathbf{n}}=\left(n_{1}, n_{2}, n_{3}\right)$ by $\phi$ radians, the action of the time reversal operator on the rotated state can be expanded in series

$$
\begin{align*}
\Theta \mathscr{D}(R)|j, m\rangle & =\Theta e^{-i \mathbf{J} \cdot \hat{\mathbf{n}} \phi / \hbar}|j, m\rangle \\
& =\sum_{k=0}^{\infty} \frac{1}{k!}\left(\frac{-i \phi}{\hbar}\right)^{k}(\mathbf{J} \cdot \hat{\mathbf{n}})^{k}|j, m\rangle  \tag{4.122}\\
& =\sum_{k=0}^{\infty} \frac{1}{k!}\left(\frac{-i \phi}{\hbar}\right)^{k} \Theta(\mathbf{J} \cdot \hat{\mathbf{n}})^{k}|j, m\rangle .
\end{align*}
$$

For any $J_{i}$, we know that $\Theta J_{i}=-J_{i} \Theta$, so

$$
\begin{align*}
\Theta\left(J_{m} n_{m}\right)^{k} & =-J_{r} n_{r} \Theta\left(J_{m} n_{m}\right)^{k-1} \\
& =\left(-J_{r} n_{r}\right)^{2} \Theta\left(J_{m} n_{m}\right)^{k-2}  \tag{4.123}\\
& =\cdots \\
& =\left(-J_{r} n_{r}\right)^{k} \Theta
\end{align*}
$$

This effectively flips the sign of the rotation angle, so we have

$$
\begin{equation*}
\Theta \mathscr{D}(R)|j, m\rangle=\mathscr{D}\left(R^{-1}\right) \Theta|j, m\rangle . \tag{4.124}
\end{equation*}
$$

Part b. For general total angular momentum states $\mathbf{J}+\mathbf{L}+\mathbf{S}$, we can utilize the postulated properties of the time reversal operator. In particular

$$
\begin{equation*}
\Theta J_{i}=-J_{i} \Theta . \tag{4.125}
\end{equation*}
$$

Considering each of $J_{z}$ and $J_{ \pm}$in turn, we first have

$$
\begin{align*}
\Theta J_{z}|j, m\rangle & =\hbar m \Theta|j, m\rangle  \tag{4.126}\\
& =-J_{z} \Theta|j, m\rangle
\end{align*}
$$

or

$$
\begin{equation*}
J_{z}(\Theta|j, m\rangle)=-m \hbar(\Theta|j, m\rangle), \tag{4.127}
\end{equation*}
$$

This means that $\Theta|j, m\rangle$ is an eigenket of $J_{z}$ with eigenvalue $-m \hbar$, so that we must have

$$
\begin{equation*}
\Theta|j, m\rangle=c_{m}|j,-m\rangle . \tag{4.128}
\end{equation*}
$$

The interplay of the ladder and time-reversal operators can be used to determine the factor $c_{m}$. Note that

$$
\begin{align*}
\Theta J_{+} \Theta^{-1} & =\Theta\left(J_{x}+i J_{y}\right) \Theta^{-1} \\
& =-J_{x}-i \Theta J_{y} \Theta^{-1} \\
& =-J_{x}+i J_{y}  \tag{4.129}\\
& =-\left(J_{x}-i J_{y}\right) \\
& =-J_{-},
\end{align*}
$$

and

$$
\begin{align*}
\Theta J_{-} \Theta^{-1} & =\Theta\left(J_{x}-i J_{y}\right) \Theta^{-1} \\
& =-J_{x}+i \Theta J_{y} \Theta^{-1} \\
& =-J_{x}-i J_{y}  \tag{4.130}\\
& =-\left(J_{x}+i J_{y}\right) \\
& =-J_{+},
\end{align*}
$$

or

$$
\begin{align*}
& J_{-} \Theta=-\Theta J_{+}  \tag{4.131}\\
& J_{+} \Theta=-\Theta J_{-}
\end{align*}
$$

Acting with the raising operator on a time reversed state we have

$$
\begin{align*}
J_{+} \Theta|j, m\rangle & =J_{+} c_{m}|j,-m\rangle \\
& =\hbar \sqrt{(j-(-m))(j+(-m)+1)} c_{m}|j,-m+1\rangle . \tag{4.132}
\end{align*}
$$

But we must also have

$$
\begin{align*}
J_{+} \Theta|j, m\rangle & =-\Theta J_{-}|j, m\rangle \\
& =-\Theta \hbar \sqrt{(j+m)(j-m+1)}|j, m-1\rangle  \tag{4.133}\\
& =-\hbar \sqrt{(j+m)(j-m+1)} c_{m-1}|j,-m+1\rangle
\end{align*}
$$

This provides a recurrence relation for the undetermined $c_{m}$ factors

$$
\begin{equation*}
c_{m}=-c_{m-1}=(-1)^{k} c_{m-k} \tag{4.134}
\end{equation*}
$$

In particular for $m-k=-j$, this is

$$
\begin{equation*}
c_{m}=(-1)^{m+j} c_{-j} . \tag{4.135}
\end{equation*}
$$

Acting twice with the time reversal operator

$$
\begin{align*}
\Theta^{2}|j, m\rangle & =\Theta(-1)^{m+j} c_{-j}|j,-m\rangle \\
& =(-1)^{m-m+2 j} c_{-j}^{*} c_{-j}|j, m\rangle  \tag{4.136}\\
& =(-1)^{2 j}\left|c_{-j}\right|^{2}|j, m\rangle
\end{align*}
$$

The $c_{-j}$ factor can be absorbed into the definition of the time reversal operator, so that, up to a phase factor, we have

$$
\begin{equation*}
\Theta|j, m\rangle=(-1)^{m}|j, m\rangle \tag{4.137}
\end{equation*}
$$

and

$$
\begin{equation*}
\Theta^{2}=(-1)^{2 j} \tag{4.138}
\end{equation*}
$$

Exercise 4.5 Non-interacting particles in a box. ([15] pr. 4.1)
Calculate the three lowest energy levels and their degeneracies for equal mass distinguishable spin half particles in a box of length $L$.

Consider
a. Two particles.
b. Three particles.
c. Four particles.

Answer for Exercise 4.5

Part $a$. The problem statement doesn't include the dimensionality of the box. The simplest case is the one dimensional box, for which the wave function of one particle is

$$
\begin{equation*}
\psi_{1}(x)=\sqrt{\frac{2}{L}} \sin \left(\frac{n \pi x}{L}\right) \tag{4.139}
\end{equation*}
$$

and the energy of that particle is

$$
\begin{equation*}
E=\frac{1}{2 m}\left(\frac{\hbar \pi}{L}\right)^{2} n^{2} . \tag{4.140}
\end{equation*}
$$

If the box is two dimensional the energy is

$$
\begin{equation*}
E=\frac{1}{2 m}\left(\frac{\hbar \pi}{L}\right)^{2}\left(n_{1}^{2}+n_{2}^{2}\right), \tag{4.141}
\end{equation*}
$$

and if it's a 3D box, we have

$$
\begin{equation*}
E=\frac{1}{2 m}\left(\frac{\hbar \pi}{L}\right)^{2}\left(n_{1}^{2}+n_{2}^{2}+n_{3}^{2}\right) . \tag{4.142}
\end{equation*}
$$

Suppose we are considering the 3D box. In statistical mechanics when we are considering particles Fermions, they are indistinguishable, and thus not allowed to share the same spin state at a given energy level. However, for distinguishable particles, that restriction doesn't exist, and we can have two (or more) such particles in the lowest order energy state. The lowest such energy is

$$
\begin{align*}
E_{1,1,1 ; 1,1,1} & =\frac{1}{2 m}\left(\frac{\hbar \pi}{L}\right)^{2}\left(6 \times 1^{2}\right) \\
& =\frac{6}{2 m}\left(\frac{\hbar \pi}{L}\right)^{2} \tag{4.143}
\end{align*}
$$

The particle spin states can be any of $|++\rangle,|+-\rangle,|-+\rangle,|--\rangle$, so there is a four way degeneracy in the ground state.
the next lowest energy level is

$$
\begin{align*}
E_{1,1,2 ; 1,1,1} & =\frac{1}{2 m}\left(\frac{\hbar \pi}{L}\right)^{2}\left(5 \times 1^{2}+2^{2}\right) \\
& =\frac{9}{2 m}\left(\frac{\hbar \pi}{L}\right)^{2}, \tag{4.144}
\end{align*}
$$

where there are $\binom{6}{1}=6$ ways to pick such a state for each variation of spin, for a total $6 \times 4=24$ way degeneracy. Finally, since $2^{2}+2^{2}<3^{2}+1^{2}$, the next lowest energy level is

$$
\begin{align*}
E_{1,2,2 ; 1,1,1} & =\frac{1}{2 m}\left(\frac{\hbar \pi}{L}\right)^{2}\left(4 \times 1^{2}+2 \times 2^{2}\right) \\
& =\frac{12}{2 m}\left(\frac{\hbar \pi}{L}\right)^{2} \tag{4.145}
\end{align*}
$$

with a $\binom{6}{2} \times 4=15 \times 4=60$ way degeneracy for this energy level.
Part $b$. For three particles (the two particle case wasn't actually in the problem statement, but seemed an easier starting place), the lowest energy state for a 3D box is

$$
\begin{align*}
E & =\frac{1}{2 m}\left(\frac{\hbar \pi}{L}\right)^{2}\left(9 \times 1^{2}\right) \\
& =\frac{9}{2 m}\left(\frac{\hbar \pi}{L}\right)^{2} \tag{4.146}
\end{align*}
$$

There are now $2^{3}=8$ variations of spin $|+++\rangle,|++-\rangle, \cdots$, so the ground state is 8 -way degenerate. Next up is

$$
\begin{align*}
E & =\frac{1}{2 m}\left(\frac{\hbar \pi}{L}\right)^{2}\left(8 \times 1^{2}+2^{2}\right) \\
& =\frac{12}{2 m}\left(\frac{\hbar \pi}{L}\right)^{2} \tag{4.147}
\end{align*}
$$

where there is a $\binom{9}{1} \times 8=9 \times 8=72$ way degeneracy in this energy level. Finally, the next lowest energy level is

$$
\begin{align*}
E & =\frac{1}{2 m}\left(\frac{\hbar \pi}{L}\right)^{2}\left(7 \times 1^{2}+2 \times 2^{2}\right) \\
& =\frac{15}{2 m}\left(\frac{\hbar \pi}{L}\right)^{2} \tag{4.148}
\end{align*}
$$

with a $\binom{9}{2} \times 8=36 \times 8=288$ way degeneracy for this energy level.

Part c. For four particles the lowest energy state for a 3D box is

$$
\begin{align*}
E & =\frac{1}{2 m}\left(\frac{\hbar \pi}{L}\right)^{2}\left(12 \times 1^{2}\right) \\
& =\frac{12}{2 m}\left(\frac{\hbar \pi}{L}\right)^{2} . \tag{4.149}
\end{align*}
$$

There are now $2^{4}=16$ variations of spin $|++++\rangle,|+++-\rangle, \cdots$, so the ground state is 16 -way degenerate. For the second level

$$
\begin{align*}
E & =\frac{1}{2 m}\left(\frac{\hbar \pi}{L}\right)^{2}\left(11 \times 1^{2}+2^{2}\right) \\
& =\frac{15}{2 m}\left(\frac{\hbar \pi}{L}\right)^{2}, \tag{4.150}
\end{align*}
$$

where there is a $\binom{12}{1} \times 16=12 \times 16=192$ way degeneracy in this energy level. Finally, the next lowest energy level is

$$
\begin{align*}
E & =\frac{1}{2 m}\left(\frac{\hbar \pi}{L}\right)^{2}\left(7 \times 1^{2}+2 \times 2^{2}\right) \\
& =\frac{15}{2 m}\left(\frac{\hbar \pi}{L}\right)^{2}, \tag{4.151}
\end{align*}
$$

with a $\binom{12}{2} \times 16=66 \times 16=1056$ way degeneracy for this energy level.
Exercise 4.6 Commutators for some symmetry operators. ([15] pr. 4.2)
If $\mathcal{T}_{\mathbf{d}}, \mathscr{D}(\hat{\mathbf{n}}, \phi)$, and $\pi$ denote the translation, rotation, and parity operators respectively. Which of the following commute and why
a. $\mathscr{T}_{\mathbf{d}}$ and $\mathscr{T}_{\mathbf{d}^{\prime}}$, translations in different directions.
b. $\mathscr{D}(\hat{\mathbf{n}}, \phi)$ and $\mathscr{D}\left(\hat{\mathbf{n}}^{\prime}, \phi^{\prime}\right)$, rotations in different directions.
c. $\mathcal{T}_{\mathrm{d}}$ and $\pi$.
d. $\mathscr{D}(\hat{\mathbf{n}}, \phi)$ and $\pi$.

Answer for Exercise 4.6

Part a. Consider

$$
\begin{align*}
\mathcal{T}_{\mathbf{d}} \mathcal{T}_{\mathbf{d}^{\prime}}|\mathbf{x}\rangle & =\mathscr{T}_{\mathbf{d}}\left|\mathbf{x}+\mathbf{d}^{\prime}\right\rangle  \tag{4.152}\\
& =\left|\mathbf{x}+\mathbf{d}^{\prime}+\mathbf{d}\right\rangle,
\end{align*}
$$

and the reverse application of the translation operators

$$
\begin{align*}
\mathcal{T}_{\mathbf{d}^{\prime}} \mathscr{T}_{\mathbf{d}}|\mathbf{x}\rangle & =\mathscr{T}_{\mathbf{d}^{\prime}}|\mathbf{x}+\mathbf{d}\rangle \\
& =\left|\mathbf{x}+\mathbf{d}+\mathbf{d}^{\prime}\right\rangle  \tag{4.153}\\
& =\left|\mathbf{x}+\mathbf{d}^{\prime}+\mathbf{d}\right\rangle .
\end{align*}
$$

so we see that

$$
\begin{equation*}
\left[\mathcal{T}_{\mathbf{d}}, \mathcal{T}_{\mathbf{d}^{\prime}}\right]|\mathbf{x}\rangle=0 \tag{4.154}
\end{equation*}
$$

for any position state $|\mathbf{x}\rangle$, and therefore in general they commute.
Part b. That rotations do not commute when they are in different directions (like any two orthogonal directions) need not be belaboured.

Partc. We have

$$
\begin{align*}
\mathcal{T}_{\mathbf{d}} \pi|\mathbf{x}\rangle & =\mathcal{T}_{\mathbf{d}}|-\mathbf{x}\rangle  \tag{4.155}\\
& =|-\mathbf{x}+\mathbf{d}\rangle,
\end{align*}
$$

yet

$$
\begin{align*}
\pi \mathscr{T}_{\mathbf{d}}|\mathbf{x}\rangle & =\pi|\mathbf{x}+\mathbf{d}\rangle \\
& =|-\mathbf{x}-\mathbf{d}\rangle \neq|-\mathbf{x}+\mathbf{d}\rangle \tag{4.156}
\end{align*}
$$

so, in general $\left[\mathcal{T}_{\mathbf{d}}, \pi\right] \neq 0$.

Part d. We have

$$
\begin{align*}
\pi \mathscr{D}(\hat{\mathbf{n}}, \phi)|\mathbf{x}\rangle & =\pi \mathscr{D}(\hat{\mathbf{n}}, \phi) \pi^{\dagger} \pi|\mathbf{x}\rangle \\
& =\pi \mathscr{D}(\hat{\mathbf{n}}, \phi) \pi^{\dagger} \pi|\mathbf{x}\rangle \\
& =\pi\left(\sum_{k=0}^{\infty} \frac{(-i \mathbf{J} \cdot \hat{\mathbf{n}})^{k}}{k!}\right) \pi^{\dagger} \pi|\mathbf{x}\rangle \\
& =\sum_{k=0}^{\infty} \frac{\left(-i\left(\pi \mathbf{J} \pi^{\dagger}\right) \cdot\left(\pi \hat{\mathbf{n}} \pi^{\dagger}\right)\right)^{k}}{k!} \pi|\mathbf{x}\rangle  \tag{4.157}\\
& =\sum_{k=0}^{\infty} \frac{(-i \mathbf{J} \cdot \hat{\mathbf{n}})^{k}}{k!} \pi|\mathbf{x}\rangle \\
& =\mathscr{D}(\hat{\mathbf{n}}, \phi) \pi|\mathbf{x}\rangle
\end{align*}
$$

so $[\mathscr{D}(\hat{\mathbf{n}}, \phi), \pi]|\mathbf{x}\rangle=0$, for any position state $|\mathbf{x}\rangle$, and therefore these operators commute in general.

## Exercise 4.7 Plane wave and spinor under time reversal. ([15] pr.4.7)

a. Find the time reversed form of a spinless plane wave state in three dimensions.
b. For the eigenspinor of $\sigma \cdot \hat{\mathbf{n}}$ expressed in terms of polar and azimuthal angles $\beta$ and $\gamma$, show that $-i \sigma_{y} \chi^{*}(\hat{\mathbf{n}})$ has the reversed spin direction.

Answer for Exercise 4.7

Part a. The Hamiltonian for a plane wave is

$$
\begin{equation*}
H=\frac{\mathbf{p}^{2}}{2 m}=i \frac{\partial}{\partial t} \tag{4.158}
\end{equation*}
$$

Under time reversal the momentum side transforms as

$$
\begin{align*}
\Theta \frac{\mathbf{p}^{2}}{2 m} \Theta^{-1} & =\frac{\left(\Theta \mathbf{p} \Theta^{-1}\right) \cdot\left(\Theta \mathbf{p} \Theta^{-1}\right)}{2 m} \\
& =\frac{(-\mathbf{p}) \cdot(-\mathbf{p})}{2 m}  \tag{4.159}\\
& =\frac{\mathbf{p}^{2}}{2 m}
\end{align*}
$$

The time derivative side of the equation is also time reversal invariant

$$
\begin{align*}
\Theta i \frac{\partial}{\partial t} \Theta^{-1} & =\Theta i \Theta^{-1} \Theta \frac{\partial}{\partial t} \Theta^{-1} \\
& =-i \frac{\partial}{\partial(-t)}=i \frac{\partial}{\partial t} \tag{4.160}
\end{align*}
$$

Solutions to this equation are linear combinations of

$$
\begin{equation*}
\psi(\mathbf{x}, t)=e^{i \mathbf{k} \cdot \mathbf{x}-i E t / \hbar} \tag{4.161}
\end{equation*}
$$

where $\hbar^{2} \mathbf{k}^{2} / 2 m=E$, the energy of the particle. Under time reversal we have

$$
\begin{align*}
\psi(\mathbf{x}, t) \rightarrow e^{-i \mathbf{k} \cdot \mathbf{x}+i E(-t) / \hbar} & =\left(e^{i \mathbf{k} \cdot \mathbf{x}-i E(-t) / \hbar}\right)^{*}  \tag{4.162}\\
& =\psi^{*}(\mathbf{x},-t)
\end{align*}
$$

Part b. The text uses a requirement for time reversal of spin states to show that the Pauli matrix form of the time reversal operator is

$$
\begin{equation*}
\Theta=-i \sigma_{y} \eta K \tag{4.163}
\end{equation*}
$$

where $K$ is a complex conjugating operator, and $\eta$ is a phase factor with $|\eta|^{2}=1$. The form of the spin up state used in that demonstration was

$$
\begin{align*}
|\hat{\mathbf{n}} ;+\rangle & =e^{-i S_{z} \beta / \hbar} e^{-i S_{y} \gamma / \hbar}|+\rangle \\
& =e^{-i \sigma_{z} \beta / 2} e^{-i \sigma_{y} \gamma / 2}|+\rangle \\
& =\left(\cos (\beta / 2)-i \sigma_{z} \sin (\beta / 2)\right)\left(\cos (\gamma / 2)-i \sigma_{y} \sin (\gamma / 2)\right)|+\rangle \\
& =\left(\begin{array}{cc}
\cos (\beta / 2)-i \\
& \left.=\left[\begin{array}{cc}
1 & 0 \\
0 & -1
\end{array}\right] \sin (\beta / 2)\right)\left(\cos (\gamma / 2)-i\left[\begin{array}{cc}
0 & -i \\
i & 0
\end{array}\right] \sin (\gamma / 2)\right)|+\rangle \\
0 & e^{-i \beta / 2}
\end{array}\right]\left[\begin{array}{cc}
\cos (\gamma / 2) & -\sin (\gamma / 2) \\
\sin (\gamma / 2) & \cos (\gamma / 2)
\end{array}\right]\left[\begin{array}{l}
1 \\
0
\end{array}\right] \\
& =\left[\begin{array}{cc}
e^{-i \beta / 2} & 0 \\
0 & e^{i \beta / 2}
\end{array}\right]\left[\begin{array}{l}
\cos (\gamma / 2) \\
\sin (\gamma / 2)
\end{array}\right] \\
& =\left[\begin{array}{cc}
\cos (\gamma / 2) e^{-i \beta / 2} \\
\sin (\gamma / 2) e^{i \beta / 2}
\end{array}\right] .
\end{align*}
$$

The state orthogonal to this one is claimed to be

$$
\begin{align*}
|\hat{\mathbf{n}} ;-\rangle & =e^{-i S_{z} \beta / \hbar} e^{-i S_{y}(\gamma+\pi) / \hbar}|+\rangle  \tag{4.165}\\
& =e^{-i \sigma_{z} \beta / 2} e^{-i \sigma_{y}(\gamma+\pi) / 2}|+\rangle .
\end{align*}
$$

We have

$$
\begin{align*}
\cos ((\gamma+\pi) / 2) & =\operatorname{Re} e^{i(\gamma+\pi) / 2} \\
& =\operatorname{Re} i e^{i \gamma / 2}  \tag{4.166}\\
& =-\sin (\gamma / 2)
\end{align*}
$$

and

$$
\begin{align*}
\sin ((\gamma+\pi) / 2) & =\operatorname{Im} e^{i(\gamma+\pi) / 2} \\
& =\operatorname{Im} i e^{i \gamma / 2}  \tag{4.167}\\
& =\cos (\gamma / 2)
\end{align*}
$$

so we should have

$$
|\hat{\mathbf{n}} ;-\rangle=\left[\begin{array}{c}
-\sin (\gamma / 2) e^{-i \beta / 2}  \tag{4.168}\\
\cos (\gamma / 2) e^{i \beta / 2}
\end{array}\right]
$$

This looks right, but we can sanity check orthogonality

$$
\begin{align*}
\langle\hat{\mathbf{n}} ;-\mid \hat{\mathbf{n}} ;+\rangle & =\left[\begin{array}{ll}
-\sin (\gamma / 2) e^{i \beta / 2} & \cos (\gamma / 2) e^{-i \beta / 2}
\end{array}\right]\left[\begin{array}{c}
\cos (\gamma / 2) e^{-i \beta / 2} \\
\sin (\gamma / 2) e^{i \beta / 2}
\end{array}\right]  \tag{4.169}\\
& =0
\end{align*}
$$

as expected.
The task at hand appears to be the operation on the column representation of $|\hat{\mathbf{n}} ;+\rangle$ using the Pauli representation of the time reversal operator. With the phase factor $\eta=1$ the time reversal action on the spin up state is

$$
\begin{align*}
\Theta|\hat{\mathbf{n}} ;+\rangle & =-i \sigma_{y} K\left[\begin{array}{c}
e^{-i \beta / 2} \cos (\gamma / 2) \\
e^{i \beta / 2} \sin (\gamma / 2)
\end{array}\right] \\
& =-i\left[\begin{array}{cc}
0 & -i \\
i & 0
\end{array}\right]\left[\begin{array}{c}
e^{i \beta / 2} \cos (\gamma / 2) \\
e^{-i \beta / 2} \sin (\gamma / 2)
\end{array}\right] \\
& =\left[\begin{array}{cc}
0 & -1 \\
1 & 0
\end{array}\right]\left[\begin{array}{c}
e^{i \beta / 2} \cos (\gamma / 2) \\
e^{-i \beta / 2} \sin (\gamma / 2)
\end{array}\right]  \tag{4.170}\\
& =\left[\begin{array}{cc}
-e^{-i \beta / 2} \sin (\gamma / 2) \\
e^{i \beta / 2} \cos (\gamma / 2)
\end{array}\right] \\
& =|\hat{\mathbf{n}} ;-\rangle . \quad \square
\end{align*}
$$

Observe that we need $\eta=i$ to have this match eq. (4.79) in the text where $\Theta=i^{2 m}|j,-m\rangle$.

Exercise 4.8 Totally asymmetric potential. ([15] pr. 4.11)
a. Given a time reversal invariant Hamiltonian, show that for any energy eigenket

$$
\begin{equation*}
\langle\mathbf{L}\rangle=0 \tag{4.171}
\end{equation*}
$$

b. If the wave function of such a state is expanded as

$$
\begin{equation*}
\sum_{l, m} F_{l m} Y_{l m}(\theta, \phi) \tag{4.172}
\end{equation*}
$$

what are the phase restrictions on $F_{l m}$ ?
Answer for Exercise 4.8

Part a. For a time reversal invariant Hamiltonian $H$ we have

$$
\begin{equation*}
H \Theta=\Theta H \tag{4.173}
\end{equation*}
$$

If $|\psi\rangle$ is an energy eigenstate with eigenvalue $E$, we have

$$
\begin{align*}
H \Theta|\psi\rangle & =\Theta H|\psi\rangle  \tag{4.174}\\
& =\lambda \Theta|\psi\rangle
\end{align*}
$$

so $\Theta|\psi\rangle$ is also an eigenvalue of $H$, so can only differ from $|\psi\rangle$ by a phase factor. That is

$$
\begin{align*}
\left|\psi^{\prime}\right\rangle & =\Theta|\psi\rangle  \tag{4.175}\\
& =e^{i \delta}|\psi\rangle
\end{align*}
$$

Now consider the expectation of $\mathbf{L}$ with respect to a time reversed state

$$
\begin{align*}
\left\langle\psi^{\prime}\right| \mathbf{L}\left|\psi^{\prime}\right\rangle & =\langle\psi| \Theta^{-1} \mathbf{L} \Theta|\psi\rangle  \tag{4.176}\\
& =\langle\psi|(-\mathbf{L})|\psi\rangle
\end{align*}
$$

however, we also have

$$
\begin{align*}
\left\langle\psi^{\prime}\right| \mathbf{L}\left|\psi^{\prime}\right\rangle & =\left(\langle\psi| e^{-i \delta}\right) \mathbf{L}\left(e^{i \delta}|\psi\rangle\right)  \tag{4.177}\\
& =\langle\psi| \mathbf{L}|\psi\rangle
\end{align*}
$$

so we have $\langle\psi| \mathbf{L}|\psi\rangle=-\langle\psi| \mathbf{L}|\psi\rangle$ which is only possible if $\langle\mathbf{L}\rangle=\langle\psi| \mathbf{L}|\psi\rangle=$ 0 .

Part b. Consider the expansion of the wave function of a time reversed energy eigenstate

$$
\begin{align*}
\langle\mathbf{x}| \Theta|\psi\rangle & =\langle\mathbf{x}| e^{i \delta}|\psi\rangle \\
& =e^{i \delta}\langle\mathbf{x} \mid \psi\rangle, \tag{4.178}
\end{align*}
$$

and then consider the same state expanded in the position basis

$$
\begin{align*}
\langle\mathbf{x}| \Theta|\psi\rangle & =\langle\mathbf{x}| \Theta \int d^{3} \mathbf{x}^{\prime}\left(\left|\mathbf{x}^{\prime}\right\rangle\left\langle\mathbf{x}^{\prime}\right|\right)|\psi\rangle \\
& =\langle\mathbf{x}| \Theta \int d^{3} \mathbf{x}^{\prime}\left(\left\langle\mathbf{x}^{\prime} \mid \psi\right\rangle\right)\left|\mathbf{x}^{\prime}\right\rangle \\
& =\langle\mathbf{x}| \int d^{3} \mathbf{x}^{\prime}\left(\left\langle\mathbf{x}^{\prime} \mid \psi\right\rangle\right)^{*} \Theta\left|\mathbf{x}^{\prime}\right\rangle \\
& =\langle\mathbf{x}| \int d^{3} \mathbf{x}^{\prime}\left(\left\langle\mathbf{x}^{\prime} \mid \psi\right\rangle\right)^{*}\left|\mathbf{x}^{\prime}\right\rangle  \tag{4.179}\\
& =\int d^{3} \mathbf{x}^{\prime}\left(\left\langle\mathbf{x}^{\prime} \mid \psi\right\rangle\right)^{*}\left\langle\mathbf{x} \mid \mathbf{x}^{\prime}\right\rangle \\
& =\int d^{3} \mathbf{x}^{\prime}\left\langle\psi \mid \mathbf{x}^{\prime}\right\rangle \delta\left(\mathbf{x}-\mathbf{x}^{\prime}\right) \\
& =\langle\psi \mid \mathbf{x}\rangle .
\end{align*}
$$

This demonstrates a relationship between the wave function and its complex conjugate

$$
\begin{equation*}
\langle\mathbf{x} \mid \psi\rangle=e^{-i \delta}\langle\psi \mid \mathbf{x}\rangle \tag{4.180}
\end{equation*}
$$

Now expand the wave function in the spherical harmonic basis

$$
\begin{align*}
\langle\mathbf{x} \mid \psi\rangle & =\int d \Omega\langle\mathbf{x} \mid \hat{\mathbf{n}}\rangle\langle\hat{\mathbf{n}} \mid \psi\rangle \\
& =\sum_{l m} F_{l m}(r) Y_{l m}(\theta, \phi) \\
& =e^{-i \delta}\left(\sum_{l m} F_{l m}(r) Y_{l m}(\theta, \phi)\right)^{*}  \tag{4.181}\\
& =e^{-i \delta} \sum_{l m}\left(F_{l m}(r)\right)^{*} Y_{l m}^{*}(\theta, \phi) \\
& =e^{-i \delta} \sum_{l m}\left(F_{l m}(r)\right)^{*}(-1)^{m} Y_{l,-m}(\theta, \phi) \\
& =e^{-i \delta} \sum_{l m}\left(F_{l,-m}(r)\right)^{*}(-1)^{m} Y_{l, m}(\theta, \phi),
\end{align*}
$$

so the $F_{l m}$ functions are constrained by

$$
\begin{equation*}
F_{l m}(r)=e^{-i \delta}\left(F_{l,-m}(r)\right)^{*}(-1)^{m} \tag{4.182}
\end{equation*}
$$

Exercise 4.9 Time reversal, crystal spin Hamiltonian. ([15] pr. 4.12)
Solve the spin 1 Hamiltonian

$$
\begin{equation*}
H=A S_{z}^{2}+B\left(S_{x}^{2}-S_{y}^{2}\right) \tag{4.183}
\end{equation*}
$$

Is this Hamiltonian invariant under time reversal? How do the eigenkets change under time reversal?
Answer for Exercise 4.9
In spinMatrices.nb the matrix representation of the Hamiltonian is found to be

$$
H=\hbar^{2}\left[\begin{array}{lll}
A & 0 & B  \tag{4.184}\\
0 & 0 & 0 \\
B & 0 & A
\end{array}\right]
$$

The eigenvalues are

$$
\begin{equation*}
\{0, A-B, A+B\} \tag{4.185}
\end{equation*}
$$

and the respective eigenvalues (unnormalized) are

$$
\left\{\left[\begin{array}{l}
0  \tag{4.186}\\
1 \\
0
\end{array}\right],\left[\begin{array}{c}
-1 \\
0 \\
1
\end{array}\right],\left[\begin{array}{l}
1 \\
0 \\
1
\end{array}\right]\right\} .
$$

Under time reversal, the Hamiltonian is

$$
\begin{equation*}
H \rightarrow A\left(-S_{z}\right)^{2}+B\left(\left(-S_{x}\right)^{2}-\left(-S_{y}\right)^{2}\right)=H \tag{4.187}
\end{equation*}
$$

so we expect the eigenkets for this Hamiltonian to vary by at most a phase factor. To check this, first recall that the time reversal action on a spin one state is

$$
\begin{equation*}
\Theta|1, m\rangle=(-1)^{m}|1,-m\rangle, \tag{4.188}
\end{equation*}
$$

or

$$
\begin{align*}
\Theta|1,1\rangle & =-|1,-1\rangle \\
\Theta|1,0\rangle & =|1,0\rangle  \tag{4.189}\\
\Theta|1,-1\rangle & =-|1,1\rangle
\end{align*}
$$

Let's write the eigenkets respectively as

$$
\begin{align*}
|0\rangle & =|1,0\rangle \\
|A-B\rangle & =-|1,-1\rangle+|1,1\rangle  \tag{4.190}\\
|A+B\rangle & =|1,-1\rangle+|1,1\rangle
\end{align*}
$$

Under the reversal operation, we should have

$$
\begin{align*}
\Theta|0\rangle & \rightarrow|1,0\rangle \\
\Theta|A-B\rangle & =+|1,-1\rangle-|1,1\rangle  \tag{4.191}\\
\Theta|A+B\rangle & =-|1,-1\rangle-|1,1\rangle
\end{align*}
$$

Up to a sign, the time reversed states match the unreversed states, which makes sense given the Hamiltonian invariance.

## THEORY OF ANGULAR MOMENTUM.

### 5.1 ANGULAR MOMENTUM.

In classical mechanics the (orbital) angular momentum is

$$
\begin{equation*}
\mathbf{L}=\mathbf{r} \times \mathbf{p} \tag{5.1}
\end{equation*}
$$

Here "orbital" is to distinguish from spin angular momentum.
In quantum mechanics, the mapping to operators, in component form, is

$$
\begin{equation*}
\hat{L}_{i}=\epsilon_{i j k} \hat{r}_{j} \hat{p}_{k} \tag{5.2}
\end{equation*}
$$

These operators do not commute

$$
\left.\begin{array}{rl}
{\left[\hat{L}_{i}, \hat{L}_{j}\right]} \tag{5.3}
\end{array}\right] .
$$

which means that we can't simultaneously determine $\hat{L}_{i}$ for all $i$.
Aside: In quantum mechanics, we define an operator $\hat{\mathbf{V}}$ to be a vector operator if

$$
\begin{equation*}
\left[\hat{L}_{i}, \hat{V}_{j}\right]=i \hbar \epsilon_{i j k} \hat{V}_{k} \tag{5.4}
\end{equation*}
$$

The commutator of the squared angular momentum operator with any $\hat{L}_{i}$, say $\hat{L}_{x}$ is zero

$$
\begin{align*}
{\left[\hat{L}_{x}^{2}+\hat{L}_{y}^{2}+\hat{L}_{z}^{2}, \hat{L}_{x}\right]=} & \hat{L}_{y} \hat{L}_{y} \hat{L}_{x}-\hat{L}_{x} \hat{L}_{y} \hat{L}_{y}+\hat{L}_{z} \hat{L}_{z} \hat{L}_{x}-\hat{L}_{x} \hat{L}_{z} \hat{L}_{z} \\
= & \hat{L}_{y}\left(\left[\hat{L}_{y}, \hat{L}_{x}\right]+\hat{L}_{x} \hat{L}_{y}\right)-\left(\left[\hat{L}_{x}, \hat{L}_{y}\right]+\hat{L}_{y} \hat{L}_{x}\right) \hat{L}_{y} \\
& +\hat{L}_{z}\left(\left[\hat{L}_{z}, \hat{L}_{x}\right]+\hat{L}_{x} \hat{L}_{z}\right)-\left(\left[\hat{L}_{x}, \hat{L}_{z}\right]+\hat{L}_{z} \hat{L}_{x}\right) \hat{L}_{z} \\
= & \hat{L}_{y}\left[\hat{L}_{y}, \hat{L}_{x}\right]-\left[\hat{L}_{x}, \hat{L}_{y}\right] \hat{L}_{y}+\hat{L}_{z}\left[\hat{L}_{z}, \hat{L}_{x}\right]-\left[\hat{L}_{x}, \hat{L}_{z}\right] \hat{L}_{z} \\
= & i \hbar\left(-\hat{L}_{y} \hat{L}_{z}-\hat{L}_{z} \hat{L}_{y}+\hat{L}_{z} \hat{L}_{y}+\hat{L}_{y} \hat{L}_{z}\right) \\
= & 0 \tag{5.5}
\end{align*}
$$

Suppose we have a state $|\Psi\rangle$ with a well defined $\hat{L}_{z}$ eigenvalue and well defined $\hat{\mathbf{L}}^{2}$ eigenvalue, written as

$$
\begin{equation*}
|\Psi\rangle=|a, b\rangle, \tag{5.6}
\end{equation*}
$$

where the label $a$ is used for the eigenvalue of $\hat{\mathbf{L}}^{2}$ and $b$ labels the eigenvalue of $\hat{L}_{z}$. Then

$$
\begin{align*}
\hat{\mathbf{L}}^{2}|a, b\rangle & =\hbar^{2} a|a, b\rangle \\
\hat{L}_{z}|a, b\rangle & =\hbar b|a, b\rangle . \tag{5.7}
\end{align*}
$$

Things aren't so nice when we act with other angular momentum operators, producing a scrambled mess

$$
\begin{align*}
& \hat{L}_{x}|a, b\rangle=\sum_{a^{\prime}, b^{\prime}} \mathcal{A}_{a, b, a^{\prime}, b^{\prime}}^{x}\left|a^{\prime}, b^{\prime}\right\rangle, \\
& \hat{L}_{y}|a, b\rangle=\sum_{a^{\prime}, b^{\prime}} \mathcal{A}_{a, b, a^{\prime}, b^{\prime}}^{y}\left|a^{\prime}, b^{\prime}\right\rangle . \tag{5.8}
\end{align*}
$$

With this representation, we have

$$
\begin{align*}
& \hat{L}_{x} \hat{\mathbf{L}}^{2}|a, b\rangle=\hat{L}_{x} \hbar^{2} a \sum_{a^{\prime}, b^{\prime}} \mathcal{A}_{a, b, a^{\prime}, b^{\prime}}^{x}\left|a^{\prime}, b^{\prime}\right\rangle,  \tag{5.9}\\
& \hat{\mathbf{L}}^{2} \hat{L}_{x}|a, b\rangle=\hbar^{2} \sum_{a^{\prime}, b^{\prime}} a^{\prime} \mathcal{A}_{a, b, a^{\prime}, b^{\prime}}^{x}\left|a^{\prime}, b^{\prime}\right\rangle . \tag{5.10}
\end{align*}
$$

Since $\hat{\mathbf{L}}^{2}, \hat{L}_{x}$ commute, we must have

$$
\begin{equation*}
\mathcal{A}_{a, b, a^{\prime}, b^{\prime}}^{x}=\delta_{a, a^{\prime}} \mathcal{A}_{a^{\prime} ; b, b^{\prime}}^{x}, \tag{5.11}
\end{equation*}
$$

and similarly

$$
\begin{equation*}
\mathcal{A}_{a, b, a^{\prime}, b^{\prime}}^{y}=\delta_{a, a^{\prime}} \mathcal{A}_{a^{\prime} ; b, b^{\prime}}^{y} . \tag{5.12}
\end{equation*}
$$

Simplifying things we can write the action of $\hat{L}_{x}, \hat{L}_{y}$ on the state as

$$
\begin{align*}
& \hat{L}_{x}|a, b\rangle=\sum_{b^{\prime}} \mathcal{A}_{a ; b, b^{\prime}}^{x}\left|a, b^{\prime}\right\rangle, \\
& \hat{L}_{y}|a, b\rangle=\sum_{b^{\prime}} \mathcal{A}_{a ; b, b^{\prime}}^{y}\left|a, b^{\prime}\right\rangle . \tag{5.13}
\end{align*}
$$

Let's define

$$
\begin{equation*}
\hat{L}_{+} \equiv \hat{L}_{x}+i \hat{L}_{y}, \hat{L}_{-} \equiv \hat{L}_{x}-i \hat{L}_{y} . \tag{5.14}
\end{equation*}
$$

Because these are sums of $\hat{L}_{x}, \hat{L}_{y}$ they must also commute with $\hat{\mathbf{L}}^{2}$

$$
\begin{equation*}
\left[\hat{\mathbf{L}}^{2}, \hat{L}_{ \pm}\right]=0 . \tag{5.15}
\end{equation*}
$$

The commutators with $\hat{L}_{z}$ are non-zero

$$
\begin{align*}
{\left[\hat{L}_{z}, \hat{L}_{ \pm}\right] } & =\hat{L}_{z}\left(\hat{L}_{x} \pm i \hat{L}_{y}\right)-\left(\hat{L}_{x} \pm i \hat{L}_{y}\right) \hat{L}_{z} \\
& =\left[\hat{L}_{z}, \hat{L}_{x}\right] \pm i\left[\hat{L}_{z}, \hat{L}_{y}\right] \\
& =i \hbar\left(\hat{L}_{y} \mp i \hat{L}_{x}\right)  \tag{5.16}\\
& =\hbar\left(i \hat{L}_{y} \pm \hat{L}_{x}\right) \\
& = \pm \hbar\left(\hat{L}_{x} \pm i \hat{L}_{y}\right) \\
& = \pm \hbar \hat{L}_{ \pm} .
\end{align*}
$$

Explicitly, that is

$$
\begin{align*}
& \hat{L}_{z} \hat{L}_{+}-\hat{L}_{+} \hat{L}_{z}=\hbar \hat{L}_{+}  \tag{5.17}\\
& \hat{L}_{z} \hat{L}_{-}-\hat{L}_{-} \hat{L}_{z}=-\hbar \hat{L}_{-}
\end{align*}
$$

Now we are set to compute actions of these (assumed) raising and lowering operators on the eigenstate of $\hat{L}_{z}, \hat{\mathbf{L}}^{2}$

$$
\begin{align*}
\hat{L}_{z} \hat{L}_{ \pm}|a, b\rangle & =\hbar \hat{L}_{ \pm}|a, b\rangle \pm \hat{L}_{ \pm} \hat{L}_{z}|a, b\rangle \\
& =\hbar \hat{L}_{ \pm}|a, b\rangle \pm \hbar b \hat{L}_{ \pm}|a, b\rangle  \tag{5.18}\\
& =\hbar(b \pm 1) \hat{L}_{ \pm}|a, b\rangle .
\end{align*}
$$

There must be a proportionality of the form

$$
\begin{equation*}
\left|\hat{L}_{ \pm}\right\rangle \propto|a, b \pm 1\rangle \tag{5.19}
\end{equation*}
$$

The products of the raising and lowering operators are

$$
\begin{align*}
\hat{L}_{-} \hat{L}_{+} & =\left(\hat{L}_{x}-i \hat{L}_{y}\right)\left(\hat{L}_{x}+i \hat{L}_{y}\right) \\
& =\hat{L}_{x}^{2}+\hat{L}_{y}^{2}+i \hat{L}_{x} \hat{L}_{y}-i \hat{L}_{y} \hat{L}_{x}  \tag{5.20}\\
& =\left(\hat{\mathbf{L}}^{2}-\hat{L}_{z}^{2}\right)+i\left[\hat{L}_{x}, \hat{L}_{y}\right] \\
& =\hat{\mathbf{L}}^{2}-\hat{L}_{z}^{2}-\hbar \hat{L}_{z},
\end{align*}
$$

and

$$
\begin{align*}
\hat{L}_{+} \hat{L}_{-} & =\left(\hat{L}_{x}+i \hat{L}_{y}\right)\left(\hat{L}_{x}-i \hat{L}_{y}\right) \\
& =\hat{L}_{x}^{2}+\hat{L}_{y}^{2}-i \hat{L}_{x} \hat{L}_{y}+i \hat{L}_{y} \hat{L}_{x} \\
& =\left(\hat{\mathbf{L}}^{2}-\hat{L}_{z}^{2}\right)-i\left[\hat{L}_{x}, \hat{L}_{y}\right]  \tag{5.21}\\
& =\hat{\mathbf{L}}^{2}-\hat{L}_{z}^{2}+\hbar \hat{L}_{z}
\end{align*}
$$

So we must have

$$
\begin{equation*}
0 \leq\langle a, b| \hat{L}_{-} \hat{L}_{+}|a, b\rangle=\langle a, b|\left(\hat{\mathbf{L}}^{2}-\hat{L}_{z}^{2}-\hbar \hat{L}_{z}\right)|a, b\rangle=\hbar^{2} a-\hbar^{2} b^{2}-\hbar^{2} b \tag{5.22}
\end{equation*}
$$

and

$$
\begin{equation*}
0 \leq\langle a, b| \hat{L}_{+} \hat{L}_{-}|a, b\rangle=\langle a, b|\left(\hat{\mathbf{L}}^{2}-\hat{L}_{z}^{2}+\hbar \hat{L}_{z}\right)|a, b\rangle=\hbar^{2} a-\hbar^{2} b^{2}+\hbar^{2} b \tag{5.23}
\end{equation*}
$$

This puts constraints on $a, b$, roughly of the form
1.

$$
\begin{equation*}
a-b(b+1) \geq 0 \tag{5.24}
\end{equation*}
$$

$$
\text { with } b_{\max }>0, b_{\max } \approx \sqrt{a}
$$

2. 

$$
\begin{equation*}
a-b(b-1) \geq 0 \tag{5.25}
\end{equation*}
$$

$$
\text { with } b_{\min }<0, b_{\min } \approx-\sqrt{a}
$$

Assuming that the $b_{\text {max }}$ and $b_{\text {min }}$ values are satisfied at the equality extremes we have

$$
\begin{align*}
b_{\max }\left(b_{\max }+1\right) & =a, \\
b_{\min }\left(b_{\min }-1\right) & =a . \tag{5.26}
\end{align*}
$$

Equating these pair of equations and rearranging, we have

$$
\begin{equation*}
\left(b_{\max }+\frac{1}{2}\right)^{2}-\frac{1}{4}=\left(b_{\min }-\frac{1}{2}\right)^{2}-\frac{1}{4} \tag{5.27}
\end{equation*}
$$

which has solutions at

$$
\begin{equation*}
b_{\max }+\frac{1}{2}= \pm\left(b_{\min }-\frac{1}{2}\right) \tag{5.28}
\end{equation*}
$$

One of the solutions is

$$
\begin{equation*}
-b_{\min }=b_{\max } \tag{5.29}
\end{equation*}
$$

as desired. The other solution is $b_{\max }=b_{\min }-1$, which we discard.
The final constraint is therefore

$$
\begin{equation*}
-b_{\max } \leq b \leq b_{\max } \tag{5.30}
\end{equation*}
$$

and

$$
\begin{align*}
\hat{L}_{+}\left|a, b_{\max }\right\rangle & =0, \\
\hat{L}_{-}\left|a, b_{\min }\right\rangle & =0 \tag{5.31}
\end{align*}
$$

If we had the sequence, which must terminate at $b_{\text {min }}$ or else it will go on forever

$$
\begin{equation*}
\left|a, b_{\max }\right\rangle \xrightarrow{\hat{L}_{-}}\left|a, b_{\max }-1\right\rangle \xrightarrow{\hat{L}_{-}}\left|a, b_{\max }-2\right\rangle \cdots \xrightarrow{\hat{L}_{-}}\left|a, b_{\min }\right\rangle, \tag{5.32}
\end{equation*}
$$

then we know that $b_{\text {max }}-b_{\text {min }} \in \mathbb{Z}$, or

$$
\begin{equation*}
b_{\max }-n=b_{\min }=-b_{\max } \tag{5.33}
\end{equation*}
$$

or

$$
\begin{equation*}
b_{\max }=\frac{n}{2} \tag{5.34}
\end{equation*}
$$

this is either an integer or a $1 / 2$ odd integer, depending on whether $n$ is even or odd. These are called "orbital" or "spin" respectively.

The convention is to write $m$ for the $\hat{L}_{z}$ eigenvalue (the magnetic quantum number), and $j$, the azimuthal quantum number, to describe the $j(j+$ 1) eigenvalue of the $\hat{\mathbf{L}}^{2}$ operator.

$$
\begin{align*}
b_{\max } & =j  \tag{5.35}\\
a & =j(j+1) .
\end{align*}
$$

so for $m \in-j,-j+1, \cdots,+j$

$$
\begin{align*}
\hat{\mathbf{L}}^{2}|j, m\rangle & =\hbar^{2} j(j+1)|j, m\rangle  \tag{5.36}\\
L_{z}|j, m\rangle & =\hbar m|j, m\rangle
\end{align*}
$$

### 5.2 SCHWINGER'S SHO ANGULAR MOMENTUM OPERATORS.

In [17] a powerful method for describing angular momentum with harmonic oscillators was introduced, which will be outlined here. The question is whether we can construct a set of harmonic oscillators that allows a mapping from

$$
\begin{equation*}
\hat{L}_{+} \leftrightarrow a^{+} ? \tag{5.37}
\end{equation*}
$$

Picture two harmonic oscillators, one with states counted from one zero towards $\infty$ and another with states counted from a different zero towards $-\infty$, as pictured in fig. 5.1. Is it possible that such an overlapping set of


Figure 5.1: Overlapping SHO domains.
harmonic oscillators can provide the properties of the angular momentum operators? Let's relabel the counting so that we have two sets of positive counted SHO systems, each counted in a positive direction as sketched in fig. 5.2. It turns out that given a constraint there the number of ways to


Figure 5.2: Relabeling the counting for overlapping SHO systems.
distribute particles between a pair of SHO systems, the process that can
be viewed as reproducing the angular momentum action is a transfer of particles from one harmonic oscillator to the other. For $\hat{L}_{z}=+j$

$$
\begin{align*}
& n_{1}=n_{\max }  \tag{5.38}\\
& n_{2}=0,
\end{align*}
$$

and for $\hat{L}_{z}=-j$

$$
\begin{align*}
& n_{1}=0  \tag{5.39}\\
& n_{2}=n_{\max }
\end{align*}
$$

We can make the identifications

$$
\begin{equation*}
\hat{L}_{z}=\left(n_{1}-n_{2}\right) \frac{\hbar}{2} \tag{5.40}
\end{equation*}
$$

and

$$
\begin{equation*}
j=\frac{1}{2} n_{\max } \tag{5.41}
\end{equation*}
$$

or

$$
\begin{equation*}
n_{1}+n_{2}=\text { fixed }=n_{\max } \tag{5.42}
\end{equation*}
$$

Changes that keep $n_{1}+n_{2}$ fixed are those that change $n_{1}, n_{2}$ by +1 or -1 respectively, as sketched in fig. 5.3.


Figure 5.3: Number conservation constraint.

Can we make an identification that takes

$$
\begin{equation*}
\left|n_{1}, n_{2}\right\rangle \xrightarrow{\hat{L}_{-}}\left|n_{1}-1, n_{2}+1\right\rangle ? \tag{5.43}
\end{equation*}
$$

What operator in the SHO problem has this effect? Let's try

$$
\begin{align*}
\hat{L}_{-} & =\hbar a_{2}^{\dagger} a_{1} \\
\hat{L}_{+} & =\hbar a_{1}^{\dagger} a_{2}  \tag{5.44}\\
\hat{L}_{z} & =\frac{\hbar}{2}\left(n_{1}-n_{2}\right)
\end{align*}
$$

Is this correct? Do we need to make any scalar adjustments? We want

$$
\begin{equation*}
\left[\hat{L}_{z}, \hat{L}_{ \pm}\right]= \pm \hbar \hat{L}_{ \pm} \tag{5.45}
\end{equation*}
$$

First check this with the $\hat{L}_{+}$commutator

$$
\begin{align*}
{\left[\hat{L}_{z}, \hat{L}_{+}\right] } & =\frac{1}{2} \hbar^{2}\left[n_{1}-n_{2}, a_{1}^{\dagger} a_{2}\right] \\
& =\frac{1}{2} \hbar^{2}\left[a_{1}^{\dagger} a_{1}-a_{2}^{\dagger} a_{2}, a_{1}^{\dagger} a_{2}\right] \\
& =\frac{1}{2} \hbar^{2}\left(\left[a_{1}^{\dagger} a_{1}, a_{1}^{\dagger} a_{2}\right]-\left[a_{2}^{\dagger} a_{2}, a_{1}^{\dagger} a_{2}\right]\right)  \tag{5.46}\\
& =\frac{1}{2} \hbar^{2}\left(a_{2}\left[a_{1}^{\dagger} a_{1}, a_{1}^{\dagger}\right]-a_{1}^{\dagger}\left[a_{2}^{\dagger} a_{2}, a_{2}\right]\right)
\end{align*}
$$

But

$$
\begin{align*}
{\left[a^{\dagger} a, a^{\dagger}\right] } & =a^{\dagger} a a^{\dagger}-a^{\dagger} a^{\dagger} a \\
& =a^{\dagger}\left(1+a^{\dagger} a\right)-a^{\dagger} a^{\dagger} a  \tag{5.47}\\
& =a^{\dagger}
\end{align*}
$$

and

$$
\begin{align*}
{\left[a^{\dagger} a, a\right] } & =a^{\dagger} a a-a a^{\dagger} a \\
& =a^{\dagger} a a-\left(1+a^{\dagger} a\right) a  \tag{5.48}\\
& =-a
\end{align*}
$$

So

$$
\begin{equation*}
\left[\hat{L}_{z}, \hat{L}_{+}\right]=\hbar^{2} a_{2} a_{1}^{\dagger}=\hbar \hat{L}_{+} \tag{5.49}
\end{equation*}
$$

as desired. Similarly

$$
\begin{align*}
{\left[\hat{L}_{z}, \hat{L}_{-}\right] } & =\frac{1}{2} \hbar^{2}\left[n_{1}-n_{2}, a_{2}^{\dagger} a_{1}\right] \\
& =\frac{1}{2} \hbar^{2}\left[a_{1}^{\dagger} a_{1}-a_{2}^{\dagger} a_{2}, a_{2}^{\dagger} a_{1}\right] \\
& =\frac{1}{2} \hbar^{2}\left(a_{2}^{\dagger}\left[a_{1}^{\dagger} a_{1}, a_{1}\right]-a_{1}\left[a_{2}^{\dagger} a_{2}, a_{2}^{\dagger}\right]\right)  \tag{5.50}\\
& =\frac{1}{2} \hbar^{2}\left(a_{2}^{\dagger}\left(-a_{1}\right)-a_{1} a_{2}^{\dagger}\right) \\
& =-\hbar^{2} a_{2}^{\dagger} a_{1} \\
& =-\hbar \hat{L}_{-}
\end{align*}
$$

With

$$
\begin{align*}
j & =\frac{n_{1}+n_{2}}{2}  \tag{5.51}\\
m & =\frac{n_{1}-n_{2}}{2}
\end{align*}
$$

we can make the identification

$$
\begin{equation*}
\left|n_{1}, n_{2}\right\rangle=|j+m, j-m\rangle \tag{5.52}
\end{equation*}
$$

## Another way With

$$
\begin{equation*}
\hat{L}_{+}|j, m\rangle=d_{j, m}^{+}|j, m+1\rangle \tag{5.53}
\end{equation*}
$$

or

$$
\begin{equation*}
\hbar a_{1}^{\dagger} a_{2}|j+m, j-m\rangle=d_{j, m}^{+}|j+m+1, j-m-1\rangle \tag{5.54}
\end{equation*}
$$

we can seek this factor $d_{j, m}^{+}$by operating with $\hat{L}_{+}$

$$
\begin{align*}
\hat{L}_{+}|j, m\rangle & =\hbar a_{1}^{\dagger} a_{2}\left|n_{1}, n_{2}\right\rangle \\
& =\hbar a_{1}^{\dagger} a_{2}|j+m, j-m\rangle \\
& =\hbar \sqrt{n+1} \sqrt{n_{2}}|j+m+1, j-m-1\rangle  \tag{5.55}\\
& =\hbar \sqrt{(j+m+1)(j-m)}|j+m+1, j-m-1\rangle
\end{align*}
$$

That gives

$$
\begin{align*}
d_{j, m}^{+} & =\hbar \sqrt{(j-m)(j+m+1)} \\
d_{j, m}^{-} & =\hbar \sqrt{(j+m)(j-m+1)} \tag{5.56}
\end{align*}
$$

This equivalence can be used to model spin interaction in crystals as harmonic oscillators. This equivalence of lattice vibrations and spin oscillations is called "spin waves".

### 5.3 REPRESENTATIONS.

It's possible to compute matrix representations of the rotation operators

$$
\begin{equation*}
\hat{R}_{\hat{\mathbf{n}}}(\phi)=e^{i \hat{\mathbf{L}} \cdot \hat{\mathbf{n}} \phi / \hbar} \tag{5.57}
\end{equation*}
$$



Figure 5.4: Block diagonal form for angular momentum matrix representation.

With respect to a ket it's possible to find

$$
\begin{equation*}
e^{i \mathbf{L} \cdot \hat{\mathbf{n}} \phi / \hbar}|j, m\rangle=\sum_{m^{\prime}} d_{m m^{\prime}}^{j}(\hat{\mathbf{n}}, \phi)\left|j, m^{\prime}\right\rangle . \tag{5.58}
\end{equation*}
$$

This has a block diagonal form that's sketched in fig. 5.4.
We can view $d_{m m^{\prime}}^{j}(\hat{\mathbf{n}}, \phi)$ as a matrix, representing the rotation. The problem of determining these matrices can be reduced to that of determining the matrix for $\hat{\mathbf{L}}$, because once we have that we can exponentiate that.

## Example 5.1: Spin half

From the eigenvalue relationships, with basis states

$$
\begin{align*}
& |\uparrow\rangle=\left[\begin{array}{l}
1 \\
0
\end{array}\right], \\
& |\downarrow\rangle=\left[\begin{array}{l}
0 \\
1
\end{array}\right], \tag{5.59}
\end{align*}
$$

we find

$$
\begin{align*}
& \hat{L}_{z}=\frac{\hbar}{2}\left[\begin{array}{cc}
1 & 0 \\
0 & -1
\end{array}\right] \\
& \hat{L}_{+}=\frac{\hbar}{2}\left[\begin{array}{ll}
0 & 1 \\
0 & 0
\end{array}\right]  \tag{5.60}\\
& \hat{L}_{-}=\frac{\hbar}{2}\left[\begin{array}{ll}
0 & 0 \\
1 & 0
\end{array}\right] .
\end{align*}
$$

Rearranging we find the Pauli matrices

$$
\begin{equation*}
\hat{L}_{k}=\frac{1}{2} \hbar \sigma_{i} . \tag{5.61}
\end{equation*}
$$

Noting that $(\boldsymbol{\sigma} \cdot \hat{\mathbf{n}})^{2}=1$, and $(\boldsymbol{\sigma} \cdot \hat{\mathbf{n}})^{3}=\boldsymbol{\sigma} \cdot \hat{\mathbf{n}}$, the rotation matrix is

$$
\begin{equation*}
e^{i \sigma \cdot \hat{\mathbf{n}} \phi / 2}\left|\frac{1}{2}, m\right\rangle=(\cos (\phi / 2)+i \sigma \cdot \hat{\mathbf{n}} \sin (\phi / 2))\left|\frac{1}{2}, m\right\rangle . \tag{5.62}
\end{equation*}
$$

The steps taken in the example above, which apply to all values of $j$ were

1. Enumerate the states.

$$
\begin{equation*}
j_{1}=\frac{1}{2} \leftrightarrow 2 \text { states (dimension of irreducible representation }=2 \text { ) } \tag{5.63}
\end{equation*}
$$

2. Construct the $\hat{\mathbf{L}}$ matrices.
3. Construct $d_{m m^{\prime}}^{j}(\hat{\mathbf{n}}, \phi)$.

## 5.4 spherical harmonics.

For $L=1$ it turns out that the rotation matrices turn out to be the 3D rotation matrices. In the space representation

$$
\begin{equation*}
\mathbf{L}=\mathbf{r} \times \mathbf{p} \tag{5.64}
\end{equation*}
$$

the coordinates of the operator are

$$
\begin{equation*}
\hat{L}_{k}=i \epsilon_{k m n} r_{m}\left(-i \hbar \frac{\partial}{\partial r_{n}}\right) . \tag{5.65}
\end{equation*}
$$

We see that scaling $\mathbf{r} \rightarrow \alpha \mathbf{r}$ does not change this operator, allowing for an angular representation $\hat{L}_{k}(\theta, \phi)$ that have the form

$$
\begin{align*}
& \hat{L}_{z}=-i \hbar \frac{\partial}{\partial \phi} \\
& \hat{L}_{ \pm}=\hbar\left( \pm \frac{\partial}{\partial \theta}+i \cot \theta \frac{\partial}{\partial \phi}\right) . \tag{5.66}
\end{align*}
$$



Figure 5.5: Spherical coordinate convention.

Here $\theta$ and $\phi$ are the polar and azimuthal angles respectively as illustrated in fig. 5.5. Introducing the spherical harmonics $Y_{l m}$, the equivalent wave function representation of the problem is

$$
\begin{align*}
\hat{\mathbf{L}} Y_{l m}(\theta, \phi) & =\hbar^{2} l(l+1) Y_{l m}(\theta, \phi) \\
\hat{L}_{z} Y_{l m}(\theta, \phi) & =\hbar m Y_{l m}(\theta, \phi) \tag{5.67}
\end{align*}
$$

One can find these functions

$$
\begin{equation*}
Y_{l m}(\theta, \phi)=P_{l, m}(\cos \theta) e^{i m \phi} \tag{5.68}
\end{equation*}
$$

where $P_{l, m}(\cos \theta)$ are called the associated Legendre polynomials. This can be applied whenever we have

$$
\begin{equation*}
\left[H, \hat{L}_{k}\right]=0 \tag{5.69}
\end{equation*}
$$

where all the eigenfunctions will have the form

$$
\begin{equation*}
\Psi(r, \theta, \phi)=R(r) Y_{l m}(\theta, \phi) \tag{5.70}
\end{equation*}
$$

### 5.5 ADDITION OF ANGULAR MOMENTUM.

Since $\hat{\mathbf{L}}$ is a vector we expect to be able to add angular momentum in some way similar to the addition of classical vectors as illustrated in fig. 5.6. When we have a potential that depends only on the difference in position $V\left(\mathbf{r}_{1}-\mathbf{r}_{2}\right)$ then we know from classical problems it is effective to work in center of mass coordinates

$$
\begin{align*}
& \hat{\mathbf{R}}_{\mathrm{cm}}=\frac{\hat{\mathbf{r}}_{1}+\hat{\mathbf{r}}_{2}}{2}  \tag{5.71}\\
& \hat{\mathbf{P}}_{\mathrm{cm}}=\hat{\mathbf{p}}_{1}+\hat{\mathbf{p}}_{2}
\end{align*}
$$



Figure 5.6: Classical vector addition.
where

$$
\begin{equation*}
\left[\hat{R}_{i}, \hat{P}_{j}\right]=i \hbar \delta_{i j} \tag{5.72}
\end{equation*}
$$

Given

$$
\begin{equation*}
\hat{\mathbf{L}}_{1}+\hat{\mathbf{L}}_{2}=\hat{\mathbf{L}}_{\mathrm{tot}} \tag{5.73}
\end{equation*}
$$

do we have

$$
\begin{equation*}
\left[\hat{L}_{\mathrm{tot}, i}, \hat{L}_{\mathrm{tot}, j}\right]=i \hbar \epsilon_{i j k} \hat{L}_{\mathrm{tot}, k} ? \tag{5.74}
\end{equation*}
$$

That is

$$
\begin{equation*}
\left[\hat{L}_{1, i}+\hat{L}_{1, j}, \hat{L}_{2, i}+\hat{L}_{2, j}\right]=i \hbar \epsilon_{i j k}\left(\hat{L}_{1, k}+\hat{L}_{1, k}\right) \tag{5.75}
\end{equation*}
$$

FIXME: Right at the end of the lecture, there was a mention of something about whether or not $\hat{\mathbf{L}}_{1}^{2}$ and $\hat{L}_{1, z}$ were sharply defined, but I missed it. Ask about this if not covered in the next lecture.

### 5.6 ADDITION OF ANGULAR MOMENTA (CONT.)

- For orbital angular momentum

$$
\begin{align*}
& \hat{\mathbf{L}}_{1}=\hat{\mathbf{r}}_{1} \times \hat{\mathbf{p}}_{1} \\
& \hat{\mathbf{L}}_{1}=\hat{\mathbf{r}}_{1} \times \hat{\mathbf{p}}_{1} \tag{5.76}
\end{align*}
$$

We can show that it is true that

$$
\begin{equation*}
\left[L_{1 i}+L_{2 i}, L_{1 j}+L_{2 j}\right]=i \hbar \epsilon_{i j k}\left(L_{1 k}+L_{2 k}\right) \tag{5.77}
\end{equation*}
$$

because the angular momentum of the independent particles commute. Given this is it fair to consider that the sum

$$
\begin{equation*}
\hat{\mathbf{L}}_{1}+\hat{\mathbf{L}}_{2} \tag{5.78}
\end{equation*}
$$

is also angular momentum.

- Given $\left|l_{1}, m_{1}\right\rangle$ and $\left|l_{2}, m_{2}\right\rangle$, if a measurement is made of $\hat{\mathbf{L}}_{1}+\hat{\mathbf{L}}_{2}$, what do we get? Specifically, what do we get for

$$
\begin{equation*}
\left(\hat{\mathbf{L}}_{1}+\hat{\mathbf{L}}_{2}\right)^{2}, \tag{5.79}
\end{equation*}
$$

and for

$$
\begin{equation*}
\left(\hat{L}_{1 z}+\hat{L}_{2 z}\right) ? \tag{5.80}
\end{equation*}
$$

For the latter, we get

$$
\begin{equation*}
\left(\hat{L}_{1 z}+\hat{L}_{2 z}\right)\left|l_{1}, m_{1} ; l_{2}, m_{2}\right\rangle=\left(\hbar m_{1}+\hbar m_{2}\right)\left|l_{1}, m_{1} ; l_{2}, m_{2}\right\rangle . \tag{5.81}
\end{equation*}
$$

Given

$$
\begin{equation*}
\hat{L}_{1 z}+\hat{L}_{2 z}=\hat{L}_{z}^{\text {tot }} \tag{5.82}
\end{equation*}
$$

we find

$$
\begin{align*}
{\left[\hat{L}_{z}^{\text {tot }}, \hat{\mathbf{L}}_{1}^{2}\right] } & =0 \\
{\left[\hat{L}_{z}^{\text {to }}, \hat{\mathbf{L}}_{2}^{2}\right] } & =0  \tag{5.83}\\
{\left[\hat{L}_{z}^{\text {tot }}, \hat{\mathbf{L}}_{1 z}\right] } & =0 \\
{\left[\hat{L}_{z}^{\text {tot }}, \hat{\mathbf{L}}_{1 z}\right] } & =0 .
\end{align*}
$$

We also find

$$
\begin{align*}
{\left[\left(\hat{\mathbf{L}}_{1}+\hat{\mathbf{L}}_{2}\right)^{2}, \hat{\mathbf{L}}_{1}^{2}\right] } & =\left[\hat{\mathbf{L}}_{1}^{2}+\hat{\mathbf{L}}_{2}^{2}+2 \hat{\mathbf{L}}_{1} \cdot \hat{\mathbf{L}}_{2}, \hat{\mathbf{L}}_{1}^{2}\right]  \tag{5.84}\\
& =0,
\end{align*}
$$

but for

$$
\begin{align*}
{\left[\left(\hat{\mathbf{L}}_{1}+\hat{\mathbf{L}}_{2}\right)^{2}, \hat{L}_{1 z}\right] } & =\left[\hat{\mathbf{L}}_{1}^{2}+\hat{\mathbf{L}}_{2}^{2}+2 \hat{\mathbf{L}}_{1} \cdot \hat{\mathbf{L}}_{2}, \hat{L}_{1 z}\right]  \tag{5.85}\\
& =2\left[\hat{\mathbf{L}}_{1} \cdot \hat{\mathbf{L}}_{2}, \hat{L}_{1 z}\right] \neq 0 .
\end{align*}
$$

Classically if we have measured $\hat{\mathbf{L}}_{1}$ and $\hat{\mathbf{L}}_{2}$ then we know the total angular momentum as sketched in fig. 5.7. In QM where we don't know all the components of the angular momentum simultaneously, things get fuzzier. For example, if the $\hat{L}_{1 z}$ and $\hat{L}_{2 z}$ components have been measured, we have the angular momentum defined within a conical region as sketched in fig. 5.8. Suppose we know $\hat{L}_{z}^{\text {tot }}$ precisely, but have imprecise information


Figure 5.7: Classical addition of angular momenta.


Figure 5.8: Addition of angular momenta given measured $\hat{L}_{z}$.
about $\left(\hat{\mathbf{L}}^{\text {tot }}\right)^{2}$. Can we determine bounds for this? Let $|\psi\rangle=\left|l_{1}, m_{2} ; l_{2}, m_{2}\right\rangle$, so

$$
\begin{align*}
\langle\psi|\left(\hat{\mathbf{L}}_{1}+\hat{\mathbf{L}}_{2}\right)^{2}|\psi\rangle & =\langle\psi| \hat{\mathbf{L}}_{1}^{2}|\psi\rangle+\langle\psi| \hat{\mathbf{L}}_{2}^{2}|\psi\rangle+2\langle\psi| \hat{\mathbf{L}}_{1} \cdot \hat{\mathbf{L}}_{2}|\psi\rangle \\
& =l_{1}\left(l_{1}+1\right) \hbar^{2}+l_{2}\left(l_{2}+1\right) \hbar^{2}+2\langle\psi| \hat{\mathbf{L}}_{1} \cdot \hat{\mathbf{L}}_{2}|\psi\rangle \tag{5.86}
\end{align*}
$$

Using the Cauchy-Schwartz inequality

$$
\begin{equation*}
|\langle\phi \mid \psi\rangle|^{2} \leq|\langle\phi \mid \phi\rangle \|\langle\psi \mid \psi\rangle| \tag{5.87}
\end{equation*}
$$

which is the equivalent of the classical relationship

$$
\begin{equation*}
(\mathbf{A} \cdot \mathbf{B})^{2} \leq \mathbf{A}^{2} \mathbf{B}^{2} \tag{5.88}
\end{equation*}
$$

Applying this to the last term, we have

$$
\begin{equation*}
\left(\langle\psi| \hat{\mathbf{L}}_{1} \cdot \hat{\mathbf{L}}_{2}|\psi\rangle\right)^{2} \leq\langle\psi| \hat{\mathbf{L}}_{1} \cdot \hat{\mathbf{L}}_{1}|\psi\rangle\langle\psi| \hat{\mathbf{L}}_{2} \cdot \hat{\mathbf{L}}_{2}|\psi\rangle=\hbar^{4} l_{1}\left(l_{1}+1\right) l_{2}\left(l_{2}+2\right) \tag{5.89}
\end{equation*}
$$

Thus for the max we have

$$
\langle\psi|\left(\hat{\mathbf{L}}_{1}+\hat{\mathbf{L}}_{2}\right)^{2}|\psi\rangle \leq \hbar^{2} l_{1}\left(l_{1}+1\right)+\hbar^{2} l_{2}\left(l_{2}+1\right)+2 \hbar^{2} \sqrt{l_{1}\left(l_{1}+1\right) l_{2}\left(l_{2}+2\right)}
$$

and for the min

$$
\begin{align*}
\langle\psi|\left(\hat{\mathbf{L}}_{1}+\hat{\mathbf{L}}_{2}\right)^{2}|\psi\rangle \geq & \hbar^{2} l_{1}\left(l_{1}+1\right)+\hbar^{2} l_{2}\left(l_{2}+1\right) \\
& -2 \hbar^{2} \sqrt{l_{1}\left(l_{1}+1\right) l_{2}\left(l_{2}+2\right)} \tag{5.91}
\end{align*}
$$

To try to pretty up these estimate, starting with the max, note that if we replace a portion of the RHS with something bigger, we are left with a strict less than relationship. That is

$$
\begin{align*}
& l_{1}\left(l_{1}+1\right)<\left(l_{1}+\frac{1}{2}\right)^{2}  \tag{5.92}\\
& l_{2}\left(l_{2}+1\right)<\left(l_{2}+\frac{1}{2}\right)^{2}
\end{align*}
$$

That is

$$
\begin{align*}
\langle\psi|\left(\hat{\mathbf{L}}_{1}+\hat{\mathbf{L}}_{2}\right)^{2}|\psi\rangle & <\hbar^{2}\left(l_{1}\left(l_{1}+1\right)+l_{2}\left(l_{2}+1\right)+2\left(l_{1}+\frac{1}{2}\right)\left(l_{2}+\frac{1}{2}\right)\right) \\
& =\hbar^{2}\left(l_{1}^{2}+l_{2}^{2}+l_{1}+l_{2}+2 l_{1} l_{2}+l_{1}+l_{2}+\frac{1}{2}\right) \\
& =\hbar^{2}\left(\left(l_{1}+l_{2}+\frac{1}{2}\right)\left(l_{1}+l_{2}+\frac{3}{2}\right)-\frac{1}{4}\right) \tag{5.93}
\end{align*}
$$

or

$$
\begin{equation*}
l_{\text {tot }}\left(l_{\text {tot }}+1\right)<\left(l_{1}+l_{2}+\frac{1}{2}\right)\left(l_{1}+l_{2}+\frac{3}{2}\right) \tag{5.94}
\end{equation*}
$$

which, gives

$$
\begin{equation*}
l_{\mathrm{tot}}<l_{1}+l_{2}+\frac{1}{2} \tag{5.95}
\end{equation*}
$$

Finally, given a quantization requirement, that is

$$
\begin{equation*}
l_{\mathrm{tot}} \leq l_{1}+l_{2} \tag{5.96}
\end{equation*}
$$

Similarly, for the min, we find

$$
\begin{align*}
\langle\psi|\left(\hat{\mathbf{L}}_{1}+\hat{\mathbf{L}}_{2}\right)^{2}|\psi\rangle & >\hbar^{2}\left(l_{1}\left(l_{1}+1\right)+l_{2}\left(l_{2}+1\right)-2\left(l_{1}+\frac{1}{2}\right)\left(l_{2}+\frac{1}{2}\right)\right) \\
& =\hbar^{2}\left(l_{1}^{2}+l_{2}^{2}-2 l_{1} l_{2}-\frac{1}{2}\right) \\
& =\hbar^{2}\left(\left(l_{1}-l_{2}-\frac{1}{2}\right)\left(l_{1}-l_{2}+\frac{1}{2}\right)-\frac{1}{4}\right) \tag{5.97}
\end{align*}
$$

The total angular momentum quantum number must then satisfy

$$
\begin{equation*}
l_{\mathrm{tot}}\left(l_{\mathrm{tot}}+1\right)>\left(l_{1}-l_{2}-\frac{1}{2}\right)\left(l_{1}-l_{2}+\frac{1}{2}\right)-\frac{1}{4} \tag{5.98}
\end{equation*}
$$

Is it true that

$$
\begin{equation*}
l_{\mathrm{tot}}\left(l_{\mathrm{tot}}+1\right)>\left(l_{1}-l_{2}-\frac{1}{2}\right)\left(l_{1}-l_{2}+\frac{1}{2}\right) ? \tag{5.99}
\end{equation*}
$$

This is true when $l_{\text {tot }}>l_{1}-l_{2}-\frac{1}{2}$, assuming that $l_{1}>l_{2}$. Suppose $l_{\text {tot }}=$ $l_{1}-l_{2}-\frac{1}{2}$, then

$$
\begin{align*}
l_{\mathrm{tot}}\left(l_{\mathrm{tot}}+1\right) & =\left(l_{1}-l_{2}-\frac{1}{2}\right)\left(l_{1}-l_{2}+\frac{1}{2}\right)  \tag{5.100}\\
& =\left(l_{1}-l_{2}\right)^{2}-\frac{1}{4}
\end{align*}
$$

So, is it true that

$$
\begin{equation*}
\left(l_{1}-l_{2}\right)^{2}-\frac{1}{4} \geq l_{1}^{2}+l_{1}+l_{2}^{2}+l_{2}-2 \sqrt{l_{1}\left(l_{1}+1\right) l_{2}\left(l_{2}+1\right)} ? \tag{5.101}
\end{equation*}
$$

If that is the case we have

$$
\begin{gather*}
\quad-2 l_{1} l_{2}-\frac{1}{4} \geq l_{1}+l_{2}-2 \sqrt{l_{1}\left(l_{1}+1\right) l_{2}\left(l_{2}+1\right)}  \tag{5.102}\\
2 \sqrt{l_{2}\left(l_{1}+1\right) l_{1}\left(l_{2}+1\right)} \geq l_{1}+l_{2}+2 l_{1} l_{2}+\frac{1}{4}=l_{1}\left(l_{2}+1\right)+l_{2}\left(l_{1}+1\right)+\frac{1}{4} . \tag{5.103}
\end{gather*}
$$

This has the structure

$$
\begin{equation*}
2 \sqrt{x y} \geq x+y+\frac{1}{4} \tag{5.104}
\end{equation*}
$$

or

$$
\begin{equation*}
4 x y \geq(x+y)^{2}+\frac{1}{16}+\frac{1}{2}(x+y) \tag{5.105}
\end{equation*}
$$

or

$$
\begin{equation*}
0 \geq(x-y)^{2}+\frac{1}{16}+\frac{1}{2}(x+y) \tag{5.106}
\end{equation*}
$$

But since $x+y \geq 0$ this inequality is not satisfied when $l_{\text {tot }}=l_{1}-l_{2}-\frac{1}{2}$. We can conclude

$$
\begin{equation*}
l_{1}-l_{2}-\frac{1}{2}<l_{\mathrm{tot}}<l_{1}+l_{2}+\frac{1}{2} . \tag{5.107}
\end{equation*}
$$

Is it true that

$$
\begin{equation*}
l_{1}-l_{2} \geq l_{\mathrm{tot}} \geq l_{1}+l_{2} ? \tag{5.108}
\end{equation*}
$$

Note that we have two separate Hilbert spaces $l_{1} \otimes l_{2}$ of dimension $2 l_{1}+1$ and $2 l_{2}+1$ respectively. The total number of states is

$$
\begin{align*}
\sum_{l_{\mathrm{tot}}=l_{1}-l_{2}}^{l_{1}+l_{2}}\left(2 l_{\mathrm{tot}}+1\right) & =2 \sum_{n=l_{1}-l_{2}}^{l_{1}+l_{2}} n+\not / 1+l_{2}-\left(\not /-l_{2}\right)+1 \\
& =2 \frac{1}{2}\left(l_{1}+l_{2}+\left(l_{1}-l_{2}\right)\right)\left(l_{1}+l_{2}-\left(l_{1}-l_{2}\right)+1\right)+2 l_{2}+1 \\
& =2 l_{1}\left(2 l_{2}+1\right)+2 l_{2}+1 \\
& =\left(2 l_{1}+1\right)\left(2 l_{2}+1\right) \tag{5.109}
\end{align*}
$$

So the end result is that given $\left|l_{1}, m_{1}\right\rangle,\left|l_{2}, m_{2}\right\rangle$, with $l_{1} \geq l_{2}$, where, in steps of 1 ,

$$
\begin{equation*}
l_{1}-l_{2} \leq l_{\mathrm{tot}} \leq l_{1}+l_{2} \tag{5.110}
\end{equation*}
$$

## 5.7 clebsch-Gordan.

How can we related total momentum states to individual momentum states in the $1 \otimes 2$ space?

$$
\begin{equation*}
\left|l_{1}, l_{2}, l, m\right\rangle=\sum_{m_{1}, m_{2}} C_{l_{1} l_{2} m_{1} m_{2}}^{l_{1} l_{2} l m}\left|l_{1} m_{1} ; l_{2} m_{2}\right\rangle . \tag{5.111}
\end{equation*}
$$

The values $C_{l_{1} l_{2} m_{1} m_{2}}^{l_{1} l m}$ are called the Clebsch-Gordan coefficients.

## Example 5.2: Example: spin one and spin one half

With individual momentum states $\left|l_{1} m_{1} ; l_{2} m_{2}\right\rangle$

$$
\begin{align*}
l_{1} & =1 \\
m_{1} & = \pm 1,0 \\
l_{2} & =\frac{1}{2}  \tag{5.112}\\
m_{2} & = \pm \frac{1}{2}
\end{align*}
$$

The total angular momentum numbers are

$$
\begin{equation*}
l_{\mathrm{tot}} \in\left[l_{1}-l_{2}, l_{1}+l_{2}\right]=[1 / 2,3 / 2] \tag{5.113}
\end{equation*}
$$

The possible states $\left|l_{\text {tot }}, m_{\text {tot }}\right\rangle$ are

$$
\begin{equation*}
\left\langle\frac{1}{2}, \frac{1}{2}\right\rangle,\left|\frac{1}{2},-\frac{1}{2}\right\rangle \tag{5.114}
\end{equation*}
$$

and

$$
\begin{equation*}
\left\langle\frac{3}{2}, \frac{3}{2}\right\rangle,\left|\frac{3}{2}, \frac{1}{2}\right\rangle,\left|\frac{3}{2},-\frac{1}{2}\right\rangle\left\langle\frac{3}{2},-\frac{3}{2}\right\rangle . \tag{5.115}
\end{equation*}
$$

The Clebsch-Gordan procedure is the search for an orthogonal angular momentum basis, built up from the individual momentum bases. For the total momentum basis we want the basis states to satisfy the ladder operators, but also have them satisfy the consistent ladder operators for the individual particle angular momenta. This procedure is sketched in fig. 5.9.


Figure 5.9: Spin one,one-half Clebsch-Gordan procedure.

Demonstrating by example, let the highest total momentum state be proportional to the highest product of individual momentum states

$$
\begin{equation*}
\left|\frac{3}{2} \frac{3}{2}\right\rangle=|11\rangle \otimes\left|\frac{1}{2} \frac{1}{2}\right\rangle . \tag{5.116}
\end{equation*}
$$

A lowered state can be constructed in two different ways, one using the total angular momentum lowering operator

$$
\begin{align*}
\left|\frac{3}{2} \frac{1}{2}\right\rangle & =\hat{L}_{-}^{\text {tot }}\left|\frac{3}{2} \frac{3}{2}\right\rangle \\
& =\hbar \sqrt{\left(\frac{3}{2}+\frac{3}{2}\right)\left(\frac{3}{2}-\frac{3}{2}+1\right)}\left|\frac{3}{2} \frac{1}{2}\right\rangle  \tag{5.117}\\
& =\hbar \sqrt{3}\left|\frac{3}{2} \frac{1}{2}\right\rangle .
\end{align*}
$$

On the other hand, the lowering operator can also be expressed as $\hat{L}_{-}^{\text {tot }}=\hat{L}_{-}^{(1)} \otimes 1+1 \otimes \hat{L}_{-}^{(2)}$. Operating with that gives

$$
\begin{align*}
\left|\frac{3}{2} \frac{1}{2}\right\rangle= & \hat{L}_{-}^{\text {tot }}|11\rangle \otimes\left|\frac{1}{2} \frac{1}{2}\right\rangle \\
= & \hat{L}_{-}^{(1)}|11\rangle \otimes\left|\frac{1}{2} \frac{1}{2}\right\rangle+|11\rangle \otimes \hat{L}_{-}^{2}\left|\frac{1}{2} \frac{1}{2}\right\rangle \\
= & \hbar \sqrt{(1+1)(1-1+1)|10\rangle \otimes\left|\frac{1}{2} \frac{1}{2}\right\rangle}  \tag{5.118}\\
& +\hbar \sqrt{\left(\frac{1}{2}+\frac{1}{2}\right)\left(\frac{1}{2}-\frac{1}{2}+1\right)}|11\rangle \otimes\left|\frac{1}{2}-\frac{1}{2}\right\rangle \\
= & \hbar \sqrt{2}|10\rangle \otimes\left|\frac{1}{2} \frac{1}{2}\right\rangle+\hbar|11\rangle \otimes\left|\frac{1}{2}-\frac{1}{2}\right\rangle .
\end{align*}
$$

Equating both sides and dispensing with the direct product notation, this is

$$
\begin{equation*}
\sqrt{3}\left|\frac{3}{2} \frac{1}{2}\right\rangle=\sqrt{2}\left|10 ; \frac{1}{2} \frac{1}{2}\right\rangle+\left|11 ; \frac{1}{2}-\frac{1}{2}\right\rangle \tag{5.119}
\end{equation*}
$$

or

$$
\begin{equation*}
\left|\frac{3}{2} \frac{1}{2}\right\rangle=\sqrt{\frac{2}{3}}\left|10 ; \frac{1}{2} \frac{1}{2}\right\rangle+\frac{1}{\sqrt{3}}\left|11 ; \frac{1}{2}-\frac{1}{2}\right\rangle \tag{5.120}
\end{equation*}
$$

This is clearly both a unit ket, and normal to $\left|\frac{3}{2} \frac{3}{2}\right\rangle$. We can continue operating with the lowering operator for the total angular momentum to construct all the states down to $\left|\frac{3}{2} \frac{-3}{2}\right\rangle$. Working with $\hbar=1$ since we see it cancel out, the next lower state follows from

$$
\begin{align*}
\hat{L}_{-}^{\mathrm{tot}}\left|\frac{3}{2} \frac{1}{2}\right\rangle & =\sqrt{2 \times 2}\left|\frac{3}{2} \frac{-1}{2}\right\rangle \\
& =2\left|\frac{3}{2} \frac{-1}{2}\right\rangle \tag{5.121}
\end{align*}
$$

and from the individual lowering operators on the components of $\left|\frac{3}{2} \frac{1}{2}\right\rangle$.

$$
\begin{equation*}
\hat{L}_{-}^{(1)}\left|10 ; \frac{1}{2} \frac{1}{2}\right\rangle=\sqrt{1 \times 2}\left|1-1 ; \frac{1}{2} \frac{1}{2}\right\rangle \tag{5.122}
\end{equation*}
$$

and

$$
\begin{equation*}
\hat{L}_{-}^{(2)}\left|10 ; \frac{1}{2} \frac{1}{2}\right\rangle=\sqrt{1 \times 1}\left|10 ; \frac{1}{2} \frac{-1}{2}\right\rangle \tag{5.123}
\end{equation*}
$$

and

$$
\begin{equation*}
\hat{L}_{-}^{1}\left|11 ; \frac{1}{2}-\frac{1}{2}\right\rangle=\sqrt{2 \times 1}\left|10 ; \frac{1}{2}-\frac{1}{2}\right\rangle \tag{5.124}
\end{equation*}
$$

This gives

$$
\begin{equation*}
2\left|\frac{3}{2} \frac{-1}{2}\right\rangle=\sqrt{\frac{2}{3}}\left(\sqrt{2}\left|1-1 ; \frac{1}{2} \frac{1}{2}\right\rangle+\left|10 ; \frac{1}{2} \frac{-1}{2}\right\rangle\right)+\frac{1}{\sqrt{3}} \sqrt{2}\left|10 ; \frac{1}{2}-\frac{1}{2}\right\rangle \tag{5.125}
\end{equation*}
$$

or

$$
\begin{equation*}
\left|\frac{3}{2} \frac{-1}{2}\right\rangle=\frac{1}{\sqrt{3}}\left|1-1 ; \frac{1}{2} \frac{1}{2}\right\rangle+\sqrt{\frac{2}{3}}\left|10 ; \frac{1}{2} \frac{-1}{2}\right\rangle \tag{5.126}
\end{equation*}
$$

There's one more possible state with total angular momentum $\frac{3}{2}$. This time

$$
\begin{align*}
\hat{L}_{-}^{\mathrm{tot}}\left|\frac{3}{2} \frac{-1}{2}\right\rangle & =\sqrt{1 \times 3}\left|\frac{3}{2} \frac{-3}{2}\right\rangle \\
& =\frac{1}{\sqrt{3}} \hat{L}_{-}^{(2)}\left|1-1 ; \frac{1}{2} \frac{1}{2}\right\rangle+\sqrt{\frac{2}{3}} \hat{L}_{-}^{(1)}\left|10 ; \frac{1}{2} \frac{-1}{2}\right\rangle \\
& =\frac{1}{\sqrt{3}} \sqrt{1 \times 1}\left|1-1 ; \frac{1}{2} \frac{-1}{2}\right\rangle+\sqrt{\frac{2}{3}} \sqrt{1 \times 2}\left|1-1 ; \frac{1}{2} \frac{-1}{2}\right\rangle \tag{5.127}
\end{align*}
$$

or

$$
\begin{equation*}
\left|\frac{3}{2} \frac{-3}{2}\right\rangle=\left|1-1 ; \frac{1}{2} \frac{-1}{2}\right\rangle \tag{5.128}
\end{equation*}
$$

The $\left|\frac{1}{2} \frac{1}{2}\right\rangle$ state is constructed as normal to $\left|\frac{3}{2} \frac{1}{2}\right\rangle$, or

$$
\begin{equation*}
\left|\frac{1}{2} \frac{1}{2}\right\rangle=\sqrt{\frac{1}{3}}\left|10 ; \frac{1}{2} \frac{1}{2}\right\rangle-\sqrt{\frac{2}{3}}\left|11 ; \frac{1}{2}-\frac{1}{2}\right\rangle, \tag{5.129}
\end{equation*}
$$

and $\left|\frac{1}{2}-\frac{1}{2}\right\rangle$ by lowering that. With

$$
\begin{equation*}
\hat{L}_{-}^{\text {tot }}\left|\frac{1}{2} \frac{1}{2}\right\rangle=\sqrt{1 \times 1}\left|\frac{1}{2}-\frac{1}{2}\right\rangle \tag{5.130}
\end{equation*}
$$

we have

$$
\begin{align*}
\left|\frac{1}{2}-\frac{1}{2}\right\rangle= & \sqrt{\frac{1}{3}}\left(\sqrt{1 \times 2}\left|1-1 ; \frac{1}{2} \frac{1}{2}\right\rangle+\left|10 ; \frac{1}{2}-\frac{1}{2}\right\rangle\right) \\
& -\sqrt{\frac{2}{3}} \sqrt{2 \times 1}\left|10 ; \frac{1}{2}-\frac{1}{2}\right\rangle \tag{5.131}
\end{align*}
$$

or

$$
\begin{equation*}
\left|\frac{1}{2}-\frac{1}{2}\right\rangle=\sqrt{\frac{2}{3}}\left|1-1 ; \frac{1}{2} \frac{1}{2}\right\rangle-\frac{1}{\sqrt{3}}\left|10 ; \frac{1}{2}-\frac{1}{2}\right\rangle . \tag{5.132}
\end{equation*}
$$

Observe that further lowering this produces zero

$$
\begin{align*}
\hat{L}_{-}^{\text {tot }}\left|\frac{1}{2}-\frac{1}{2}\right\rangle & =\sqrt{\frac{2}{3}} \sqrt{1 \times 1}\left|1-1 ; \frac{1}{2}-\frac{1}{2}\right\rangle-\frac{1}{\sqrt{3}} \sqrt{1 \times 2}\left|1-1 ; \frac{1}{2}-\frac{1}{2}\right\rangle . \\
& =0 . \tag{5.133}
\end{align*}
$$

All the basis elements have been determined. In summary

$$
\begin{align*}
\left|\frac{3}{2} \frac{3}{2}\right\rangle & =\left|11 ; \frac{1}{2} \frac{1}{2}\right\rangle \\
\sqrt{3}\left|\frac{3}{2} \frac{1}{2}\right\rangle & =\sqrt{2}\left|10 ; \frac{1}{2} \frac{1}{2}\right\rangle+\left|11 ; \frac{1}{2}-\frac{1}{2}\right\rangle \\
\sqrt{3}\left|\frac{1}{2} \frac{1}{2}\right\rangle & =\left|10 ; \frac{1}{2} \frac{1}{2}\right\rangle-\sqrt{2}\left|11 ; \frac{1}{2}-\frac{1}{2}\right\rangle \\
\sqrt{3}\left|\frac{3}{2} \frac{-1}{2}\right\rangle & =\left|1-1 ; \frac{1}{2} \frac{1}{2}\right\rangle+\sqrt{2}\left|10 ; \frac{1}{2} \frac{-1}{2}\right\rangle  \tag{5.134}\\
\sqrt{3}\left|\frac{1}{2}-\frac{1}{2}\right\rangle & =\sqrt{2}\left|1-1 ; \frac{1}{2} \frac{1}{2}\right\rangle-\left|10 ; \frac{1}{2}-\frac{1}{2}\right\rangle \\
\left|\frac{3}{2} \frac{-3}{2}\right\rangle & =\left|1-1 ; \frac{1}{2} \frac{-1}{2}\right\rangle .
\end{align*}
$$

## Example 5.3: Spin two, spin one.

With $j_{1}=2$ and $j_{2}=1$, we have $j \in 1,2,3$, and can proceed the same way as sketched in fig. 5.10.


Figure 5.10: Spin two,one Clebsch-Gordan procedure.

Working through the details for this example is left to the problem set.

### 5.8 PROBLEMS.

## Exercise 5.1 Angular momentum commutators.

Using $\hat{L}_{i}=\epsilon_{i j k} \hat{r}_{j} \hat{p}_{k}$, show that

$$
\begin{equation*}
\left[\hat{L}_{i}, \hat{L}_{j}\right]=i \hbar \epsilon_{i j k} \hat{L}_{k} \tag{5.135}
\end{equation*}
$$

Answer for Exercise 5.1
Let's start without using abstract index expressions, computing the commutator for $\hat{L}_{1}, \hat{L}_{2}$, which should show the basic steps required

$$
\begin{align*}
{\left[\hat{L}_{1}, \hat{L}_{2}\right] } & =\left[\hat{r}_{2} \hat{p}_{3}-\hat{r}_{3} \hat{p}_{2}, \hat{r}_{3} \hat{p}_{1}-\hat{r}_{1} \hat{p}_{3}\right] \\
& =\left[\hat{r}_{2} \hat{p}_{3}, \hat{r}_{3} \hat{p}_{1}\right]-\left[\hat{r}_{2} \hat{p}_{3}, \hat{r}_{1} \hat{p}_{3}\right]-\left[\hat{r}_{3} \hat{p}_{2}, \hat{r}_{3} \hat{p}_{1}\right]+\left[\hat{r}_{3} \hat{p}_{2}, \hat{r}_{1} \hat{p}_{3}\right] . \tag{5.136}
\end{align*}
$$

The first of these commutators is

$$
\begin{align*}
{\left[\hat{r}_{2} \hat{p}_{3}, \hat{r}_{3} \hat{p}_{1}\right] } & =\hat{r}_{2} \hat{p}_{3} \hat{r}_{3} \hat{p}_{1}-\hat{r}_{3} \hat{p}_{1} \hat{r}_{2} \hat{p}_{3} \\
& =\hat{r}_{2} \hat{p}_{1}\left[\hat{p}_{3}, \hat{r}_{3}\right]  \tag{5.137}\\
& =-i \hbar \hat{r}_{2} \hat{p}_{1} .
\end{align*}
$$

We see that any factors in the commutator don't have like indexes (i.e. $\hat{r}_{k}, \hat{p}_{k}$ ) on both position and momentum terms, can be pulled out of the commutator. This leaves

$$
\begin{align*}
{\left[\hat{L}_{1}, \hat{L}_{2}\right] } & =\hat{r}_{2} \hat{p}_{1}\left[\hat{p}_{3}, \hat{r}_{3}\right]-\left[\hat{r}_{2} \hat{p}_{3}, \hat{r}_{1} \hat{p}_{3}\right]-\left[\hat{r}_{3} \hat{p}_{2}, \hat{r}_{3} \hat{p}_{1}\right]+\hat{r}_{1} \hat{p}_{2}\left[\hat{r}_{3}, \hat{p}_{3}\right] \\
& =i \hbar\left(\hat{r}_{1} \hat{p}_{2}-\hat{r}_{2} \hat{p}_{1}\right) \\
& =i \hbar \hat{L}_{3} . \tag{5.138}
\end{align*}
$$

With cyclic permutation this is really enough to consider eq. (5.135) proven. However, can we do this in the general case with the abstract index expression? The quantity to simplify looks forbidding

$$
\begin{equation*}
\left[\hat{L}_{i}, \hat{L}_{j}\right]=\epsilon_{i a b} \epsilon_{j s t}\left[\hat{r}_{a} \hat{p}_{b}, \hat{r}_{s} \hat{p}_{t}\right] . \tag{5.139}
\end{equation*}
$$

Because there are no repeated indexes, this doesn't submit to any of the normal reduction identities. Note however, since we only care about the $i \neq j$ case, that one of the indexes $a, b$ must be $j$ for this quantity to be non-zero. Therefore (for $i \neq j$ )

$$
\begin{align*}
{\left[\hat{L}_{i}, \hat{L}_{j}\right] } & =\epsilon_{i j b} \epsilon_{j s t}\left[\hat{r}_{j} \hat{p}_{b}, \hat{r}_{s} \hat{p}_{t}\right]+\epsilon_{i a j} \epsilon_{j s t}\left[\hat{r}_{a} \hat{p}_{j}, \hat{r}_{s} \hat{p}_{t}\right] \\
& =\epsilon_{i j b} \epsilon_{j s t}\left(\left[\hat{r}_{j} \hat{p}_{b}, \hat{r}_{s} \hat{p}_{t}\right]-\left[\hat{r}_{b} \hat{p}_{j}, \hat{r}_{s} \hat{p}_{t}\right]\right) \\
& =-\delta_{[i b]}^{s t}\left[\hat{r}_{j} \hat{p}_{b}-\hat{r}_{b} \hat{p}_{j}, \hat{r}_{s} \hat{p}_{t}\right] \\
& =\left[\hat{r}_{j} \hat{p}_{b}-\hat{r}_{b} \hat{p}_{j}, \hat{r}_{b} \hat{p}_{i}-\hat{r}_{i} \hat{p}_{b}\right]  \tag{5.140}\\
& =\left[\hat{r}_{j} \hat{p}_{b}, \hat{r}_{b} \hat{p}_{i}\right]-\left[\hat{r}_{j} \hat{p}_{b}, \hat{r}_{i} \hat{p}_{b}\right]-\left[\hat{r}_{b} \hat{p}_{j}, \hat{r}_{b} \hat{p}_{i}\right]+\left[\hat{r}_{b} \hat{p}_{j}, \hat{r}_{i} \hat{p}_{b}\right] \\
& =\hat{r}_{j} \hat{p}_{i}\left[\hat{p}_{b}, \hat{r}_{b}\right]+\hat{r}_{i} \hat{p}_{j}\left[\hat{r}_{b}, \hat{p}_{b}\right] \\
& =i \hbar\left(\hat{r}_{i} \hat{p}_{j}-\hat{r}_{j} \hat{p}_{i}\right) \\
& =i \hbar \epsilon_{i j k} \hat{r}_{i} \hat{p}_{j} .
\end{align*}
$$

Exercise $5.2 \quad S_{y}$ eigenvectors. ([15] pr. 4.1)
Find the eigenvectors of $\sigma_{y}$, and then find the probability that a measurement of $S_{y}$ will be $\hbar / 2$ when the state is initially $\left[\begin{array}{l}\alpha \\ \beta\end{array}\right]$.
Answer for Exercise 5.2
The eigenvalues should be $\pm 1$, which is easily checked

$$
\begin{align*}
0 & =\left|\sigma_{y}-\lambda\right| \\
& =\left|\begin{array}{cc}
-\lambda & -i \\
i & -\lambda
\end{array}\right|  \tag{5.141}\\
& =\lambda^{2}-1 .
\end{align*}
$$

For $|+\rangle=(a, b)^{\mathrm{T}}$ we must have

$$
\begin{equation*}
-1 a-i b=0 \tag{5.142}
\end{equation*}
$$

so
or

For $|-\rangle$ we must have

$$
\begin{equation*}
a-i b=0 \tag{5.145}
\end{equation*}
$$

so
or

The normalized eigenvectors are

For the probability question we are interested in

$$
\begin{align*}
\left\lvert\,\left.\left\langle S_{y} ;+\right|\left[\begin{array}{l}
\alpha \\
\beta
\end{array}\right]\right|^{2}\right. & =\frac{1}{2}\left|\left[\begin{array}{ll}
1 & -i
\end{array}\right]\left[\begin{array}{l}
\alpha \\
\beta
\end{array}\right]\right|^{2} \\
& =\frac{1}{2}\left(|\alpha|^{2}+|\beta|^{2}\right)  \tag{5.149}\\
& =\frac{1}{2} .
\end{align*}
$$

There is a $50 \%$ chance of finding the particle in the $\left|S_{x} ;+\right\rangle$ state, independent of the initial state.

Exercise 5.3 Magnetic Hamiltonian eigenvectors. ([15] pr. 3.2)
Using Pauli matrices, find the eigenvectors for the magnetic spin interaction Hamiltonian

$$
\begin{equation*}
H=-\frac{1}{\hbar} 2 \mu \mathbf{S} \cdot \mathbf{B} \tag{5.150}
\end{equation*}
$$

Answer for Exercise 5.3

$$
\begin{align*}
H & =-\mu \sigma \cdot \mathbf{B} \\
& =-\mu\left(B_{x}\left[\begin{array}{ll}
0 & 1 \\
1 & 0
\end{array}\right]+B_{y}\left[\begin{array}{cc}
0 & -i \\
i & 0
\end{array}\right]+B_{z}\left[\begin{array}{cc}
1 & 0 \\
0 & -1
\end{array}\right]\right)  \tag{5.151}\\
& =-\mu\left[\begin{array}{cc}
B_{z} & B_{x}-i B_{y} \\
B_{x}+i B_{y} & -B_{z}
\end{array}\right] .
\end{align*}
$$

The characteristic equation is

$$
\begin{align*}
0 & =\left|\begin{array}{cc}
-\mu B_{z}-\lambda & -\mu\left(B_{x}-i B_{y}\right) \\
-\mu\left(B_{x}+i B_{y}\right) & \mu B_{z}-\lambda
\end{array}\right|  \tag{5.152}\\
& =-\left(\left(\mu B_{z}\right)^{2}-\lambda^{2}\right)-\mu^{2}\left(B_{x}^{2}-\left(i B_{y}\right)^{2}\right) \\
& =\lambda^{2}-\mu^{2} \mathbf{B}^{2}
\end{align*}
$$

That is

$$
\begin{equation*}
\lambda= \pm \mu B . \tag{5.153}
\end{equation*}
$$

Now for the eigenvectors. We are looking for $| \pm\rangle=(a, b)^{\mathrm{T}}$ such that

$$
\begin{equation*}
0=\left(-\mu B_{z} \mp \mu B\right) a-\mu\left(B_{x}-i B_{y}\right) b \tag{5.154}
\end{equation*}
$$

or

This squares to

$$
\begin{equation*}
B_{x}^{2}+B_{y}^{2}+B_{z}^{2}+B^{2} \pm 2 B B_{z}=2 B\left(B \pm B_{z}\right) \tag{5.156}
\end{equation*}
$$

so the normalized eigenkets are

## Exercise 5.4 Unimodular transformation. ([15] pr. 3.3)

Given the matrix

$$
\begin{equation*}
U=\frac{a_{0}+i \sigma \cdot \mathbf{a}}{a_{0}-i \sigma \cdot \mathbf{a}} \tag{5.158}
\end{equation*}
$$

where $a_{0}$, a are real valued constant and vector respectively.
a. Show that this is a unimodular and unitary transformation.
b. A unitary transformation can represent an arbitrary rotation. Determine the rotation angle and direction in terms of $a_{0}, \mathbf{a}$.

Answer for Exercise 5.4

Part a. Let's call these factors $A_{ \pm}$, which expand to

$$
\begin{align*}
A_{ \pm} & =a_{0} \pm i \sigma \cdot \mathbf{a} \\
& =\left[\begin{array}{cc}
a_{0} \pm i a_{z} & \pm\left(a_{y}+i a_{x}\right) \\
\mp\left(a_{y}-i a_{x}\right) & a_{0} \mp i a_{z}
\end{array}\right] \tag{5.159}
\end{align*}
$$

or with $z=a_{0}+i a_{z}$, and $w=a_{y}+i a_{x}$, these are

$$
A_{+}=\left[\begin{array}{cc}
z & w  \tag{5.160}\\
-w^{*} & z^{*}
\end{array}\right]
$$

$$
A_{-}=\left[\begin{array}{cc}
z^{*} & -w  \tag{5.161}\\
w^{*} & z
\end{array}\right] .
$$

These both have a determinant of

$$
\begin{align*}
|z|^{2}+|w|^{2} & =\left|a_{0}+i a_{z}\right|^{2}+\left|a_{y}+i a_{x}\right|^{2}  \tag{5.162}\\
& =a_{0}^{2}+\mathbf{a}^{2}
\end{align*}
$$

The inverse of the latter is

$$
A_{-}^{-1}=\frac{1}{a_{0}^{2}+\mathbf{a}^{2}}\left[\begin{array}{cc}
z & w  \tag{5.163}\\
-w^{*} & z^{*}
\end{array}\right] .
$$

Noting that the numerator and denominator commute the inverse can be applied in either order. Picking one, the transformation of interest, after writing $A=a_{0}^{2}+\mathbf{a}^{2}$, is

$$
\begin{align*}
U & =\frac{1}{A}\left[\begin{array}{cc}
z & w \\
-w^{*} & z^{*}
\end{array}\right]\left[\begin{array}{cc}
z & w \\
-w^{*} & z^{*}
\end{array}\right]  \tag{5.164}\\
& =\frac{1}{A}\left[\begin{array}{cc}
z^{2}-|w|^{2} & w\left(z+z^{*}\right) \\
-w^{*}\left(z^{*}+z\right) & \left(z^{*}\right)^{2}-|w|^{2}
\end{array}\right] .
\end{align*}
$$

Recall that a unimodular transformation is one that has the form

$$
\left[\begin{array}{cc}
z & w  \tag{5.165}\\
-w^{*} & z^{*}
\end{array}\right]
$$

provided $|z|^{2}+|w|^{2}=1$, so eq. (5.164) is unimodular if the following sum is unity, which is the case

$$
\begin{align*}
& \frac{\left|z^{2}-|w|^{2}\right|^{2}}{\left(|z|^{2}+|w|^{2}\right)^{2}}+|w|^{2} \frac{\left|z+z^{*}\right|^{2}}{\left(|z|^{2}+|w|^{2}\right)^{2}} \\
& =\frac{\left(z^{2}-|w|^{2}\right)\left(\left(z^{*}\right)^{2}-|w|^{2}\right)+|w|^{2}\left(z+z^{*}\right)^{2}}{\left(|z|^{2}+|w|^{2}\right)^{2}}  \tag{5.166}\\
& =\frac{|z|^{4}+|w|^{4}-|w|^{2}\left(z^{2}+\left(z^{*}\right)^{2}\right)+|w|^{2}\left(z^{2}+\left(z^{*}\right)^{2}+2|z|^{2}\right)}{\left(|z|^{2}+|w|^{2}\right)^{2}} \\
& =1 .
\end{align*}
$$

Part b. The most general rotation of a vector $\mathbf{a}$, described by Pauli matrices is

$$
\begin{equation*}
e^{i \boldsymbol{\sigma} \cdot \hat{\mathbf{n}} \theta / 2} \boldsymbol{\sigma} \cdot \mathbf{a} e^{-i \boldsymbol{\sigma} \cdot \hat{\mathbf{n}} \theta / 2}=\boldsymbol{\sigma} \cdot \hat{\mathbf{n}}+(\boldsymbol{\sigma} \cdot \mathbf{a}-(\mathbf{a} \cdot \hat{\mathbf{n}}) \boldsymbol{\sigma} \cdot \hat{\mathbf{n}}) \cos \theta+\boldsymbol{\sigma} \cdot(\mathbf{a} \times \hat{\mathbf{n}}) \sin \theta \tag{5.167}
\end{equation*}
$$

If the unimodular matrix above, applied as $\sigma \cdot \mathbf{a}^{\prime}=U^{\dagger} \boldsymbol{\sigma} \cdot \mathbf{a} U$ is to also describe this rotation, we want the equivalence

$$
\begin{equation*}
U=e^{-i \sigma \cdot \hat{\mathbf{n}} \theta / 2} \tag{5.168}
\end{equation*}
$$

or

$$
\begin{align*}
& \frac{1}{a_{0}^{2}+\mathbf{a}^{2}}\left[\begin{array}{cc}
a_{0}^{2}-\mathbf{a}^{2}+2 i a_{0} a_{z} & 2 a_{0}\left(a_{y}+i a_{x}\right) \\
-2 a_{0}\left(a_{y}-i a_{x}\right) & a_{0}^{2}-\mathbf{a}^{2}-2 i a_{0} a_{z}
\end{array}\right]  \tag{5.169}\\
& \quad=\left[\begin{array}{cc}
\cos (\theta / 2)-i n_{z} \sin (\theta / 2) & \left(-n_{y}-i n_{x}\right) \sin (\theta / 2) \\
-\left(-n_{y}+i n_{x}\right) \sin (\theta / 2) & \cos (\theta / 2)+i n_{z} \sin (\theta / 2)
\end{array}\right]
\end{align*}
$$

Equating components, that is

$$
\begin{align*}
\cos (\theta / 2) & =\frac{a_{0}^{2}-\mathbf{a}^{2}}{a_{0}^{2}+\mathbf{a}^{2}} \\
-n_{x} \sin (\theta / 2) & =\frac{2 a_{0} a_{x}}{a_{0}^{2}+\mathbf{a}^{2}}  \tag{5.170}\\
-n_{y} \sin (\theta / 2) & =\frac{2 a_{0} a_{y}}{a_{0}^{2}+\mathbf{a}^{2}} \\
-n_{z} \sin (\theta / 2) & =\frac{2 a_{0} a_{y}}{a_{0}^{2}+\mathbf{a}^{2}}
\end{align*}
$$

Noting that

$$
\begin{align*}
\sin (\theta / 2) & =\sqrt{1-\frac{\left(a_{0}^{2}-\mathbf{a}^{2}\right)^{2}}{\left(a_{0}^{2}+\mathbf{a}^{2}\right)^{2}}} \\
& =\frac{\sqrt{\left(a_{0}^{2}+\mathbf{a}^{2}\right)^{2}-\left(a_{0}^{2}-\mathbf{a}^{2}\right)^{2}}}{a_{0}^{2}+\mathbf{a}^{2}}  \tag{5.171}\\
& =\frac{\sqrt{4 a_{0}^{2} \mathbf{a}^{2}}}{a_{0}^{2}+\mathbf{a}^{2}} \\
& =\frac{2 a_{0}|\mathbf{a}|}{a_{0}^{2}+\mathbf{a}^{2}}
\end{align*}
$$

The vector normal direction can be written

$$
\begin{equation*}
\mathbf{n}=-\frac{2 a_{0}}{\left(a_{0}^{2}+\mathbf{a}^{2}\right) \sin (\theta / 2)} \mathbf{a} \tag{5.172}
\end{equation*}
$$

or

$$
\begin{equation*}
\mathbf{n}=-\frac{\mathbf{a}}{|\mathbf{a}|} \tag{5.173}
\end{equation*}
$$

The angle of rotation is

$$
\begin{equation*}
\theta=2 \operatorname{atan} \frac{2 a_{0}|\mathbf{a}|}{a_{0}^{2}-\mathbf{a}^{2}} \tag{5.174}
\end{equation*}
$$

## Exercise 5.5 Rotation through Euler angles. ([15] pr. 3.9)

Consider a sequence of Euler rotations represented by

$$
\begin{equation*}
\mathscr{D}^{1 / 2}(\alpha, \beta, \gamma)=e^{-i \sigma_{z} \alpha / 2} e^{-i \sigma_{y} \beta / 2} e^{-i \sigma_{z} \gamma_{2}} \tag{5.175}
\end{equation*}
$$

Because rotations form a group, this sequence of rotations corresponds to a single rotation by an angle $\theta$ about a new axis $\hat{\mathbf{n}}$. What is $\theta$ ? What is $\hat{\mathbf{n}}$ ? Answer for Exercise 5.5

First expand the z-axis rotations into their Pauli matrix form

$$
\begin{align*}
e^{-i \sigma_{z} \mu / 2} & =\cos (\mu / 2)-i\left[\begin{array}{cc}
1 & 0 \\
0 & -1
\end{array}\right] \sin (\mu / 2) \\
& =\left[\begin{array}{cc}
\cos (\mu / 2)-i \sin (\mu / 2) & 0 \\
0 & \cos (\mu / 2)+i \sin (\mu / 2)
\end{array}\right]  \tag{5.176}\\
& =\left[\begin{array}{cc}
e^{-i \mu / 2} & 0 \\
0 & e^{i \mu / 2}
\end{array}\right]
\end{align*}
$$

For the $y$-axis rotation we have

$$
\begin{align*}
e^{-i \sigma_{y} \beta / 2} & =\cos (\beta / 2)-i\left[\begin{array}{cc}
0 & -i \\
i & 0
\end{array}\right] \sin (\beta / 2) \\
& =\cos (\beta / 2)+\left[\begin{array}{cc}
0 & -1 \\
1 & 0
\end{array}\right] \sin (\beta / 2)  \tag{5.177}\\
& =\left[\begin{array}{cc}
\cos (\beta / 2) & -\sin (\beta / 2) \\
\sin (\beta / 2) & \cos (\beta / 2)
\end{array}\right]
\end{align*}
$$

The composition of rotations is therefore

$$
\begin{align*}
\mathscr{D}^{1 / 2}(\alpha, \beta, \gamma) & =\left[\begin{array}{cc}
e^{-i \alpha / 2} & 0 \\
0 & e^{i \alpha / 2}
\end{array}\right]\left[\begin{array}{cc}
\cos (\beta / 2) & -\sin (\beta / 2) \\
\sin (\beta / 2) & \cos (\beta / 2)
\end{array}\right]\left[\begin{array}{cc}
e^{-i \gamma / 2} & 0 \\
0 & e^{i \gamma / 2}
\end{array}\right] \\
& =\left[\begin{array}{cc}
e^{-i \alpha / 2} & 0 \\
0 & e^{i \alpha / 2}
\end{array}\right]\left[\begin{array}{cc}
e^{-i \gamma / 2} \cos (\beta / 2) & -e^{i \gamma / 2} \sin (\beta / 2) \\
e^{-i \gamma / 2} \sin (\beta / 2) & e^{i \gamma / 2} \cos (\beta / 2)
\end{array}\right] \\
& =\left[\begin{array}{cc}
e^{-i \alpha / 2} e^{-i \gamma / 2} \cos (\beta / 2) & -e^{-i \alpha / 2} e^{i \gamma / 2} \sin (\beta / 2) \\
e^{i \alpha / 2} e^{-i \gamma / 2} \sin (\beta / 2) & e^{+i \alpha / 2} e^{i \gamma / 2} \cos (\beta / 2)
\end{array}\right] \\
& =\left[\begin{array}{cc}
e^{-i(\alpha+\gamma) / 2} \cos \frac{\beta}{2} & -e^{-i(\alpha-\gamma) / 2} \sin \frac{\beta}{2} \\
e^{i(\alpha-\gamma) / 2} \sin \frac{\beta}{2} & e^{i(\alpha+\gamma) / 2} \cos \frac{\beta}{2}
\end{array}\right] \tag{5.178}
\end{align*}
$$

Compare this to the matrix for a rotation (double sided) about a normal, given by

$$
\begin{align*}
\mathscr{R} & =e^{-i \boldsymbol{\sigma} \cdot \hat{\mathbf{n}} \theta / 2} \\
& =\cos \frac{\theta}{2} I-i \sigma \cdot \hat{\mathbf{n}} \sin \frac{\theta}{2} . \tag{5.179}
\end{align*}
$$

With $\hat{\mathbf{n}}=(\sin \Theta \cos \Phi, \sin \Theta \sin \Phi, \cos \Theta)$, the normal direction in its Pauli basis is

$$
\begin{align*}
\boldsymbol{\sigma} \cdot \hat{\mathbf{n}} & =\left[\begin{array}{cc}
\cos \Theta & \sin \Theta \cos \Phi-i \sin \Theta \sin \Phi \\
\sin \Theta \cos \Phi+i \sin \Theta \sin \Phi & -\cos \Theta
\end{array}\right] \\
& =\left[\begin{array}{cc}
\cos \Theta & \sin \Theta e^{-i \Phi} \\
\sin \Theta e^{i \Phi} & -\cos \Theta
\end{array}\right] \tag{5.180}
\end{align*}
$$

so

$$
\mathcal{R}=\left[\begin{array}{cc}
\cos \frac{\theta}{2}-i \sin \frac{\theta}{2} \cos \Theta & -i \sin \Theta e^{-i \Phi} \sin \frac{\theta}{2}  \tag{5.181}\\
-i \sin \Theta e^{i \Phi} \sin \frac{\theta}{2} & \cos \frac{\theta}{2}+i \sin \frac{\theta}{2} \cos \Theta
\end{array}\right]
$$

It's not obvious how to put this into correspondence with the matrix for the Euler rotations. Doing so certainly doesn't look fun. To solve this
problem, let's go the opposite direction, and put the matrix for the Euler rotations into the form of eq. (5.179). That is

$$
\begin{align*}
\mathscr{D}^{1 / 2}(\alpha, \beta, \gamma)= & {\left[\begin{array}{cc}
e^{-i(\alpha+\gamma) / 2} \cos \frac{\beta}{2} & -e^{-i(\alpha-\gamma) / 2} \sin \frac{\beta}{2} \\
e^{i(\alpha-\gamma) / 2} \sin \frac{\beta}{2} & e^{i(\alpha+\gamma) / 2} \cos \frac{\beta}{2}
\end{array}\right] } \\
= & {\left[\begin{array}{cc}
\cos \frac{\alpha+\gamma}{2} \cos \frac{\beta}{2} & -\cos \frac{\alpha-\gamma}{2} \sin \frac{\beta}{2} \\
\cos \frac{\alpha-\gamma}{2} \sin \frac{\beta}{2} & \cos \frac{\alpha+\gamma}{2} \cos \frac{\beta}{2}
\end{array}\right] } \\
& +i\left[\begin{array}{cc}
-\sin \frac{\alpha+\gamma}{2} \cos \frac{\beta}{2} & \sin \frac{\alpha-\gamma}{2} \sin \frac{\beta}{2} \\
\sin \frac{\alpha-\gamma}{2} \sin \frac{\beta}{2} & \sin \frac{\alpha+\gamma}{2} \cos \frac{\beta}{2}
\end{array}\right] \\
= & \left(\cos \frac{\alpha+\gamma}{2} \cos \frac{\beta}{2}\right)+\left(i \sin \frac{\alpha-\gamma}{2} \sin \frac{\beta}{2}\right) \sigma_{x} \\
& -\left(i \cos \frac{\alpha-\gamma}{2} \sin \frac{\beta}{2}\right) \sigma_{y}-\left(i \sin \frac{\alpha+\gamma}{2} \cos \frac{\beta}{2}\right) \sigma_{z} . \tag{5.182}
\end{align*}
$$

This gives us

$$
\begin{align*}
\cos \frac{\theta}{2} & =\cos \frac{\alpha+\gamma}{2} \cos \frac{\beta}{2} \\
\hat{\mathbf{n}} \sin \frac{\theta}{2} & =\left(-\sin \frac{\alpha-\gamma}{2} \sin \frac{\beta}{2}, \cos \frac{\alpha-\gamma}{2} \sin \frac{\beta}{2}, \sin \frac{\alpha+\gamma}{2} \cos \frac{\beta}{2}\right) . \tag{5.183}
\end{align*}
$$

The angle is

$$
\begin{equation*}
\theta=2 \operatorname{atan} \frac{\sqrt{\sin ^{2} \frac{\beta}{2}+\sin ^{2} \frac{\alpha+\gamma}{2} \cos ^{2} \frac{\beta}{2}}}{\cos \frac{\alpha+\gamma}{2} \cos \frac{\beta}{2}} \tag{5.184}
\end{equation*}
$$

or

$$
\begin{equation*}
\theta=2 \operatorname{atan} \frac{\sqrt{\tan ^{2} \frac{\beta}{2}+\sin ^{2} \frac{\alpha+\gamma}{2}}}{\cos \frac{\alpha+\gamma}{2}} \tag{5.185}
\end{equation*}
$$

and the normal direction is

$$
\begin{aligned}
\hat{\mathbf{n}}= & \frac{1}{\sqrt{1-\cos ^{2} \frac{\alpha+\gamma}{2} \cos ^{2} \frac{\beta}{2}}} \\
& \left(-\sin \frac{\alpha-\gamma}{2} \sin \frac{\beta}{2}, \cos \frac{\alpha-\gamma}{2} \sin \frac{\beta}{2}, \sin \frac{\alpha+\gamma}{2} \cos \frac{\beta}{2}\right) .
\end{aligned}
$$

## Exercise 5.6 Angular momentum addition. (2015 ps6 p2)

You have to add angular momenta $j_{1}=1$ and $j_{2}=2$ to form total angular momentum states with $j=1,2,3$. There are a total of $15|j, m\rangle$ states in the total angular momentum basis. Express each of them in terms of the old basis $\left|j_{1}, j_{2} ; m_{1}, m_{2}\right\rangle$ set.
Answer for Exercise 5.6
This problem was computed using ps6:clebschGordan2.nb.

## Exercise 5.7 Spin-1 rotations. (2015 ps6 p3)

Consider angular momentum $j=1$.
a. Express $\left\langle j=1, m^{\prime}\right| \hat{J}_{y}|j=1, m\rangle$ as a $3 \times 3$ matrix.
b. Show that for $j=1$

$$
\begin{equation*}
e^{-i \hat{J}_{y} \beta / \hbar}=1-i \frac{\hat{J}_{y}}{\hbar} \sin \beta-\frac{\hat{J}_{y}^{2}}{\hbar^{2}}(1-\cos \beta) . \tag{5.187}
\end{equation*}
$$

Answer for Exercise 5.7

Part a. From [15] (5.41), for $j^{\prime}=j=1$, the matrix elements for the ladder operators can be summarized as

$$
\begin{equation*}
\left\langle j, m^{\prime}\right| \hat{J}_{ \pm}|j, m\rangle=\hbar \sqrt{(1 \mp m)(2 \pm m)} \delta_{m^{\prime}, m \pm 1} \tag{5.188}
\end{equation*}
$$

With

$$
\begin{align*}
& \hat{J}_{+}=\hat{J}_{x}+i \hat{J}_{y} \\
& \hat{J}_{-}=\hat{J}_{x}-i \hat{J}_{y} \tag{5.189}
\end{align*}
$$

we have

$$
\begin{equation*}
\hat{J}_{y}=\frac{\hat{J}_{+}-\hat{J}_{-}}{2 i} \tag{5.190}
\end{equation*}
$$

so

$$
\frac{2 i}{\hbar}\left\langle j, m^{\prime}\right| \hat{J}_{y}|j, m\rangle=\sqrt{(1-m)(2+m)} \delta_{m^{\prime}, m+1}-\sqrt{(1+m)(2-m)} \delta_{m^{\prime}, m-1}
$$

We have nine matrix elements

$$
\begin{align*}
& \frac{2 i}{\hbar}\langle j, 1| \hat{J}_{y}|j, 1\rangle= \\
& \sqrt{(1-1)(2+1)} \delta_{1,1+1}-\sqrt{(1+1)(2-1)} \delta_{1,1-1}=0 \\
& \frac{2 i}{\hbar}\langle j, 1| \hat{J}_{y}|j, 0\rangle= \\
& \sqrt{(1-0)(2+0)} \delta_{1,0+1}-\sqrt{(1+0)(2-0)} \delta_{1,0-1}=\sqrt{2} \\
& \frac{2 i}{\hbar}\langle j, 1| \hat{J}_{y}|j,-1\rangle= \\
& \quad \sqrt{(1--1)(2+-1)} \delta_{1,-1+1}-\sqrt{(1+-1)(2--1)} \delta_{1,-1-1}=0 \\
& \frac{2 i}{\hbar}\langle j, 0| \hat{J}_{y}|j, 1\rangle= \\
& \quad \sqrt{(1-1)(2+1)} \delta_{0,1+1}-\sqrt{(1+1)(2-1)} \delta_{0,1-1}=-\sqrt{2} \\
& \frac{2 i}{\hbar}\langle j, 0| \hat{J}_{y}|j, 0\rangle= \\
& \quad \sqrt{(1-0)(2+0)} \delta_{0,0+1}-\sqrt{(1+0)(2-0)} \delta_{0,0-1}=0 \\
& \frac{2 i}{\hbar}\langle j, 0| \hat{J}_{y}|j,-1\rangle= \\
& \quad \sqrt{(1--1)(2+-1)} \delta_{0,-1+1}-\sqrt{(1+-1)(2--1)} \delta_{0,-1-1}=\sqrt{2} \\
& \frac{2 i}{\hbar}\langle j,-1| \hat{J}_{y}|j, 1\rangle= \\
& \quad \sqrt{(1-1)(2+1)} \delta_{-1,1+1}-\sqrt{(1+1)(2-1)} \delta_{-1,1-1}=0 \\
& \frac{2 i}{\hbar}\langle j,-1| \hat{J}_{y}|j, 0\rangle= \\
& \frac{2 i}{(1-0)(2+0)} \delta_{-1,0+1}\langle j,-1| \hat{J}_{y}|j,-1\rangle= \\
& \sqrt{(1--1)(2+-1)} \delta_{-1,-1+1}-\sqrt{(1+0)(2-0)} \delta_{-1,0-1}=-\sqrt{2}
\end{align*}
$$

Put into matrix form, that is

$$
\left\langle j, m^{\prime}\right| \hat{J}_{y}|j, m\rangle=\frac{\hbar}{2 i}\left[\begin{array}{ccc}
0 & \sqrt{2} & 0  \tag{5.193}\\
-\sqrt{2} & 0 & \sqrt{2} \\
0 & -\sqrt{2} & 0
\end{array}\right]
$$

or

$$
\hat{J}_{y}=\frac{\hbar i}{\sqrt{2}}\left[\begin{array}{ccc}
0 & -1 & 0  \tag{5.194}\\
1 & 0 & -1 \\
0 & 1 & 0
\end{array}\right] .
$$

Part b. The square of the matrix representation of $\hat{J}_{y}$ of eq. (5.194) is

$$
\begin{align*}
\hat{J}_{y}^{2} & =-\frac{\hbar^{2}}{2}\left[\begin{array}{ccc}
0 & -1 & 0 \\
1 & 0 & -1 \\
0 & 1 & 0
\end{array}\right]\left[\begin{array}{ccc}
0 & -1 & 0 \\
1 & 0 & -1 \\
0 & 1 & 0
\end{array}\right]  \tag{5.195}\\
& =-\frac{\hbar^{2}}{2}\left[\begin{array}{ccc}
-1 & 0 & 1 \\
0 & -2 & 0 \\
1 & 0 & -1
\end{array}\right],
\end{align*}
$$

and the cube is

$$
\begin{align*}
\hat{J}_{y}^{3} & =-\frac{\hbar^{2}}{2} \frac{\hbar i}{\sqrt{2}}\left[\begin{array}{ccc}
-1 & 0 & 1 \\
0 & -2 & 0 \\
1 & 0 & -1
\end{array}\right]\left[\begin{array}{ccc}
0 & -1 & 0 \\
1 & 0 & -1 \\
0 & 1 & 0
\end{array}\right] \\
& =-\frac{i \hbar^{3}}{2 \sqrt{2}}\left[\begin{array}{ccc}
0 & 2 & 0 \\
-2 & 0 & 2 \\
0 & -2 & 0
\end{array}\right]  \tag{5.196}\\
& =\hbar^{2} \frac{i \hbar}{\sqrt{2}}\left[\begin{array}{ccc}
0 & -1 & 0 \\
1 & 0 & -1 \\
0 & 1 & 0
\end{array}\right] \\
& =\hbar^{2} \hat{J}_{y} .
\end{align*}
$$

This proves (5.55) from the text

$$
\begin{equation*}
\frac{\hat{y}_{y}^{3}}{\hbar^{3}}=\frac{\hat{J}_{y}}{\hbar} . \tag{5.197}
\end{equation*}
$$

In particular

$$
\begin{align*}
& \left(\frac{\hat{J}_{y}}{\hbar}\right)^{0}=1 \\
& \left(\frac{\hat{J}_{y}}{\hbar}\right)^{1}=\left(\frac{\hat{J}_{y}}{\hbar}\right)^{1} \\
& \left(\frac{\hat{J}_{y}}{\hbar}\right)^{2}=\left(\frac{\hat{J}_{y}}{\hbar}\right)^{2} \\
& \left(\frac{\hat{J}_{y}}{\hbar}\right)^{3}=\frac{\hat{J}_{y}}{\hbar}  \tag{5.198}\\
& \left(\frac{\hat{J}_{y}}{\hbar}\right)^{4}=\left(\frac{\hat{J}_{y}}{\hbar}\right)^{2} \\
& \left(\frac{\hat{J}_{y}}{\hbar}\right)^{5}=\frac{\hat{J}_{y}}{\hbar}
\end{align*}
$$

Expanding the exponential in series we have

$$
\begin{align*}
e^{-i \beta \hat{J}_{y} / \hbar=} & 1-\frac{i \beta \hat{J}_{y} / \hbar}{1!}+\frac{\left(-i \beta \hat{J}_{y} / \hbar\right)^{2}}{2!}+\frac{\left(-i \beta \hat{J}_{y} / \hbar\right)^{3}}{3!} \\
& +\frac{\left(-i \beta \hat{J}_{y} / \hbar\right)^{4}}{4!}+\frac{\left(-i \beta \hat{J}_{y} / \hbar\right)^{5}}{5!}+\frac{\left(-i \beta \hat{J}_{y} / \hbar\right)^{6}}{6!}+\cdots \\
= & 1-i \frac{\beta \hat{J}_{y} / \hbar}{1!}-\frac{\left(\beta \hat{J}_{y} / \hbar\right)^{2}}{2!}+i \frac{\left(\beta \hat{J}_{y} / \hbar\right)^{3}}{3!} \\
& +\frac{\left(\beta \hat{J}_{y} / \hbar\right)^{4}}{4!}-i \frac{\left(\beta \hat{J}_{y} / \hbar\right)^{5}}{5!}-\frac{\left(\beta \hat{J}_{y} / \hbar\right)^{6}}{6!}+\cdots  \tag{5.199}\\
= & 1+i \frac{\hat{J}_{y}}{\hbar}\left(-\beta+\frac{\beta^{3}}{3!}-\frac{\beta^{5}}{5!}+\cdots\right) \\
+ & \frac{\hat{J}_{y}^{2}}{\hbar^{2}}\left(-\frac{\beta^{2}}{2!}+\frac{\beta^{4}}{4!}-\frac{\beta^{6}}{6!}+\cdots\right) \\
= & 1-i \frac{\hat{J}_{y}}{\hbar} \sin \beta+\frac{\hat{J}_{y}^{2}}{\hbar^{2}}(-1+\cos \beta)
\end{align*}
$$

Note that $J_{x}^{3} / \hbar^{3}=J_{x} / \hbar$ and $J_{z}^{3} / \hbar^{3}=J_{z} / \hbar$ too, so this relation generalizes to the other spin one operators as well.

Exercise $5.8 \quad L_{z}$ and $\mathbf{L}^{2}$ eigenvalues and probabilities. ([15] pr. 3.17)
Given a wave function

$$
\begin{equation*}
\psi(r, \theta, \phi)=f(r)(x+y+3 z) \tag{5.200}
\end{equation*}
$$

a. Determine if this wave function is an eigenfunction of $\mathbf{L}^{2}$, and the value of $l$ if it is an eigenfunction.
b. Determine the probabilities for the particle to be found in any given $|l, m\rangle$ state,
c. If it is known that $\psi$ is an energy eigenfunction with energy $E$ indicate how we can find $V(r)$.
Answer for Exercise 5.8

Part a. Using

$$
\begin{equation*}
\mathbf{L}^{2}=-\hbar^{2}\left(\frac{1}{\sin ^{2} \theta} \partial_{\phi \phi}+\frac{1}{\sin \theta} \partial_{\theta}\left(\sin \theta \partial_{\theta}\right)\right) \tag{5.201}
\end{equation*}
$$

and

$$
\begin{align*}
& x=r \sin \theta \cos \phi \\
& y=r \sin \theta \sin \phi  \tag{5.202}\\
& z=r \cos \theta .
\end{align*}
$$

It's a quick computation to show that

$$
\begin{equation*}
\mathbf{L}^{2} \psi=2 \hbar^{2} \psi=1(1+1) \hbar^{2} \psi \tag{5.203}
\end{equation*}
$$

so this function is an eigenket of $\mathbf{L}^{2}$ with an eigenvalue of $2 \hbar^{2}$, which corresponds to $l=1$, a p-orbital state.

Part b. Recall that the angular representation of $L_{z}$ is

$$
\begin{equation*}
L_{z}=-i \hbar \frac{\partial}{\partial \phi} \tag{5.204}
\end{equation*}
$$

so we have

$$
\begin{align*}
L_{z} x & =i \hbar y \\
L_{z} y & =-i \hbar x  \tag{5.205}\\
L_{z} z & =0
\end{align*}
$$

The $L_{z}$ action on $\psi$ is

$$
\begin{equation*}
L_{z} \psi=-i \hbar r f(r)(-y+x) \tag{5.206}
\end{equation*}
$$

This wave function is not an eigenket of $L_{z}$. Expressed in terms of the $L_{z}$ basis states $e^{i m \phi}$, this wave function is

$$
\begin{align*}
\psi & =r f(r)(\sin \theta(\cos \phi+\sin \phi)+\cos \theta) \\
& =r f(r)\left(\frac{\sin \theta}{2}\left(e^{i \phi}\left(1+\frac{1}{i}\right)+e^{-i \phi}\left(1-\frac{1}{i}\right)\right)+\cos \theta\right)  \tag{5.207}\\
& =r f(r)\left(\frac{(1-i) \sin \theta}{2} e^{1 i \phi}+\frac{(1+i) \sin \theta}{2} e^{-1 i \phi}+\cos \theta e^{0 i \phi}\right) .
\end{align*}
$$

Assuming that $\psi$ is normalized, the probabilities for measuring $m=1,-1,0$ respectively are

$$
\begin{align*}
P_{ \pm 1} & =2 \pi \rho\left|\frac{1 \mp i}{2}\right|^{2} \int_{0}^{\pi} \sin \theta d \theta \sin ^{2} \theta \\
& =-2 \pi \rho \int_{1}^{-1} d u\left(1-u^{2}\right) \\
& =\left.2 \pi \rho\left(u-\frac{u^{3}}{3}\right)\right|_{-1} ^{1}  \tag{5.208}\\
& =2 \pi \rho\left(2-\frac{2}{3}\right) \\
& =\frac{8 \pi \rho}{3}
\end{align*}
$$

and

$$
\begin{equation*}
P_{0}=2 \pi \rho \int_{0}^{\pi} \sin \theta \cos \theta=0 \tag{5.209}
\end{equation*}
$$

where

$$
\begin{equation*}
\rho=\int_{0}^{\infty} r^{4}|f(r)|^{2} d r . \tag{5.210}
\end{equation*}
$$

Because the probabilities must sum to 1 , this means the $m= \pm 1$ states are equiprobable with $P_{ \pm}=1 / 2$, fixing $\rho=3 / 16 \pi$, even without knowing $f(r)$.

Part c. The operator $r^{2} \mathbf{p}^{2}$ can be decomposed into a $\mathbf{L}^{2}$ component and some other portions, from which we can write

$$
\begin{align*}
H \psi & =\left(\frac{\mathbf{p}^{2}}{2 m}+V(r)\right) \psi  \tag{5.211}\\
& =\left(-\frac{\hbar^{2}}{2 m}\left(\partial_{r r}+\frac{2}{r} \partial_{r}-\frac{1}{\hbar^{2} r^{2}} \mathbf{L}^{2}\right)+V(r)\right) \psi
\end{align*}
$$

(See: [15] eq. 6.21) In this case where $\mathbf{L}^{2} \psi=2 \hbar^{2} \psi$ we can rearrange for $V(r)$

$$
\begin{align*}
V(r) & =E+\frac{1}{\psi} \frac{\hbar^{2}}{2 m}\left(\partial_{r r}+\frac{2}{r} \partial_{r}-\frac{2}{r^{2}}\right) \psi  \tag{5.212}\\
& =E+\frac{1}{f(r)} \frac{\hbar^{2}}{2 m}\left(\partial_{r r}+\frac{2}{r} \partial_{r}-\frac{2}{r^{2}}\right) f(r)
\end{align*}
$$

See sakuraiProblem3.17.nb for some verifications of some of the algebra for this problem.

Exercise 5.9 Angular momentum expectation values. ([15] pr. 3.18)
Compute the expectation values for the first and second powers of the angular momentum operators with respect to states $|l m\rangle$.
Answer for Exercise 5.9
We can write the expectation values for the $L_{z}$ powers immediately

$$
\begin{equation*}
\left\langle L_{z}\right\rangle=m \hbar, \tag{5.213}
\end{equation*}
$$

and

$$
\begin{equation*}
\left\langle L_{z}^{2}\right\rangle=(m \hbar)^{2} \tag{5.214}
\end{equation*}
$$

For the x and y components first express the operators in terms of the ladder operators.

$$
\begin{align*}
& L_{+}=L_{x}+i L_{y} \\
& L_{-}=L_{x}-i L_{y} \tag{5.215}
\end{align*}
$$

Rearranging gives

$$
\begin{align*}
L_{x} & =\frac{1}{2}\left(L_{+}+L_{-}\right) \\
L_{y} & =\frac{1}{2 i}\left(L_{+}-L_{-}\right) \tag{5.216}
\end{align*}
$$

The first order expectations $\left\langle L_{x}\right\rangle,\left\langle L_{y}\right\rangle$ are both zero since $\left\langle L_{+}\right\rangle=\left\langle L_{-}\right\rangle$. For the second order expectation values we have

$$
\begin{align*}
L_{x}^{2} & =\frac{1}{4}\left(L_{+}+L_{-}\right)\left(L_{+}+L_{-}\right) \\
& =\frac{1}{4}\left(L_{+} L_{+}+L_{-} L_{-}+L_{+} L_{-}+L_{-} L_{+}\right)  \tag{5.217}\\
& =\frac{1}{4}\left(L_{+} L_{+}+L_{-} L_{-}+2\left(L_{x}^{2}+L_{y}^{2}\right)\right) \\
& =\frac{1}{4}\left(L_{+} L_{+}+L_{-} L_{-}+2\left(\mathbf{L}^{2}-L_{z}^{2}\right)\right),
\end{align*}
$$

and

$$
\begin{align*}
L_{y}^{2} & =-\frac{1}{4}\left(L_{+}-L_{-}\right)\left(L_{+}-L_{-}\right) \\
& =-\frac{1}{4}\left(L_{+} L_{+}+L_{-} L_{-}-L_{+} L_{-}-L_{-} L_{+}\right) \\
& =-\frac{1}{4}\left(L_{+} L_{+}+L_{-} L_{-}-2\left(L_{x}^{2}+L_{y}^{2}\right)\right)  \tag{5.218}\\
& =-\frac{1}{4}\left(L_{+} L_{+}+L_{-} L_{-}-2\left(\mathbf{L}^{2}-L_{z}^{2}\right)\right)
\end{align*}
$$

Any expectation value $\langle l m| L_{+} L_{+}|l m\rangle$ or $\langle l m| L_{-} L_{-}|l m\rangle$ will be zero, leaving

$$
\begin{align*}
\left\langle L_{x}^{2}\right\rangle & =\left\langle L_{y}^{2}\right\rangle \\
& =\frac{1}{4}\left\langle 2\left(\mathbf{L}^{2}-L_{z}^{2}\right)\right\rangle  \tag{5.219}\\
& =\frac{1}{2}\left(\hbar^{2} l(l+1)-(\hbar m)^{2}\right) .
\end{align*}
$$

Observe that we have

$$
\begin{equation*}
\left\langle L_{x}^{2}\right\rangle+\left\langle L_{y}^{2}\right\rangle+\left\langle L_{z}^{2}\right\rangle=\hbar^{2} l(l+1)=\left\langle\mathbf{L}^{2}\right\rangle, \tag{5.220}
\end{equation*}
$$

which is the quantum mechanical analogue of the classical scalar equation $\mathbf{L}^{2}=L_{x}^{2}+L_{y}^{2}+L_{z}^{2}$.

Exercise 5.10 Spin three halves spin interaction. ([15] pr. 3.33)
A spin 3/2 nucleus subjected to an external electric field has an interaction Hamiltonian of the form

$$
\begin{equation*}
H=\frac{e Q}{2 s(s-1) \hbar^{2}}\left(\left(\frac{\partial^{2} \phi}{\partial x^{2}}\right)_{0} S_{x}^{2}+\left(\frac{\partial^{2} \phi}{\partial y^{2}}\right)_{0} S_{y}^{2}+\left(\frac{\partial^{2} \phi}{\partial z^{2}}\right)_{0} S_{z}^{2}\right) . \tag{5.221}
\end{equation*}
$$

a. Show that the interaction energy can be written as

$$
\begin{equation*}
A\left(3 S_{z}^{2}-\mathbf{S}^{2}\right)+B\left(S_{+}^{2}+S_{-}^{2}\right) \tag{5.222}
\end{equation*}
$$

b. Find the energy eigenvalues for such a Hamiltonian.

Answer for Exercise 5.10

Part a. Reordering

$$
\begin{align*}
& S_{+}=S_{x}+i S_{y}  \tag{5.223}\\
& S_{-}=S_{x}-i S_{y}
\end{align*}
$$

gives

$$
\begin{align*}
S_{x} & =\frac{1}{2}\left(S_{+}+S_{-}\right) \\
S_{y} & =\frac{1}{2 i}\left(S_{+}-S_{-}\right) . \tag{5.224}
\end{align*}
$$

The squared spin operators are

$$
\begin{align*}
S_{x}^{2} & =\frac{1}{4}\left(S_{+}^{2}+S_{-}^{2}+S_{+} S_{-}+S_{-} S_{+}\right) \\
& =\frac{1}{4}\left(S_{+}^{2}+S_{-}^{2}+2\left(S_{x}^{2}+S_{y}^{2}\right)\right)  \tag{5.225}\\
& =\frac{1}{4}\left(S_{+}^{2}+S_{-}^{2}+2\left(\mathbf{S}^{2}-S_{z}^{2}\right)\right), \\
S_{y}^{2} & =-\frac{1}{4}\left(S_{+}^{2}+S_{-}^{2}-S_{+} S_{-}-S_{-} S_{+}\right) \\
& =-\frac{1}{4}\left(S_{+}^{2}+S_{-}^{2}-2\left(S_{x}^{2}+S_{y}^{2}\right)\right)  \tag{5.226}\\
& =-\frac{1}{4}\left(S_{+}^{2}+S_{-}^{2}-2\left(\mathbf{S}^{2}-S_{z}^{2}\right)\right) .
\end{align*}
$$

This gives

$$
\begin{align*}
& H=\frac{e Q}{2 s(s-1) \hbar^{2}}\left(\frac{1}{4}\left(\frac{\partial^{2} \phi}{\partial x^{2}}\right)_{0}\left(S_{+}^{2}+S_{-}^{2}+2\left(\mathbf{S}^{2}-S_{z}^{2}\right)\right)\right. \\
&-\left(\frac{\partial^{2} \phi}{\partial y^{2}}\right)_{0}\left(S_{+}^{2}+S_{-}^{2}-2\left(\mathbf{S}^{2}-S_{z}^{2}\right)\right) \\
&\left.+\left(\frac{\partial^{2} \phi}{\partial z^{2}}\right)_{0} S_{z}^{2}\right)  \tag{5.227}\\
&=\frac{e Q}{2 s(s-1) \hbar^{2}}\left(\frac{1}{4}\left(\left(\frac{\partial^{2} \phi}{\partial x^{2}}\right)_{0}-\left(\frac{\partial^{2} \phi}{\partial y^{2}}\right)_{0}\right)\left(S_{+}^{2}+S_{-}^{2}\right)\right. \\
&+\frac{1}{2}\left(\left(\frac{\partial^{2} \phi}{\partial x^{2}}\right)_{0}+\left(\frac{\partial^{2} \phi}{\partial y^{2}}\right)_{0}\right) \mathbf{S}^{2} \\
&\left.+\left(\left(\frac{\partial^{2} \phi}{\partial z^{2}}\right)_{0}-\frac{1}{2}\left(\frac{\partial^{2} \phi}{\partial x^{2}}\right)_{0}-\frac{1}{2}\left(\frac{\partial^{2} \phi}{\partial y^{2}}\right)_{0}\right) S_{z}^{2}\right)
\end{align*}
$$

For a static electric field we have

$$
\begin{equation*}
\nabla^{2} \phi=-\frac{\rho}{\epsilon_{0}} \tag{5.228}
\end{equation*}
$$

but are evaluating it at a point away from the generating charge distribution, so $\nabla^{2} \phi=0$ at that point. This gives

$$
\begin{align*}
H=\frac{e Q}{4 s(s-1) \hbar^{2}}( & \frac{1}{2}\left(\left(\frac{\partial^{2} \phi}{\partial x^{2}}\right)_{0}-\left(\frac{\partial^{2} \phi}{\partial y^{2}}\right)_{0}\right)\left(S_{+}^{2}+S_{-}^{2}\right) \\
& \left.+\left(\left(\frac{\partial^{2} \phi}{\partial x^{2}}\right)_{0}+\left(\frac{\partial^{2} \phi}{\partial y^{2}}\right)_{0}\right)\left(\mathbf{S}^{2}-3 S_{z}^{2}\right)\right) \tag{5.229}
\end{align*}
$$

so

$$
\begin{align*}
& A=-\frac{e Q}{4 s(s-1) \hbar^{2}}\left(\left(\frac{\partial^{2} \phi}{\partial x^{2}}\right)_{0}+\left(\frac{\partial^{2} \phi}{\partial y^{2}}\right)_{0}\right)  \tag{5.230}\\
& B=\frac{e Q}{8 s(s-1) \hbar^{2}}\left(\left(\frac{\partial^{2} \phi}{\partial x^{2}}\right)_{0}-\left(\frac{\partial^{2} \phi}{\partial y^{2}}\right)_{0}\right) . \tag{5.231}
\end{align*}
$$

Part b. Using sakuraiProblem3.33.nb, matrix representations for the spin three halves operators and the Hamiltonian were constructed with respect to the basis $\{|3 / 2\rangle,|1 / 2\rangle,|-1 / 2\rangle,|-3 / 2\rangle\}$

$$
\begin{align*}
& S_{+}=\hbar\left[\begin{array}{cccc}
0 & \sqrt{3} & 0 & 0 \\
0 & 0 & 2 & 0 \\
0 & 0 & 0 & \sqrt{3} \\
0 & 0 & 0 & 0
\end{array}\right] \\
& S_{-}=\hbar\left[\begin{array}{cccc}
0 & 0 & 0 & 0 \\
\sqrt{3} & 0 & 0 & 0 \\
0 & 2 & 0 & 0 \\
0 & 0 & \sqrt{3} & 0
\end{array}\right] \\
& S_{x}=\hbar\left[\begin{array}{cccc}
0 & \sqrt{3} / 2 & 0 & 0 \\
\sqrt{3} / 2 & 0 & 1 & 0 \\
0 & 1 & 0 & \sqrt{3} / 2 \\
0 & 0 & \sqrt{3} / 2 & 0
\end{array}\right] \\
& S_{y}=i \hbar\left[\begin{array}{cccc}
0 & -\sqrt{3} / 2 & 0 & 0 \\
\sqrt{3} / 2 & 0 & -1 & 0 \\
0 & 1 & 0 & -\sqrt{3} / 2 \\
0 & 0 & \sqrt{3} / 2 & 0
\end{array}\right]  \tag{5.232}\\
& S_{z}=\frac{\hbar}{2}\left[\begin{array}{cccc}
3 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & -1 & 0 \\
0 & 0 & 0 & -3
\end{array}\right] \\
& H=\left[\begin{array}{cccc}
3 A & 0 & 2 \sqrt{3} B & 0 \\
0 & -3 A & 0 & 2 \sqrt{3} B \\
2 \sqrt{3} B & 0 & -3 A & 0 \\
0 & 2 \sqrt{3} B & 0 & 3 A
\end{array}\right] .
\end{align*}
$$

The energy eigenvalues are found to be

$$
\begin{equation*}
E= \pm \hbar^{2} \sqrt{9 A^{2}+12 B^{2}} \tag{5.233}
\end{equation*}
$$

with two fold degeneracies for each eigenvalue.

## 6

### 6.1 APPROXIMATION METHODS.

Suppose we have a perturbed Hamiltonian

$$
\begin{equation*}
H=H_{0}+\lambda V \tag{6.1}
\end{equation*}
$$

where $\lambda=0$ represents a solvable (perhaps known) system, and $\lambda=1$ is the case of interest. There are two approaches of interest

1. Direct solution of $H$ with $\lambda=1$.
2. Take $\lambda$ small, and do a series expansion. This is perturbation theory.

### 6.2 VARIATIONAL METHODS.

Given

$$
\begin{equation*}
H\left|\phi_{n}\right\rangle=E_{n}\left|\phi_{n}\right\rangle, \tag{6.2}
\end{equation*}
$$

where we don't know $\left|\phi_{n}\right\rangle$, we can compute the expectation with respect to an arbitrary state $|\psi\rangle$

$$
\begin{align*}
\langle\psi| H|\psi\rangle & =\langle\psi| H\left(\sum_{n}\left|\phi_{n}\right\rangle\left\langle\phi_{n}\right|\right)|\psi\rangle \\
& =\sum_{n} E_{n}\left\langle\psi \mid \phi_{n}\right\rangle\left\langle\phi_{n} \mid \psi\right\rangle  \tag{6.3}\\
& =\sum_{n} E_{n}\left|\left\langle\psi \mid \phi_{n}\right\rangle\right|^{2} .
\end{align*}
$$

Define

$$
\begin{equation*}
\bar{E}=\frac{\langle\psi| H|\psi\rangle}{\langle\psi \mid \psi\rangle} \tag{6.4}
\end{equation*}
$$

Assuming that it is possible to express the state in the Hamiltonian energy basis

$$
\begin{equation*}
|\psi\rangle=\sum_{n} a_{n}\left|\phi_{n}\right\rangle \tag{6.5}
\end{equation*}
$$

this average energy is

$$
\begin{align*}
\bar{E} & =\frac{\sum_{m, n}\left\langle\phi_{m}\right| a_{m}^{*} H a_{n}\left|\phi_{n}\right\rangle}{\sum_{n}\left|a_{n}\right|^{2}} \\
& =\frac{\sum_{n}\left|a_{n}\right|^{2} E_{n}}{\sum_{n}\left|a_{n}\right|^{2}} . \\
& =\sum_{n} \frac{\left|a_{n}\right|^{2}}{\sum_{n}\left|a_{n}\right|^{2}} E_{n}  \tag{6.6}\\
& =\sum_{n} \frac{P_{n}}{\sum_{m} P_{m}} E_{n},
\end{align*}
$$

where $P_{m}=\left|a_{m}\right|^{2}$, which has the structure of a probability coefficient once divided by $\sum_{m} P_{m}$, as sketched in fig. 6.1. This average energy is a


Figure 6.1: A decreasing probability distribution.
probability weighted average of the individual energy basis states. One of those energies is the ground state energy $E_{1}$, so we necessarily have

$$
\begin{equation*}
\bar{E} \geq E_{1} \tag{6.7}
\end{equation*}
$$

## Example 6.1: Particle in an offset box.

For the infinite potential box sketched in fig. 6.2.


Figure 6.2: Infinite potential $[0, L]$ box.
The exact solutions for such a system are found to be

$$
\begin{equation*}
\psi(x)=\sqrt{\frac{2}{L}} \sin \left(\frac{n \pi}{L} x\right) \tag{6.8}
\end{equation*}
$$

where the energies are

$$
\begin{equation*}
E=\frac{\hbar^{2}}{2 m} \frac{n^{2} \pi^{2}}{L^{2}} \tag{6.9}
\end{equation*}
$$

The function $\psi^{\prime}=x(L-x)$ also satisfies the boundary value constraints? How close in energy is that function to the ground state?

$$
\begin{align*}
\bar{E} & =-\frac{\hbar^{2}}{2 m} \frac{\int_{0}^{L} d x x(L-x) \frac{d^{2}}{d x^{2}}(x(L-x))}{\int_{0}^{L} d x x^{2}(L-x)^{2}} \\
& =\frac{\hbar^{2}}{2 m} \frac{\frac{2 L^{3}}{6}}{\frac{L^{5}}{30}}  \tag{6.10}\\
& =\frac{\hbar^{2}}{2 m} \frac{10}{L^{2}}
\end{align*}
$$

This average energy is quite close to the ground state energy

$$
\begin{equation*}
\frac{\bar{E}}{E_{1}}=\frac{10}{\pi^{2}}=1.014 \tag{6.11}
\end{equation*}
$$

## Example 6.2: Particle in a symmetric box.

\section*{|  |  |  |
| :--- | :--- | :--- |
| -42 | 0 |  |
|  |  |  |
|  |  |  |
|  |  |  |
| $1 / 2$ |  |  |}

Figure 6.3: Infinite potential $[-L / 2, L / 2]$ box.
Shifting the boundaries, as sketched in fig. 6.3 doesn't change the energy levels. For this potential let's try a shifted trial function

$$
\begin{equation*}
\psi(x)=\left(x-\frac{L}{2}\right)\left(x+\frac{L}{2}\right)=x^{2}-\frac{L^{2}}{4} \tag{6.12}
\end{equation*}
$$

without worrying about the form of the exact solution. This produces the same result as above

$$
\begin{align*}
\bar{E} & =-\frac{\hbar^{2}}{2 m} \frac{\int_{0}^{L} d x\left(x^{2}-\frac{L^{2}}{4}\right) \frac{d^{2}}{d x^{2}}\left(x^{2}-\frac{L^{2}}{4}\right)}{\int_{0}^{L} d x\left(x^{2}-\frac{L^{2}}{4}\right)^{2}} \\
& =-\frac{\hbar^{2}}{2 m} \frac{-2 L^{3} / 6}{\frac{L^{5}}{30}}  \tag{6.13}\\
& =\frac{\hbar^{2}}{2 m} \frac{10}{L^{2}} .
\end{align*}
$$

Summary (Nishant) The above example is that of a particle in a box. The actual wave function is a sin as shown. But we can come up with a guess wave function that meets the boundary conditions and ask how accurate it is compared to the actual one.

Basically we are assuming a wave function form and then seeing how it differs from the exact form. We cannot do this if we have nothing to compare it against. But, we note that the variance of the number operator in the systems eigenstate is zero. So we can still calculate the variance and try to minimize it. This is one way of coming up with an approximate wave function. This does not necessarily give the ground state wave function though. For this we need to minimize the energy itself.

### 6.3 VARIATIONAL METHOD.

Today we want to use the variational degree of freedom to try to solve some problems that we don't have analytic solutions for.

## Anharmonic oscillator

$$
\begin{equation*}
V(x)=\frac{1}{2} m \omega^{2} x^{2}+\lambda x^{4}, \quad \lambda \geq 0 \tag{6.14}
\end{equation*}
$$

With the potential growing faster than the harmonic oscillator, which had a ground state solution

$$
\begin{equation*}
\psi(x)=\frac{1}{\pi^{1 / 4}} \frac{1}{a_{0}^{1 / 2}} e^{-x^{2} / 2 a_{0}^{2}} \tag{6.15}
\end{equation*}
$$

where

$$
\begin{equation*}
a_{0}=\sqrt{\frac{\hbar}{m \omega}} \tag{6.16}
\end{equation*}
$$

Let's try allowing $a_{0} \rightarrow a$, to be a variational degree of freedom

$$
\begin{align*}
& \psi_{a}(x)=\frac{1}{\pi^{1 / 4}} \frac{1}{a^{1 / 2}} e^{-x^{2} / 2 a^{2}}  \tag{6.17}\\
& \left\langle\psi_{a}\right| H\left|\psi_{a}\right\rangle=\left\langle\psi_{a}\right| \frac{p^{2}}{2 m}+\frac{1}{2} m \omega^{2} x^{2}+\lambda x^{4}\left|\psi_{a}\right\rangle \tag{6.18}
\end{align*}
$$

We can find

$$
\begin{align*}
& \left\langle x^{2}\right\rangle=\frac{1}{2} a^{2},  \tag{6.19}\\
& \left\langle x^{4}\right\rangle=\frac{3}{4} a^{4} . \tag{6.20}
\end{align*}
$$

Define

$$
\begin{equation*}
\tilde{\omega}=\frac{\hbar}{m a^{2}}, \tag{6.21}
\end{equation*}
$$

so that

$$
\begin{align*}
\bar{E}_{a} & =\left\langle\psi_{a}\right|\left(\frac{p^{2}}{2 m}+\frac{1}{2} m \tilde{\omega}^{2} x^{2}\right)+\left(\frac{1}{2} m\left(\omega^{2}-\tilde{\omega}^{2}\right) x^{2}+\lambda x^{4}\right)\left|\psi_{a}\right\rangle  \tag{6.22}\\
& =\frac{1}{2} \hbar \tilde{\omega}+\frac{1}{2} m\left(\omega^{2}-\tilde{\omega}^{2}\right) \frac{1}{2} a^{2}+\frac{3}{4} \lambda a^{4} .
\end{align*}
$$

Write this as

$$
\begin{equation*}
\bar{E}_{\tilde{\omega}}=\frac{1}{2} \hbar \tilde{\omega}+\frac{1}{4} \frac{\hbar}{\tilde{\omega}}\left(\omega^{2}-\tilde{\omega}^{2}\right)+\frac{3}{4} \lambda \frac{\hbar^{2}}{m^{2} \tilde{\omega}^{2}} . \tag{6.23}
\end{equation*}
$$

This might look something like fig. 6.4. Demand that


Figure 6.4: Energy after perturbation.

$$
\begin{align*}
0 & =\frac{\partial \bar{E}_{\tilde{\omega}}}{\partial \tilde{\omega}} \\
& =\frac{\hbar}{2}-\frac{\hbar}{4} \frac{\omega^{2}}{\tilde{\omega}^{2}}-\frac{\hbar}{4}+\frac{3}{4}(-2) \frac{\lambda \hbar^{2}}{m^{2} \tilde{\omega}^{3}}  \tag{6.24}\\
& =\frac{\hbar}{4}\left(1-\frac{\omega^{2}}{\tilde{\omega}^{2}}-6 \frac{\lambda \hbar}{m^{2} \tilde{\omega}^{3}}\right),
\end{align*}
$$

or

$$
\begin{equation*}
\tilde{\omega}^{3}-\omega^{2} \tilde{\omega}-\frac{6 \lambda \hbar}{m^{2}}=0 . \tag{6.25}
\end{equation*}
$$

for $\lambda a_{0}^{4} \ll \hbar \omega$, we have something like $\tilde{\omega}=\omega+\epsilon$. Expanding eq. (6.25) to first order in $\epsilon$, this gives

$$
\begin{equation*}
\omega^{3}+3 \omega^{2} \epsilon-\omega^{2}(\omega+\epsilon)-\frac{6 \lambda \hbar}{m^{2}}=0 \tag{6.26}
\end{equation*}
$$

so that

$$
\begin{equation*}
2 \omega^{2} \epsilon=\frac{6 \lambda \hbar}{m^{2}} \tag{6.27}
\end{equation*}
$$

and

$$
\begin{equation*}
\hbar \epsilon=\frac{3 \lambda \hbar^{2}}{m^{2} \omega^{2}}=3 \lambda a_{0}^{4} . \tag{6.28}
\end{equation*}
$$

Table 6.1: Comparing numeric and variational solutions

| $\hbar / \omega$ | numeric | variational |
| :--- | :--- | :--- |
| 100 | 3.13 | 3.16 |
| 1000 | 6.69 | 6.81 |

Plugging into

$$
\begin{align*}
\bar{E}_{\omega+\epsilon} & =\frac{1}{2} \hbar(\omega+\epsilon)+\frac{1}{4} \frac{\hbar}{\omega}\left(-2 \omega \epsilon+\epsilon^{2}\right)+\frac{3}{4} \lambda \frac{\hbar^{2}}{m^{2} \omega^{2}} \\
& \approx \frac{1}{2} \hbar(\omega+\epsilon)-\frac{1}{2} \hbar \epsilon+\frac{3}{4} \lambda \frac{\hbar^{2}}{m^{2} \omega^{2}}  \tag{6.29}\\
& =\frac{1}{2} \hbar \omega+\frac{3}{4} \lambda a_{0}^{4} .
\end{align*}
$$

With eq. (6.28), that is

$$
\begin{equation*}
\bar{E}_{\tilde{\omega}=\omega+\epsilon} \approx \frac{1}{2} \hbar\left(\omega+\frac{\epsilon}{2}\right) . \tag{6.30}
\end{equation*}
$$

The energy levels are shifted slightly for each shift in the Hamiltonian frequency.

What do we have in the extreme anharmonic limit, where $\lambda a_{0}^{4} \gg \hbar \omega$ ? Now we get

$$
\begin{equation*}
\tilde{\omega}^{*}=\left(\frac{6 \hbar \lambda}{m^{2}}\right)^{1 / 3} \tag{6.31}
\end{equation*}
$$

and

$$
\begin{equation*}
\bar{E}_{\tilde{\omega}^{*}}=\frac{\hbar^{4 / 3} \lambda^{1 / 3}}{m^{2 / 3}} \frac{3}{8} 6^{1 / 3} \tag{6.32}
\end{equation*}
$$

(this last result is pulled from a web treatment somewhere of the anharmonic oscillator). Note that the first factor in this energy, with $\hbar^{4} \lambda / m^{2}$ travelling together could have been worked out on dimensional grounds.

This variational method tends to work quite well in these limits. For a system where $m=\omega=\hbar=1$, for this problem, we have

Example: (sketch) double well potential

$$
\begin{equation*}
V(x)=\frac{m \omega^{2}}{8 a^{2}}(x-a)^{2}(x+a)^{2} \tag{6.33}
\end{equation*}
$$



Figure 6.5: Double well potential.

Note that this potential, and the Hamiltonian, both commute with parity. We are interested in the regime where $a_{0}^{2}=\frac{\hbar}{m \omega} \ll a^{2}$. Near $x= \pm a$, this will be approximately

$$
\begin{equation*}
V(x)=\frac{1}{2} m \omega^{2}(x \pm a)^{2} . \tag{6.34}
\end{equation*}
$$

Guessing a wave function that is an eigenstate of parity

$$
\begin{equation*}
\Psi_{ \pm}=g_{ \pm}\left(\phi_{\mathrm{R}}(x) \pm \phi_{\mathrm{L}}(x)\right) . \tag{6.35}
\end{equation*}
$$

Perhaps this looks like the even and odd functions sketched in fig. 6.6, and fig. 6.7. Using harmonic oscillator functions


Figure 6.6: Even double well function.


Figure 6.7: Odd double well function.

$$
\begin{align*}
\phi_{\mathrm{L}}(x) & =\Psi_{\text {Н.О. }}(x+a),  \tag{6.36}\\
\phi_{\mathrm{R}}(x) & =\Psi_{\text {H.O. }}(x-a) .
\end{align*}
$$

After doing a lot of integral (i.e. in the problem set), we will see a splitting of the variational energy levels as sketched in fig. 6.8. This sort of level


Figure 6.8: Splitting for double well potential.
splitting was what was used in the very first mazers.

### 6.4 PERTURBATION THEORY (OUTLINE).

Given

$$
\begin{equation*}
H=H_{0}+\lambda V \tag{6.37}
\end{equation*}
$$

where $\lambda V$ is "small". We want to figure out the eigenvalues and eigenstates of this Hamiltonian

$$
\begin{equation*}
H|n\rangle=E_{n}|n\rangle . \tag{6.38}
\end{equation*}
$$

We don't know what these are, but do know that

$$
\begin{equation*}
H_{0}\left|n^{(0)}\right\rangle=E_{n}^{(0)}\left|n^{(0)}\right\rangle \tag{6.39}
\end{equation*}
$$

We are hoping that the level transitions have adiabatic transitions between the original and perturbed levels as sketched in fig. 6.9. and not crossed


Figure 6.9: Adiabatic transitions.
level transitions as sketched in fig. 6.10. If we have level crossings (which can in general occur), as opposed to adiabatic transitions, then we have no hope of using perturbation theory.


Figure 6.10: Crossed level transitions.

### 6.5 SIMPLEST PERTURBATION EXAMPLE.

Given a $2 \times 2$ Hamiltonian $H=H_{0}+V$, where

$$
H=\left[\begin{array}{ll}
a & c  \tag{6.40}\\
c^{*} & b
\end{array}\right]
$$

note that if $c=0$ is

$$
H=H_{0}=\left[\begin{array}{ll}
a & 0  \tag{6.41}\\
0 & b
\end{array}\right]
$$

The off diagonal terms can be considered to be a perturbation

$$
V=\left[\begin{array}{ll}
0 & c  \tag{6.42}\\
c^{*} & 0
\end{array}\right]
$$

with $H=H_{0}+V$.

Energy levels after perturbation We can solve for the eigenvalues of $H$ easily, finding

$$
\begin{equation*}
\lambda_{ \pm}=\frac{a+b}{2} \pm \sqrt{\left(\frac{a-b}{2}\right)^{2}+|c|^{2}} \tag{6.43}
\end{equation*}
$$

Plots of a few $a, b$ variations of $\lambda_{ \pm}$are shown in fig. 6.11. The quadratic (non-degenerate) domain is found near the $c=0$ points of all but the first $(a=b)$ plot, and the degenerate (linear in $\left.|c|^{2}\right)$ regions are visible for larger values of $c$.

Some approximations $\quad$ Suppose that $|c| \ll|a-b|$, then

$$
\begin{equation*}
\lambda_{ \pm} \approx \frac{a+b}{2} \pm\left|\frac{a-b}{2}\right|\left(1+2 \frac{|c|^{2}}{|a-b|^{2}}\right) \tag{6.44}
\end{equation*}
$$



Figure 6.11: $\lambda_{ \pm}$for $(a, b) \in\{(1,1),(1,0),(1,5),(-8,8)\}$.

If $a>b$, then

$$
\begin{align*}
\lambda_{ \pm} & \approx \frac{a+b}{2} \pm \frac{a-b}{2}\left(1+2 \frac{|c|^{2}}{(a-b)^{2}}\right)  \tag{6.45}\\
\lambda_{+} & =\frac{a+b}{2}+\frac{a-b}{2}\left(1+2 \frac{|c|^{2}}{(a-b)^{2}}\right) \\
& =a+(a-b) \frac{|c|^{2}}{(a-b)^{2}}  \tag{6.46}\\
& =a+\frac{|c|^{2}}{a-b}
\end{align*}
$$

and

$$
\begin{align*}
\lambda_{-} & =\frac{a+b}{2}-\frac{a-b}{2}\left(1+2 \frac{|c|^{2}}{(a-b)^{2}}\right) \\
& =b+(a-b) \frac{|c|^{2}}{(a-b)^{2}}  \tag{6.47}\\
& =b+\frac{|c|^{2}}{a-b}
\end{align*}
$$

This adiabatic evolution displays a "level repulsion", quadratic in $|c|$, and is described as a non-degenerate permutation.

If $|c| \gg|a-b|$, then

$$
\begin{align*}
\lambda_{ \pm} & =\frac{a+b}{2} \pm|c| \sqrt{1+\frac{1}{|c|^{2}}\left(\frac{a-b}{2}\right)^{2}} \approx \frac{a+b}{2} \pm|c|\left(1+\frac{1}{2|c|^{2}}\left(\frac{a-b}{2}\right)^{2}\right) \\
& =\frac{a+b}{2} \pm|c| \pm \frac{(a-b)^{2}}{8|c|} . \tag{6.48}
\end{align*}
$$

Here we loose the adiabaticity, and have "level repulsion" that is linear in $|c|$. We no longer have the sign of $a-b$ in the expansion. This is described as a degenerate permutation.

### 6.6 General non-degenerate perturbation.

Given an unperturbed system with solutions of the form

$$
\begin{equation*}
H_{0}\left|n^{(0)}\right\rangle=E_{n}^{(0)}\left|n^{(0)}\right\rangle \tag{6.49}
\end{equation*}
$$

we want to solve the perturbed Hamiltonian equation

$$
\begin{equation*}
\left(H_{0}+\lambda V\right)|n\rangle=\left(E_{n}^{(0)}+\Delta n\right)|n\rangle . \tag{6.50}
\end{equation*}
$$

Here $\Delta n$ is an energy shift as that goes to zero as $\lambda \rightarrow 0$. We can write this as

$$
\begin{equation*}
\left(E_{n}^{(0)}-H_{0}\right)|n\rangle=\left(\lambda V-\Delta_{n}\right)|n\rangle . \tag{6.51}
\end{equation*}
$$

We are hoping to iterate with application of the inverse to an initial estimate of $|n\rangle$

$$
\begin{equation*}
|n\rangle=\left(E_{n}^{(0)}-H_{0}\right)^{-1}\left(\lambda V-\Delta_{n}\right)|n\rangle . \tag{6.52}
\end{equation*}
$$

This gets us into trouble if $\lambda \rightarrow 0$, which can be fixed by using

$$
\begin{equation*}
|n\rangle=\left(E_{n}^{(0)}-H_{0}\right)^{-1}\left(\lambda V-\Delta_{n}\right)|n\rangle+\left|n^{(0)}\right\rangle, \tag{6.53}
\end{equation*}
$$

which can be seen to be a solution to eq. (6.51). We want to ask if

$$
\begin{equation*}
\left(\lambda V-\Delta_{n}\right)|n\rangle, \tag{6.54}
\end{equation*}
$$

contains a bit of $\left|n^{(0)}\right\rangle$ ? To determine this act with $\left\langle n^{(0)}\right|$ on the left

$$
\begin{align*}
\left\langle n^{(0)}\right|\left(\lambda V-\Delta_{n}\right)|n\rangle & =\left\langle n^{(0)}\right|\left(E_{n}^{(0)}-H_{0}\right)|n\rangle \\
& =\left(E_{n}^{(0)}-E_{n}^{(0)}\right)\left\langle n^{(0)} \mid n\right\rangle  \tag{6.55}\\
& =0 .
\end{align*}
$$

This shows that $|n\rangle$ is entirely orthogonal to $\left|n^{(0)}\right\rangle$. First define a projection operator

$$
\begin{equation*}
P_{n}=\left|n^{(0)}\right\rangle\left\langle n^{(0)}\right| \tag{6.56}
\end{equation*}
$$

which has the idempotent property $P_{n}^{2}=P_{n}$ that we expect of a projection operator. Now define a rejection operator

$$
\begin{align*}
\bar{P}_{n} & =1-\left|n^{(0)}\right\rangle\left\langle n^{(0)}\right| \\
& =\sum_{m \neq n}\left|m^{(0)}\right\rangle\left\langle m^{(0)}\right| . \tag{6.57}
\end{align*}
$$

Because $|n\rangle$ has no component in the direction $\left|n^{(0)}\right\rangle$, the rejection operator can be inserted much like we normally do with the identity operator, yielding

$$
\begin{equation*}
|n\rangle^{\prime}=\left(E_{n}^{(0)}-H_{0}\right)^{-1} \bar{P}_{n}\left(\lambda V-\Delta_{n}\right)|n\rangle+\left|n^{(0)}\right\rangle \tag{6.58}
\end{equation*}
$$

valid for any initial $|n\rangle$.

Power series perturbation expansion Instead of iterating, suppose that the unknown state and unknown energy difference operator can be expanded in a $\lambda$ power series, say

$$
\begin{equation*}
|n\rangle=\left|n_{0}\right\rangle+\lambda\left|n_{1}\right\rangle+\lambda^{2}\left|n_{2}\right\rangle+\lambda^{3}\left|n_{3}\right\rangle+\cdots \tag{6.59}
\end{equation*}
$$

and

$$
\begin{equation*}
\Delta_{n}=\Delta_{n_{0}}+\lambda \Delta_{n_{1}}+\lambda^{2} \Delta_{n_{2}}+\lambda^{3} \Delta_{n_{3}}+\cdots \tag{6.60}
\end{equation*}
$$

We usually interpret functions of operators in terms of power series expansions. In the case of $\left(E_{n}^{(0)}-H_{0}\right)^{-1}$, we have a concrete interpretation when acting on one of the unperturbed eigenstates

$$
\begin{equation*}
\frac{1}{E_{n}^{(0)}-H_{0}}\left|m^{(0)}\right\rangle=\frac{1}{E_{n}^{(0)}-E_{m}^{0}}\left|m^{(0)}\right\rangle \tag{6.61}
\end{equation*}
$$

This gives

$$
\begin{equation*}
|n\rangle=\frac{1}{E_{n}^{(0)}-H_{0}} \sum_{m \neq n}\left|m^{(0)}\right\rangle\left\langle m^{(0)}\right|\left(\lambda V-\Delta_{n}\right)|n\rangle+\left|n^{(0)}\right\rangle, \tag{6.62}
\end{equation*}
$$

or

$$
\begin{equation*}
|n\rangle=\left|n^{(0)}\right\rangle+\sum_{m \neq n} \frac{\left|m^{(0)}\right\rangle\left\langle m^{(0)}\right|}{E_{n}^{(0)}-E_{m}^{(0)}}\left(\lambda V-\Delta_{n}\right)|n\rangle \tag{6.63}
\end{equation*}
$$

From eq. (6.51), note that

$$
\begin{equation*}
\Delta_{n}=\frac{\left\langle n^{(0)}\right| \lambda V|n\rangle}{\left\langle n^{0} \mid n\right\rangle} \tag{6.64}
\end{equation*}
$$

however, we will normalize by setting $\left\langle n^{0} \mid n\right\rangle=1$, so

$$
\begin{equation*}
\Delta_{n}=\left\langle n^{(0)}\right| \lambda V|n\rangle \tag{6.65}
\end{equation*}
$$

to $O\left(\lambda^{0}\right)$ If all $\lambda^{n}, n>0$ are zero, then we have

$$
\begin{equation*}
\left|n_{0}\right\rangle=\left|n^{(0)}\right\rangle+\sum_{m \neq n} \frac{\left|m^{(0)}\right\rangle\left\langle m^{(0)}\right|}{E_{n}^{(0)}-E_{m}^{(0)}}\left(-\Delta_{n_{0}}\right)\left|n_{0}\right\rangle \tag{6.66a}
\end{equation*}
$$

$$
\begin{equation*}
\Delta_{n_{0}}\left\langle n^{(0)} \mid n_{0}\right\rangle=0, \tag{6.66b}
\end{equation*}
$$

SO

$$
\begin{align*}
\left|n_{0}\right\rangle & =\left|n^{(0)}\right\rangle  \tag{6.67}\\
\Delta_{n_{0}} & =0 .
\end{align*}
$$

to $O\left(\lambda^{1}\right) \quad$ Requiring identity for all $\lambda^{1}$ terms means

$$
\begin{equation*}
\left|n_{1}\right\rangle \lambda=\sum_{m \neq n} \frac{\left|m^{(0)}\right\rangle\left\langle m^{(0)}\right|}{E_{n}^{(0)}-E_{m}^{(0)}}\left(\lambda V-\Delta_{n_{1}} \lambda\right)\left|n_{0}\right\rangle, \tag{6.68}
\end{equation*}
$$

so

$$
\begin{equation*}
\left|n_{1}\right\rangle=\sum_{m \neq n} \frac{\left|m^{(0)}\right\rangle\left\langle m^{(0)}\right|}{E_{n}^{(0)}-E_{m}^{(0)}}\left(V-\Delta_{n_{1}}\right)\left|n_{0}\right\rangle . \tag{6.69}
\end{equation*}
$$

With the assumption that $\left|n^{(0)}\right\rangle$ is normalized, and with the shorthand

$$
\begin{equation*}
V_{m n}=\left\langle m^{(0)}\right| V\left|n^{(0)}\right\rangle, \tag{6.70}
\end{equation*}
$$

that is

$$
\begin{align*}
& \left|n_{1}\right\rangle=\sum_{m \neq n} \frac{\left|m^{(0)}\right\rangle}{E_{n}^{(0)}-E_{m}^{(0)}} V_{m n},  \tag{6.71}\\
& \Delta_{n_{1}}=\left\langle n^{(0)}\right| V\left|n^{(0)}\right\rangle=V_{n n} .
\end{align*}
$$

to $O\left(\lambda^{2}\right)$ The second order perturbation states are found by selecting only the $\lambda^{2}$ contributions to

$$
\begin{equation*}
\lambda^{2}\left|n_{2}\right\rangle=\sum_{m \neq n} \frac{\left|m^{(0)}\right\rangle\left\langle m^{(0)}\right|}{E_{n}^{(0)}-E_{m}^{(0)}}\left(\lambda V-\left(\lambda \Delta_{n_{1}}+\lambda^{2} \Delta_{n_{2}}\right)\right)\left(\left|n_{0}\right\rangle+\lambda\left|n_{1}\right\rangle\right) . \tag{6.72}
\end{equation*}
$$

Because $\left|n_{0}\right\rangle=\left|n^{(0)}\right\rangle$, the $\lambda^{2} \Delta_{n_{2}}$ is killed, leaving

$$
\begin{align*}
\left|n_{2}\right\rangle & =\sum_{m \neq n} \frac{\left|m^{(0)}\right\rangle\left\langle m^{(0)}\right|}{E_{n}^{(0)}-E_{m}^{(0)}}\left(V-\Delta_{n_{1}}\right)\left|n_{1}\right\rangle \\
& =\sum_{m \neq n} \frac{\left|m^{(0)}\right\rangle\left\langle m^{(0)}\right|}{E_{n}^{(0)}-E_{m}^{(0)}}\left(V-\Delta_{n_{1}}\right) \sum_{l \neq n} \frac{\left|l^{(0)}\right\rangle}{E_{n}^{(0)}-E_{l}^{(0)}} V_{l n}, \tag{6.73}
\end{align*}
$$

which can be written as

$$
\begin{equation*}
\left|n_{2}\right\rangle=\sum_{l, m \neq n}\left|m^{(0)}\right\rangle \frac{V_{m l} V_{l n}}{\left(E_{n}^{(0)}-E_{m}^{(0)}\right)\left(E_{n}^{(0)}-E_{l}^{(0)}\right)}-\sum_{m \neq n}\left|m^{(0)}\right\rangle \frac{V_{n n} V_{m n}}{\left(E_{n}^{(0)}-E_{m}^{(0)}\right)^{2}} . \tag{6.74}
\end{equation*}
$$

For the second energy perturbation we have

$$
\begin{equation*}
\lambda^{2} \Delta_{n_{2}}=\left\langle n^{(0)}\right| \lambda V\left(\lambda\left|n_{1}\right\rangle\right), \tag{6.75}
\end{equation*}
$$

or

$$
\begin{align*}
\Delta_{n_{2}} & =\left\langle n^{(0)}\right| V\left|n_{1}\right\rangle \\
& =\left\langle n^{(0)}\right| V \sum_{m \neq n} \frac{\left|m^{(0)}\right\rangle}{E_{n}^{(0)}-E_{m}^{(0)}} V_{m n} \tag{6.76}
\end{align*}
$$

That is

$$
\begin{equation*}
\Delta_{n_{2}}=\sum_{m \neq n} \frac{V_{n m} V_{m n}}{E_{n}^{(0)}-E_{m}^{(0)}} . \tag{6.77}
\end{equation*}
$$

to $O\left(\lambda^{3}\right)$ Similarly, it can be shown that

$$
\begin{equation*}
\Delta_{n_{3}}=\sum_{l, m \neq n} \frac{V_{n m} V_{m l} V_{l n}}{\left(E_{n}^{(0)}-E_{m}^{(0)}\right)\left(E_{n}^{(0)}-E_{l}^{(0)}\right)}-\sum_{m \neq n} \frac{V_{n m} V_{n n} V_{m n}}{\left(E_{n}^{(0)}-E_{m}^{(0)}\right)^{2}} . \tag{6.78}
\end{equation*}
$$

In general, the energy perturbation is given by

$$
\begin{equation*}
\Delta_{n}^{(l)}=\left\langle n^{(0)}\right| V\left|n_{l-1}\right\rangle . \tag{6.79}
\end{equation*}
$$

## 6.7 stark effect.

$$
\begin{equation*}
H=H_{\text {atom }}+e \varepsilon \varepsilon z, \tag{6.80}
\end{equation*}
$$

where $H_{\text {atom }}$ is assumed to be Hydrogen-like with Hamiltonian

$$
\begin{equation*}
H_{\text {atom }}=\frac{\mathbf{p}^{2}}{2 m}-\frac{e^{2}}{4 \pi \epsilon_{0} r}, \tag{6.81}
\end{equation*}
$$

and wave functions

$$
\begin{equation*}
\left\langle\mathbf{r} \mid \psi_{n l m}\right\rangle=R_{n l}(r) Y_{l m}(\theta, \phi) \tag{6.82}
\end{equation*}
$$

Referring to eq. (A.117), the first level correction to the energy

$$
\begin{align*}
\Delta_{1} & =\left\langle\psi_{100}\right| e \varepsilon z\left|\psi_{100}\right\rangle \\
& =e \& \int \frac{d \Omega}{4 \pi} \cos \theta \int d r r^{2} R_{100}^{2}(r) . \tag{6.83}
\end{align*}
$$

The cosine integral is obliterated, so we have $\Delta_{1}=0$. How about the second order energy correction? That is

$$
\begin{equation*}
\Delta_{2}=\sum_{n l m \neq 100} \frac{\left.\left|\left\langle\psi_{100}\right| e \varepsilon z\right| n l m\right\rangle\left.\right|^{2}}{E_{100}^{(0)}-E_{n l m}} . \tag{6.84}
\end{equation*}
$$

The matrix element in the numerator is the absolute square of

$$
\begin{equation*}
V_{100, n l m}=e \S \int d \Omega \frac{1}{\sqrt{4 \pi}} \cos \theta Y_{l m}(\theta, \phi) \int d r r^{3} R_{100}(r) R_{n l}(r) . \tag{6.85}
\end{equation*}
$$

For all $m \neq 0, Y_{l m}$ includes a $e^{i m \phi}$ factor, so this cosine integral is zero. For $m=0$, each of the $Y_{l m}$ functions appears to contain either even or odd powers of cosines (see: eq. (A.115)). This shows that for even $2 k=l$, the cosine integral is zero

$$
\begin{equation*}
\int_{0}^{\pi} \sin \theta \cos \theta \sum_{k} a_{k} \cos ^{2 k} \theta d \theta=0 \tag{6.86}
\end{equation*}
$$

since $\cos ^{2 k}(\theta)$ is even and $\sin \theta \cos \theta$ is odd over the same interval. We find zero for $\int_{0}^{\pi} \sin \theta \cos \theta Y_{30}(\theta, \phi) d \theta$, and Mathematica appears to show that the rest of these integrals for $l>1$ are also zero.

FIXME: find the property of the spherical harmonics that can be used to prove that this is true in general for $l>1$.

This leaves

$$
\begin{align*}
\Delta_{2} & =\sum_{n \neq 1} \frac{\left.\left|\left\langle\psi_{100}\right| e \delta z\right| n 10\right\rangle\left.\right|^{2}}{E_{100}^{(0)}-E_{n 10}} \\
& =-e^{2} \mathcal{E}^{2} \sum_{n \neq 1} \frac{\left.\left|\left\langle\psi_{100}\right| z\right| n 10\right\rangle\left.\right|^{2}}{E_{n 10}-E_{100}^{(0)}} . \tag{6.87}
\end{align*}
$$

This is sometimes written in terms of a polarizability $\alpha$

$$
\begin{equation*}
\Delta_{2}=-\frac{\mathcal{E}^{2}}{2} \alpha \tag{6.88}
\end{equation*}
$$

where

$$
\begin{equation*}
\alpha=2 e^{2} \sum_{n \neq 1} \frac{\left.\left|\left\langle\psi_{100}\right| z\right| n 10\right\rangle\left.\right|^{2}}{E_{n 10}-E_{100}^{(0)}} \tag{6.89}
\end{equation*}
$$

With

$$
\begin{equation*}
\mathbf{P}=\alpha \mathcal{E} \tag{6.90}
\end{equation*}
$$

the energy change upon turning on the electric field from $0 \rightarrow \mathcal{E}$ is simply $-\mathbf{P} \cdot d \mathscr{E}$ integrated from $0 \rightarrow \mathcal{E}$. Putting $\mathbf{P}=\alpha \mathcal{E} \hat{\mathbf{z}}$, we have

$$
\begin{align*}
-\int_{0}^{\delta} P_{z} d \S & =-\int_{0}^{\mathscr{E}} \alpha \S d \S  \tag{6.91}\\
& =-\frac{1}{2} \alpha \mathcal{E}^{2}
\end{align*}
$$

leading to an energy change $-\alpha \mathcal{E}^{2} / 2$, so we can directly compute $\langle\mathbf{P}\rangle$ or we can compute change in energy, and both contain information about the polarization factor $\alpha$. There is an exact answer to the sum eq. (6.89), but we aren't going to try to get it here. Instead let's look for bounds

$$
\begin{align*}
& \Delta_{2}^{\min }<\Delta_{2}<\Delta_{2}^{\max }  \tag{6.92}\\
& \alpha^{\min }=2 e^{2} \frac{\left.\left|\left\langle\psi_{100}\right| z\right| \psi_{210}\right\rangle\left.\right|^{2}}{E_{210}^{(0)}-E_{100}^{(0)}} \tag{6.93}
\end{align*}
$$

For the hydrogen atom we have

$$
\begin{equation*}
E_{n}=-\frac{e^{2}}{2 n^{2} a_{0}}, \tag{6.94}
\end{equation*}
$$

allowing any difference of energy levels to be expressed as a fraction of the ground state energy, such as

$$
\begin{equation*}
E_{210}^{(0)}=\frac{1}{4} E_{100}^{(0)}=\frac{1}{4} \frac{-\hbar^{2}}{2 m a_{0}^{2}} \tag{6.95}
\end{equation*}
$$

So

$$
\begin{equation*}
E_{210}^{(0)}-E_{100}^{(0)}=\frac{3}{4} \frac{\hbar^{2}}{2 m a_{0}^{2}} \tag{6.96}
\end{equation*}
$$

In the numerator we have

$$
\begin{align*}
\left\langle\psi_{100}\right| z\left|\psi_{210}\right\rangle & =\int r^{2} d \Omega\left(\frac{1}{\sqrt{\pi} a_{0}^{3 / 2}} e^{-r / a_{0}}\right) r \cos \theta\left(\frac{1}{4 \sqrt{2 \pi} a_{0}^{3 / 2}} \frac{r}{a_{0}} e^{-r / 2 a_{0}} \cos \theta\right) \\
& =(2 \pi) \frac{1}{\sqrt{\pi}} \frac{1}{4 \sqrt{2 \pi}} a_{0} \int_{0}^{\pi} d \theta \sin \theta \cos ^{2} \theta \int_{0}^{\infty} \frac{d r}{a_{0}} \frac{r^{4}}{a_{0}^{4}} e^{-r / a_{0}-r / 2 a_{0}} \\
& =(2 \not \pi) \frac{1 /}{\sqrt{\pi}} \frac{1}{4 \sqrt{2 \pi}} a_{0}\left(-\left.\frac{u^{3}}{3}\right|_{1} ^{-1}\right) \int_{0}^{\infty} s^{4} d s e^{-3 s / 2} \\
& =\frac{1}{2 \sqrt{2}} \frac{2}{3} a_{0} \frac{256}{81} \\
& =\frac{1}{3 \sqrt{2}} \frac{256}{81} a_{0} \approx 0.75 a_{0} . \tag{6.97}
\end{align*}
$$

This gives

$$
\begin{align*}
\alpha^{\min } & =\frac{2 e^{2}(0.75)^{2} a_{0}^{2}}{\frac{3}{4} \frac{\hbar^{2}}{2 m a_{0}^{2}}} \\
& =\frac{6}{4} \frac{2 m e^{2} a_{0}^{4}}{\hbar^{2}}  \tag{6.98}\\
& =3 \frac{m e^{2} a_{0}^{4}}{\hbar^{2}} \\
& =3 \frac{4 \pi \epsilon_{0}}{a_{0}} a_{0}^{4} \approx 4 \pi \epsilon_{0} a_{0}^{3} \times 3
\end{align*}
$$

The factor $4 \pi \epsilon_{0} a_{0}^{3}$ are the natural units for the polarizability. There is a neat trick that generalizes to many problems to find the upper bound. Recall that the general polarizability was

$$
\begin{equation*}
\alpha=2 e^{2} \sum_{n l m \neq 100} \frac{|\langle 100| z| n l m\rangle\left.\right|^{2}}{E_{n l m}-E_{100}^{(0)}} \tag{6.99}
\end{equation*}
$$

If we are looking for the upper bound, and replace the denominator by the smallest energy difference that will be encountered, it can be brought out of the sum, for

$$
\begin{equation*}
\alpha^{\max }=2 e^{2} \frac{1}{E_{210}-E_{100}^{(0)}} \sum_{n l m \neq 100}\langle 100| z|n l m\rangle\langle n l m| z|100\rangle . \tag{6.100}
\end{equation*}
$$

Because $\langle n l m| z|100\rangle=0$, the constraint in the sum can be removed, and the identity summation evaluated

$$
\begin{align*}
\alpha^{\max } & =2 e^{2} \frac{1}{E_{210}-E_{100}^{(0)}} \sum_{n l m}\langle 100| z|n l m\rangle\langle n l m| z|100\rangle \\
& =\frac{2 e^{2}}{\frac{3}{4} \frac{\hbar^{2}}{2 m a_{0}^{2}}}\langle 100| z^{2}|100\rangle  \tag{6.101}\\
& =\frac{16 e^{2} m a_{0}^{2}}{3 \hbar^{2}} \times a_{0}^{2} \\
& =4 \pi \epsilon_{0} a_{0}^{3} \times \frac{16}{3}
\end{align*}
$$

The bounds are

$$
\begin{equation*}
3 \geq \frac{\alpha}{\alpha^{\mathrm{at}}}<\frac{16}{3} \tag{6.102}
\end{equation*}
$$

where

$$
\begin{equation*}
\alpha^{\text {at }}=4 \pi \epsilon_{0} a_{0}^{3} \tag{6.103}
\end{equation*}
$$

The actual value is

$$
\begin{equation*}
\frac{\alpha}{\alpha^{\mathrm{at}}}=\frac{9}{2} . \tag{6.104}
\end{equation*}
$$

See lecture21someSphericalHarmonicsAndTheirIntegrals.nb, for some of the integrals above, and for spherical harmonic tables.

## Example: Computing the dipole moment

$$
\begin{equation*}
\left\langle P_{z}\right\rangle=\alpha \mathscr{E}=\left\langle\psi_{100}\right| e z\left|\psi_{100}\right\rangle . \tag{6.105}
\end{equation*}
$$

Without any perturbation this is zero. After perturbation, retaining only the terms that are first order in $\delta \psi_{100}$ we have

$$
\begin{equation*}
\left\langle\psi_{100}+\delta \psi_{100}\right| e z\left|\psi_{100}+\delta \psi_{100}\right\rangle \approx\left\langle\psi_{100}\right| e z\left|\delta \psi_{100}\right\rangle+\left\langle\delta \psi_{100}\right| e z\left|\psi_{100}\right\rangle . \tag{6.106}
\end{equation*}
$$

Next time: van der Walls We will look at two hyrdogenic atomic systems interacting where the pair of nuclei are supposed to be infinitely heavy and stationary. The wave functions each set of atoms are individually known, but we can consider the problem of the interactions of atom 1 's electrons with atom 2's nucleus and atom 2's electrons, and also the opposite interactions of atom 2's electrons with atom 1's nucleus and its electrons. This leads to a result that is linear in the electric field (unlike the above result, which is called the quadratic Stark effect).

Another approach (for last time?) Imagine we perturb a potential, say a harmonic oscillator with an electric field

$$
\begin{align*}
& V_{0}(x)=\frac{1}{2} k x^{2},  \tag{6.107}\\
& V(x)=\delta e x . \tag{6.108}
\end{align*}
$$

After minimizing the energy, using $\partial V / \partial x=0$, we get

$$
\begin{align*}
& \frac{1}{2} k x^{2}+\varepsilon e x \rightarrow k x^{*}=-e \varepsilon,  \tag{6.109}\\
& p^{*}=-e x^{*}=-\frac{e^{2} \varepsilon}{k} . \tag{6.110}
\end{align*}
$$

For such a system the polarizability is

$$
\begin{align*}
& \alpha=\frac{e^{2}}{k},  \tag{6.111}\\
& \begin{aligned}
\frac{1}{2} k\left(-\frac{e \delta}{k}\right)^{2}+\mathcal{E} e\left(-\frac{e \mathcal{E}}{k}\right) & =-\frac{1}{2}\left(\frac{e^{2}}{k}\right) \mathcal{E}^{2} \\
& =-\frac{1}{2} \alpha \delta^{2}
\end{aligned} \tag{6.112}
\end{align*}
$$

### 6.8 VAN DER WALLS POTENTIAL.

$$
\begin{equation*}
H_{0}=H_{01}+H_{02} \tag{6.113}
\end{equation*}
$$

where

$$
\begin{equation*}
H_{0 \alpha}=\frac{p_{\alpha}^{2}}{2 m}-\frac{e^{2}}{4 \pi \epsilon_{0}\left|\mathbf{r}_{\alpha}-\mathbf{R}_{\alpha}\right|}, \quad \alpha=1,2 \tag{6.114}
\end{equation*}
$$

The full interaction potential is

$$
\begin{equation*}
V=\frac{e^{2}}{4 \pi \epsilon_{0}}\left(\frac{1}{\left|\mathbf{R}_{1}-\mathbf{R}_{2}\right|}+\frac{1}{\left|\mathbf{r}_{1}-\mathbf{r}_{2}\right|}-\frac{1}{\left|\mathbf{r}_{1}-\mathbf{R}_{2}\right|}-\frac{1}{\left|\mathbf{r}_{2}-\mathbf{R}_{1}\right|}\right) \tag{6.115}
\end{equation*}
$$

Let

$$
\begin{equation*}
\mathbf{x}_{\alpha}=\mathbf{r}_{\alpha}-\mathbf{R}_{\alpha} \tag{6.116}
\end{equation*}
$$

$$
\begin{equation*}
\mathbf{R}=\mathbf{R}_{1}-\mathbf{R}_{2} \tag{6.117}
\end{equation*}
$$

as sketched in fig. 6.12.


Figure 6.12: Two atom interaction.

$$
\begin{equation*}
H_{0 \alpha}=\frac{\mathbf{p}^{2}}{2 m}-\frac{e^{2}}{4 \pi \epsilon_{0}\left|\mathbf{x}_{\alpha}\right|} \tag{6.118}
\end{equation*}
$$

which allows the total interaction potential to be written

$$
\begin{equation*}
V=\frac{e^{2}}{4 \pi \epsilon_{0} R}\left(1+\frac{R}{\left|\mathbf{x}_{1}-\mathbf{x}_{2}+\mathbf{R}\right|}-\frac{R}{\left|\mathbf{x}_{1}+\mathbf{R}\right|}-\frac{R}{\left|-\mathbf{x}_{2}+\mathbf{R}\right|}\right) . \tag{6.119}
\end{equation*}
$$

For $R \gg x_{1}, x_{2}$, this interaction potential, after a multipole expansion, is approximately

$$
\begin{equation*}
V=\frac{e^{2}}{4 \pi \epsilon_{0}}\left(\frac{\mathbf{x}_{1} \cdot \mathbf{x}_{2}}{|\mathbf{R}|^{3}}-3 \frac{\left(\mathbf{x}_{1} \cdot \mathbf{R}\right)\left(\mathbf{x}_{2} \cdot \mathbf{R}\right)}{|\mathbf{R}|^{5}}\right) \tag{6.120}
\end{equation*}
$$

Showing this is left as a exercise.

$$
\begin{align*}
1 . O(\lambda) \quad \text {. With } \\
\psi_{0}=|1 s, 1 s\rangle \tag{6.121}
\end{align*}
$$

$$
\begin{equation*}
\Delta E^{(1)}=\left\langle\psi_{0}\right| V\left|\psi_{0}\right\rangle \tag{6.122}
\end{equation*}
$$

the two particle wave functions are of the form

$$
\begin{equation*}
\left\langle\mathbf{x}_{1}, \mathbf{x}_{2} \mid \psi_{0}\right\rangle=\psi_{1 s}\left(\mathbf{x}_{1}\right) \psi_{1 s}\left(\mathbf{x}_{2}\right) \tag{6.123}
\end{equation*}
$$

so braket integrals must be evaluated over a six-fold space. Recall that

$$
\begin{equation*}
\psi_{1 s}=\frac{1}{\sqrt{\pi} a_{0}^{3 / 2}} e^{-r / a_{0}} \tag{6.124}
\end{equation*}
$$

so

$$
\begin{equation*}
\left\langle\psi_{1 s}\right| x_{i}\left|\psi_{1 s}\right\rangle \propto \int_{0}^{\pi} \sin \theta d \theta \int_{0}^{2 \pi} d \phi x_{i} \tag{6.125}
\end{equation*}
$$

where

$$
\begin{equation*}
x_{i} \in\{r \sin \theta \cos \phi, r \sin \theta \sin \phi, r \cos \theta\} . \tag{6.126}
\end{equation*}
$$

The $x, y$ integrals are zero because of the $\phi$ integral, and the $z$ integral is proportional to $\int_{0}^{\pi} \sin (2 \theta) d \theta$, which is also zero. This leads to zero averages

$$
\begin{equation*}
\left\langle\mathbf{x}_{1}\right\rangle=0=\left\langle\mathbf{x}_{2}\right\rangle, \tag{6.127}
\end{equation*}
$$

so

$$
\begin{equation*}
\Delta E^{(1)}=0 \tag{6.128}
\end{equation*}
$$

2. $O\left(\lambda^{2}\right)$.

$$
\begin{align*}
\Delta E^{(2)} & =\sum_{n \neq 0} \frac{\left.\left|\left\langle\psi_{n}\right| V\right| \psi_{0}\right\rangle\left.\right|^{2}}{E_{0}-E_{n}} \\
& =\sum_{n \neq 0} \frac{\left\langle\psi_{0}\right| V\left|\psi_{n}\right\rangle\left\langle\psi_{n}\right| V\left|\psi_{0}\right\rangle}{E_{0}-E_{n}} \tag{6.129}
\end{align*}
$$

This is a sum over all excited states. We expect that this will be of the form

$$
\begin{equation*}
\Delta E^{(2)}=-\left(\frac{e^{2}}{4 \pi \epsilon_{0}}\right)^{2} \frac{C_{6}}{R^{6}}, \tag{6.130}
\end{equation*}
$$

where $\mathbf{x}_{1}$ and $\mathbf{x}_{2}$ are dipole operators. The first time this has a non-zero expectation is when we go from the 1 s to the 2 p states (both 1 s and 2 s states are spherically symmetric). Noting that $E_{n}=-e^{2} / 2 n^{2} a_{0}$, we can compute a minimum bound for the energy denominator

$$
\begin{align*}
\left(E_{n}-E_{0}\right)^{\min } & =2\left(E_{2 p}-E_{1 s}\right) \\
& =2 E_{1 s}\left(\frac{1}{4}-1\right) \\
& =2 \frac{3}{4}\left|E_{1 s}\right|  \tag{6.131}\\
& =\frac{3}{2}\left|E_{1 s}\right| .
\end{align*}
$$

Note that the factor of two above comes from summing over the energies for both electrons. This gives us

$$
\begin{equation*}
C_{6}=\frac{3}{2}\left|E_{1 s}\right|\left\langle\psi_{0}\right| \tilde{V}\left|\psi_{0}\right\rangle, \tag{6.132}
\end{equation*}
$$

where

$$
\begin{equation*}
\tilde{V}=\left(\mathbf{x}_{1} \cdot \mathbf{x}_{2}-3\left(\mathbf{x}_{1} \cdot \hat{\mathbf{R}}\right)\left(\mathbf{x}_{2} \cdot \hat{\mathbf{R}}\right)\right) . \tag{6.133}
\end{equation*}
$$

What about degeneracy?

$$
\begin{equation*}
\Delta E_{n}^{(2)}=\sum_{m \neq n} \frac{\left.\left|\left\langle\psi_{n}\right| V\right| \psi_{0}\right\rangle\left.\right|^{2}}{E_{0}-E_{n}} . \tag{6.134}
\end{equation*}
$$

If $\left\langle\psi_{n}\right| V\left|\psi_{m}\right\rangle \propto \delta_{n m}$ then it's okay. In general the we can't expect the matrix element will be anything but fully populated, say

$$
V=\left[\begin{array}{llll}
V_{11} & V_{12} & V_{13} & V_{14}  \tag{6.135}\\
V_{21} & V_{22} & V_{23} & V_{24} \\
V_{31} & V_{32} & V_{33} & V_{34} \\
V_{41} & V_{42} & V_{43} & V_{44}
\end{array}\right] .
$$

If we choose a basis so that

$$
V=\left[\begin{array}{llll}
V_{11} & & &  \tag{6.136}\\
& V_{22} & & \\
& & V_{33} & \\
& & & V_{44}
\end{array}\right]
$$

When this is the case, we have no mixing of elements in the sum of eq. (6.134)

Degeneracy in the Stark effect

$$
\begin{equation*}
H=H_{0}+e \varepsilon \varepsilon z \tag{6.137}
\end{equation*}
$$

where

$$
\begin{equation*}
H_{0}=\frac{\mathbf{p}^{2}}{2 m}-\frac{e}{4 \pi \epsilon_{0}} \frac{1}{|\mathbf{x}|} \tag{6.138}
\end{equation*}
$$

Consider the states $2 s, 2 p_{x}, 2 p_{y}, 2 p_{z}$, for which $E_{n}^{(0)} \equiv E_{2 s}$, as sketched in fig. 6.13. Because of spherical symmetry


Figure 6.13: 2 s 2 p degeneracy.

$$
\begin{align*}
\langle 2 s| e \mathcal{E} z|2 s\rangle & =0 \\
\left\langle 2 p_{x}\right| e \mathcal{E} z\left|2 p_{x}\right\rangle & =0 \\
\left\langle 2 p_{y}\right| e \mathcal{E} z\left|2 p_{y}\right\rangle & =0  \tag{6.139}\\
\left\langle 2 p_{z}\right| e \mathcal{E} z\left|2 p_{z}\right\rangle & =0
\end{align*}
$$

Looking at odd and even properties, it turns out that the only off-diagonal matrix element is

$$
\begin{equation*}
\langle 2 s| e \mathcal{E} z\left|2 p_{z}\right\rangle=V_{1}=-3 e \mathscr{E} a_{0} \tag{6.140}
\end{equation*}
$$

With a $\left\{2 s, 2 p_{x}, 2 p_{y}, 2 p_{z}\right\}$ basis the potential matrix is

$$
\left[\begin{array}{cccc}
0 & 0 & 0 & V_{1}  \tag{6.141}\\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
V_{1}^{*} & 0 & 0 & 0
\end{array}\right]
$$

which has the block structure

$$
\left[\begin{array}{cc}
0 & -\left|V_{1}\right|  \tag{6.142}\\
-\left|V_{1}\right| & 0
\end{array}\right]
$$

This implies that the energy splitting goes as

$$
\begin{equation*}
E_{2 s} \rightarrow E_{2 s} \pm\left|V_{1}\right|, \tag{6.1.13}
\end{equation*}
$$

as sketched in fig. 6.14. The diagonalizing states corresponding to eigen-


Figure 6.14: Stark effect energy level splitting.
values $\pm 3 a_{0} \mathcal{E}$, are $\left(|2 s\rangle \mp\left|2 p_{z}\right\rangle\right) / \sqrt{2}$. The matrix element above is calculated explicitly in lecture22Integrals.nb. The degeneracy that is left unsplit here, and has to be accounted for should we attempt higher order perturbation calculations.
6.9 Problems.

Exercise $6.1 \quad$ van der Walls multipole expansion.
Prove eq. (6.120).

## Answer for Exercise 6.1

Noting that

$$
\begin{align*}
(1+\epsilon)^{-1 / 2} & =1-\frac{1}{2} \epsilon-\frac{1}{2}\left(\frac{-3}{2}\right) \frac{1}{2!} \epsilon^{2}  \tag{6.144}\\
& =1-\frac{1}{2} \epsilon+\frac{3}{8} \epsilon^{2},
\end{align*}
$$

we have

$$
\begin{align*}
\frac{R}{|\boldsymbol{\epsilon}+\mathbf{R}|} & =\frac{1}{\left|\frac{\boldsymbol{\epsilon}}{R}+\hat{\mathbf{R}}\right|} \\
& =\left(1+2 \frac{\boldsymbol{\epsilon}}{R} \cdot \hat{\mathbf{R}}+\left(\frac{\boldsymbol{\epsilon}}{R}\right)^{2}\right)^{-1 / 2} \\
& =1-\frac{\boldsymbol{\epsilon}}{R} \cdot \hat{\mathbf{R}}-\frac{1}{2}\left(\frac{\boldsymbol{\epsilon}}{R}\right)^{2}+\frac{3}{8}\left(2 \frac{\boldsymbol{\epsilon}}{R} \cdot \hat{\mathbf{R}}+\left(\frac{\boldsymbol{\epsilon}}{R}\right)^{2}\right)^{2} \\
& =1-\frac{\boldsymbol{\epsilon}}{R} \cdot \hat{\mathbf{R}}-\frac{1}{2}\left(\frac{\boldsymbol{\epsilon}}{R}\right)^{2}+\frac{3}{8}\left(4\left(\frac{\boldsymbol{\epsilon}}{R} \cdot \hat{\mathbf{R}}\right)^{2}+\left(\frac{\boldsymbol{\epsilon}}{R}\right)^{4}+4 \frac{\boldsymbol{\epsilon}}{R} \cdot \hat{\mathbf{R}}\left(\frac{\boldsymbol{\epsilon}}{R}\right)^{2}\right) \\
& \approx 1-\frac{\boldsymbol{\epsilon}}{R} \cdot \hat{\mathbf{R}}-\frac{1}{2}\left(\frac{\boldsymbol{\epsilon}}{R}\right)^{2}+\frac{3}{2}\left(\frac{\boldsymbol{\epsilon}}{R} \cdot \hat{\mathbf{R}}\right)^{2} . \tag{6.145}
\end{align*}
$$

Inserting the values from the brackets of eq. (6.119) we have

$$
\begin{aligned}
1+\frac{R}{\left|\mathbf{x}_{1}-\mathbf{x}_{2}+\mathbf{R}\right|}- & \frac{R}{\left|\mathbf{x}_{1}+\mathbf{R}\right|}-\frac{R}{\left|-\mathbf{x}_{2}+\mathbf{R}\right|} \\
= & -\frac{\left(\mathbf{x}_{1}-\mathbf{x}_{2}\right)}{R} \cdot \hat{\mathbf{R}}-\frac{1}{2}\left(\frac{\left(\mathbf{x}_{1}-\mathbf{x}_{2}\right)}{R}\right)^{2}+\frac{3}{2}\left(\frac{\left(\mathbf{x}_{1}-\mathbf{x}_{2}\right)}{R} \cdot \hat{\mathbf{R}}\right)^{2} \\
& +\frac{\mathbf{x}_{1}}{R} \cdot \hat{\mathbf{R}}+\frac{1}{2}\left(\frac{\mathbf{x}_{1}}{R}\right)^{2}-\frac{3}{2}\left(\frac{\mathbf{x}_{1}}{R} \cdot \hat{\mathbf{R}}\right)^{2} \\
& -\frac{\mathbf{x}_{2}}{R} \cdot \hat{\mathbf{R}}+\frac{1}{2}\left(\frac{\mathbf{x}_{2}}{R}\right)^{2}-\frac{3}{2}\left(\frac{\mathbf{x}_{2}}{R} \cdot \hat{\mathbf{R}}\right)^{2} \\
= & \frac{\mathbf{x}_{1}}{R} \cdot \frac{\mathbf{x}_{2}}{R}+\frac{3}{2}\left(\frac{\left(\mathbf{x}_{1}-\mathbf{x}_{2}\right)}{R} \cdot \hat{\mathbf{R}}\right)^{2} \\
& -\frac{3}{2}\left(\frac{\mathbf{x}_{1}}{R} \cdot \hat{\mathbf{R}}\right)^{2} \\
& -\frac{3}{2}\left(\frac{\mathbf{x}_{2}}{R} \cdot \hat{\mathbf{R}}\right)^{2} \\
= & \frac{\mathbf{x}_{1}}{R} \cdot \frac{\mathbf{x}_{2}}{R}-3 \frac{\mathbf{x}_{1}}{R} \cdot \hat{\mathbf{R}} \frac{\mathbf{x}_{2}}{R} \cdot \hat{\mathbf{R}} .
\end{aligned}
$$

This proves eq. (6.120).
Exercise 6.2 Harmonic oscillator with energy shift. ([15] pr. 5.1)
Given a perturbed 1D SHO Hamiltonian

$$
\begin{equation*}
H=\frac{1}{2 m} p^{2}+\frac{1}{2} m \omega^{2} x^{2}+\lambda b x \tag{6.147}
\end{equation*}
$$

calculate the first non-zero perturbation to the ground state energy. Then solve for that energy directly and compare.
Answer for Exercise 6.2
The first order energy shift is seen to be zero

$$
\begin{align*}
\Delta_{0}^{(0)} & =V_{00} \\
& =\langle 0| b x|0\rangle \\
& =\frac{x_{0}}{\sqrt{2}}\langle 0| a+a^{\dagger}|0\rangle  \tag{6.148}\\
& =\frac{x_{0}}{\sqrt{2}}\langle 0 \mid 1\rangle \\
& =0
\end{align*}
$$

The first order perturbation to the ground state is

$$
\begin{align*}
\left|0^{(1)}\right\rangle & =\sum_{m \neq 0} \frac{|m\rangle\langle m| b x|0\rangle}{\hbar \omega / 2-\hbar \omega(m-1 / 2)} \\
& =-b \frac{x_{0}}{\sqrt{2} \hbar \omega} \sum_{m \neq 0} \frac{|m\rangle\langle m \mid 1\rangle}{m}  \tag{6.149}\\
& =-b \frac{x_{0}}{\sqrt{2} \hbar \omega}|1\rangle
\end{align*}
$$

The second order ground state energy perturbation is

$$
\begin{align*}
\Delta_{0}^{(2)} & =\langle 0| b x\left|0^{(1)}\right\rangle \\
& =\frac{b x_{0}}{\sqrt{2}}\langle 0| a+a^{\dagger}\left(-b \frac{x_{0}}{\sqrt{2} \hbar \omega}|1\rangle\right) \\
& =\frac{b x_{0}}{\sqrt{2}}\left(-b \frac{x_{0}}{\sqrt{2} \hbar \omega}\right) \\
& =-\frac{b^{2} x_{0}^{2}}{2 \hbar \omega}  \tag{6.150}\\
& =-\frac{b^{2}}{2 \hbar \omega} \frac{\hbar}{m \omega} \\
& =-\frac{b^{2}}{2 m \omega^{2}}
\end{align*}
$$

so the total energy perturbation up to second order is

$$
\begin{equation*}
\Delta_{0}=-\lambda^{2} \frac{b^{2}}{2 m \omega^{2}} \tag{6.151}
\end{equation*}
$$

To compare to the exact result, rewrite the Hamiltonian as

$$
\begin{align*}
H & =\frac{1}{2 m} p^{2}+\frac{1}{2} m \omega^{2}\left(x^{2}+\frac{2 \lambda b x}{m \omega^{2}}\right) \\
& =\frac{1}{2 m} p^{2}+\frac{1}{2} m \omega^{2}\left(x+\frac{\lambda b}{m \omega^{2}}\right)^{2}-\frac{1}{2} m \omega^{2}\left(\frac{\lambda b}{m \omega^{2}}\right)^{2} \tag{6.152}
\end{align*}
$$

The Hamiltonian is subject to a constant energy shift

$$
\begin{align*}
\Delta E & =-\frac{1}{2} m \omega^{2} \frac{\lambda^{2} b^{2}}{m^{2} \omega^{4}} \\
& =-\frac{\lambda^{2} b^{2}}{2 m \omega^{2}} \tag{6.153}
\end{align*}
$$

This matches the second order perturbation result eq. (6.151) exactly.

## Exercise $6.3 \quad$ Double well potential. (2015 ps 7 pl)

Consider a particle in the double well potential

$$
\begin{equation*}
V(x)=\frac{m \omega^{2}}{8 a^{2}}(x+a)^{2}(x-a)^{2} \tag{6.154}
\end{equation*}
$$

Expanding $V(x)$ around $x= \pm a$ leads to a harmonic potential with frequency $\omega$. Construct variational states with even/odd parity as $\psi_{ \pm}(x)=$
$g_{ \pm}(\phi(x-a) \pm \phi(x+a))$ where $\phi(x)$ is the normalized ground state of the usual harmonic oscillator with frequency $\omega$, i.e.,

$$
\begin{equation*}
\phi(x)=\left(\frac{1}{\pi a_{0}^{2}}\right)^{1 / 4} e^{-\frac{x^{2}}{2 a_{0}^{2}}} ; \quad a_{0}=\sqrt{\frac{\hbar}{m \omega}} \tag{6.155}
\end{equation*}
$$

a. Determine the normalization constants $g_{ \pm}$. Next using these wavefunctions, determine the variational energies of these two states. Hence determine the 'tunnel splitting' between the two states, induced by the tunnelling through the barrier region. In your calculations, you can assume $a \gg a_{0}$, so retain only the leading terms in any polynomials you might encounter when you do the integrals.
b. If we pay attention to these lowest two states (left well and right well) in the full Hilbert space, we can write a phenomenological $2 \times 2$ Hamiltonian

$$
H=\left[\begin{array}{cc}
\epsilon_{0} & -\gamma  \tag{6.156}\\
-\gamma & \epsilon_{0}
\end{array}\right]
$$

where $\epsilon_{0}$ is the energy on each side, and $\gamma$ leads to tunnelling, so if we start off in the left well, $t$ leads to a nonzero amplitude to find it in the right well at a later time. Find its eigenvalues and eigenvectors. Comparing with your variational result for the energy splitting, determine the 'tunnel coupling' $\gamma$.

Answer for Exercise 6.3

Part $a$. The integration grunt work for this problem can be found in ps7:doubleWellPotential.nb. This yields an energy difference of

$$
\begin{equation*}
\bar{E}_{+}-\bar{E}_{-}=\frac{\hbar \omega}{8}\left(\left(\frac{4 a^{2}}{a_{0}^{2}}+\left(\frac{a^{2}}{a_{0}^{2}}-5\right) \exp \left(\frac{a^{2}}{a_{0}^{2}}\right)-2\right)\left(\operatorname{coth}\left(\frac{a^{2}}{a_{0}^{2}}\right)-1\right)\right) \tag{6.157}
\end{equation*}
$$

With $u=a^{2} / a_{0}^{2}$, in the $a \gg a_{0}$ limit, the almost zero coth $u-1$ difference can be approximated as an exponential

$$
\begin{align*}
\operatorname{coth} u-1 & =\frac{e^{2 u}+1}{e^{2 u}-1}-1 \\
& =\frac{e^{2 u}+1-e^{2 u}+1}{e^{2 u}-1}  \tag{6.158}\\
& =\frac{2}{e^{2 u}-1} \\
& \approx 2 e^{-2 u},
\end{align*}
$$

so the energy difference is approximately

$$
\begin{equation*}
\bar{E}_{+}-\bar{E}_{-} \approx \frac{\hbar \omega}{4} \frac{a^{2}}{a_{0}^{2}} \exp \left(-\frac{a^{2}}{a_{0}^{2}}\right) . \tag{6.159}
\end{equation*}
$$

Part b. Now lets compare to the energy levels of the phenomenological Hamiltonian, which are given by

$$
\begin{equation*}
0=\left(\epsilon_{0}-E\right)^{2}-\gamma^{2}, \tag{6.160}
\end{equation*}
$$

with eigenvalues

$$
\begin{equation*}
E_{ \pm}=\epsilon_{0} \pm \gamma . \tag{6.161}
\end{equation*}
$$

If the eigenvectors are proportional to the column vector given by
then we must have

$$
\begin{align*}
0 & =\left(\epsilon_{0}-\left(\epsilon_{0} \pm \gamma\right)\right) a-\gamma b \\
& =\gamma(\mp a-b), \tag{6.163}
\end{align*}
$$

or

The energy level difference for this Hamiltonian is

$$
\begin{align*}
\Delta E & =E_{+}-E_{-} \\
& =\epsilon_{0}+\gamma-\left(\epsilon_{0}-\gamma\right)  \tag{6.165}\\
& =2 \gamma .
\end{align*}
$$

Equating this difference with eq. (6.159), we have

$$
\begin{equation*}
\gamma=\frac{\hbar \omega}{8} \frac{a^{2}}{a_{0}^{2}} \exp \left(-\frac{a^{2}}{a_{0}^{2}}\right) \tag{6.166}
\end{equation*}
$$

Exercise $6.4 \quad$ Helium-4 atom. (2015 ps7 p2)
Consider the Helium atom with atomic number $Z=2$, which leads to the nuclear charge $Z=2 e$, and two electrons with charge $-e$ each.
a. Show that ignoring electron-electron interactions leads to a ground state energy $E_{\mathrm{He}}=4 E_{\mathrm{H}}$ where $E_{\mathrm{H}}$ is the ground state energy of the hydrogen atom.
b. Consider the full problem which retains the Coulomb interaction between the electrons, i.e.

$$
\begin{equation*}
H=\frac{1}{2 m}\left(\mathbf{p}_{1}^{2}+\mathbf{p}_{2}^{2}\right)-2 e^{2} \frac{1}{4 \pi \epsilon_{0}}\left(\frac{1}{r_{1}}+\frac{1}{r_{2}}\right)+e^{2} \frac{1}{4 \pi \epsilon_{0}} \frac{1}{\left|\mathbf{r}_{1}-\mathbf{r}_{2}\right|} \tag{6.167}
\end{equation*}
$$

and consider the variational wavefunction

$$
\begin{equation*}
\psi\left(\mathbf{r}_{1}, \mathbf{r}_{2}\right)=N e^{-\frac{1}{a}\left(r_{1}+r_{2}\right)} \tag{6.168}
\end{equation*}
$$

where $N$ is the normalization constant, and $a$ is a variational parameter. Determine the variational ground state energy, and minimize with respect to a to find the best estimate for the ground state energy of Helium. Compare with numerical estimates of the energy.

## Answer for Exercise 6.4

Part a. Without the electron-electron interaction term, the Helium Hamiltonian is separable. Assuming a wave function of the form

$$
\begin{equation*}
\psi\left(r_{1}, r_{2}\right)=\psi_{1}\left(r_{1}\right) \psi_{2}\left(r_{2}\right) \tag{6.169}
\end{equation*}
$$

The Hamiltonian action on this wave function is

$$
\begin{aligned}
E \psi_{1}\left(r_{1}\right) \psi_{2}\left(r_{2}\right)= & \left(\frac{1}{2 m} \mathbf{p}_{1}^{2} \psi_{1}\left(r_{1}\right)\right) \psi_{2}\left(r_{2}\right)+\left(\frac{1}{2 m} \mathbf{p}_{2}^{2} \psi_{2}\left(r_{1}\right)\right) \psi_{1}\left(r_{2}\right) \\
& -2 e^{2} \frac{1}{4 \pi \epsilon_{0}} \frac{1}{r_{1}} \psi_{1}\left(r_{1}\right) \psi_{2}\left(r_{2}\right)-2 e^{2} \frac{1}{4 \pi \epsilon_{0}} \frac{1}{r_{1}} \psi_{1}\left(r_{1}\right) \psi_{2}\left(r_{2}\right)
\end{aligned}
$$

or

$$
\begin{align*}
E= & \left(\frac{1}{\psi_{1}(r)}\left(\frac{1}{2 m} \mathbf{p}_{1}^{2} \psi_{1}\left(r_{1}\right)\right)-2 e^{2} \frac{1}{4 \pi \epsilon_{0}} \frac{1}{r_{1}}\right)+\left(\frac{1}{\psi_{2}(r)}\left(\frac{1}{2 m} \mathbf{p}_{2}^{2} \psi_{2}\left(r_{2}\right)\right)\right.  \tag{6.171}\\
& \left.-2 e^{2} \frac{1}{4 \pi \epsilon_{0}} \frac{1}{r_{2}}\right) .
\end{align*}
$$

This can be written in separated form as

$$
\begin{align*}
E_{1} \psi_{1}\left(r_{1}\right) & =\frac{1}{2 m} \mathbf{p}_{1}^{2} \psi_{1}\left(r_{1}\right)-2 e^{2} \frac{1}{4 \pi \epsilon_{0}} \frac{1}{r_{1}} \psi_{1}\left(r_{1}\right), \\
E_{2} \psi_{2}\left(r_{2}\right) & =\frac{1}{2 m} \mathbf{p}_{2}^{2} \psi_{2}\left(r_{2}\right)-2 e^{2} \frac{1}{4 \pi \epsilon_{0}} \frac{1}{r_{2}} \psi_{2}\left(r_{2}\right),  \tag{6.172}\\
E & =E_{1}+E_{2} .
\end{align*}
$$

Observe that each of these separated Hamiltonians have (with $Z=2$ ) the form

$$
\begin{equation*}
E \psi(r)=\frac{1}{2 m} \mathbf{p}^{2} \psi(r)-Z e^{2} \frac{1}{4 \pi \epsilon_{0}} \frac{1}{r} \psi(r) \tag{6.173}
\end{equation*}
$$

With $Z=1$ that is precisely the Hamiltonian for the Hydrogen atom. If the wavefunction for this Hamiltonian is assumed to be $\psi(r)=e^{-r / a}$, we find

$$
\begin{equation*}
\frac{\langle\psi| H|\psi\rangle}{\langle\psi \mid \psi\rangle}=\frac{\hbar^{2}}{2 m a^{2}}-\frac{Z e^{2}}{4 \pi \epsilon_{0} a}, \tag{6.174}
\end{equation*}
$$

which has its minimum at

$$
\begin{equation*}
a_{\min }=\frac{a_{0}}{Z}, \tag{6.175}
\end{equation*}
$$

where

$$
\begin{equation*}
a_{0}=\frac{4 \pi \epsilon_{0} \hbar^{2}}{m e^{2}} . \tag{6.176}
\end{equation*}
$$

The minimum energy is found to be

$$
\begin{equation*}
E_{\min }=-\frac{1}{2} \frac{e^{2} Z^{2}}{4 \pi \epsilon_{0} a_{0}} . \tag{6.177}
\end{equation*}
$$

With $Z=1$, the Hydrogen ground state energy is

$$
\begin{equation*}
E_{\mathrm{H}}=-\frac{1}{2} \frac{e^{2}}{4 \pi \epsilon_{0} a_{0}}, \tag{6.178}
\end{equation*}
$$

a value of about -13.6 eV . The Helium ground state energy is

$$
\begin{align*}
E_{\mathrm{He}} & =\left.E_{1}\right|_{Z=2}+\left.E_{2}\right|_{Z=2} \\
& =-\frac{1}{2}\left(2^{2}+2^{2}\right) \frac{e^{2}}{4 \pi \epsilon_{0} a_{0}}  \tag{6.179}\\
& =-4 \frac{e^{2}}{4 \pi \epsilon_{0} a_{0}} .
\end{align*}
$$

This is $E_{\mathrm{He}}=8 E_{\mathrm{H}}$, a value of about -109 eV . The computations above can be found in ps7:heliumAtomGroundStateWithInteraction.nb.

Part b. The Laplacian of an exponentially decreasing trial function $e^{-r / a}$ is

$$
\begin{align*}
\boldsymbol{\nabla}^{2} e^{-r / a} & =\frac{1}{r^{2}} \frac{\partial}{\partial r}\left(r^{2} \frac{\partial e^{-r / a}}{\partial r}\right) \\
& =\frac{1}{r^{2}} \frac{\partial}{\partial r}\left(-\frac{r^{2}}{a} e^{-r / a}\right)  \tag{6.180}\\
& =-\frac{1}{r^{2} a}\left(2 r-\frac{r^{2}}{a}\right) e^{-r / a}
\end{align*}
$$

or

$$
\begin{equation*}
\boldsymbol{\nabla}^{2} e^{-r / a}=\frac{1}{a}\left(\frac{1}{a}-\frac{2}{r}\right) e^{-r / a} . \tag{6.181}
\end{equation*}
$$

For Helium without electron-electron interaction the kinetic portion of the Hamiltonian action on this trial function $\psi=e^{-\left(r_{1}+r_{2}\right) / a}$ is

$$
\begin{align*}
H \psi\left(r_{1}, r_{2}\right) & =\frac{\mathbf{p}_{1}^{2}}{2 m} \psi+\frac{\mathbf{p}_{2}^{2}}{2 m} \psi-2 e^{2} \frac{1}{4 \pi \epsilon_{0}}\left(\frac{1}{r_{1}}+\frac{1}{r_{2}}\right) \psi \\
& =-\frac{\hbar^{2}}{2 m a}\left(\frac{2}{a}-\frac{2}{r_{1}}-\frac{2}{r_{2}}\right) \psi-2 e^{2} \frac{1}{4 \pi \epsilon_{0}}\left(\frac{1}{r_{1}}+\frac{1}{r_{2}}\right) \psi  \tag{6.182}\\
& =\left(-\frac{\hbar^{2}}{m a^{2}}+\left(\frac{\hbar^{2}}{m a}-\frac{e^{2}}{2 \pi \epsilon_{0}}\right)\left(\frac{1}{r_{1}}+\frac{1}{r_{2}}\right)\right) e^{-\left(r_{1}+r_{2}\right) / a} .
\end{align*}
$$

Now, assuming that $\psi=e^{-\left(r_{1}+r_{2}\right) / a}$ is the unnormalized ground state wavefunction for the Helium atom without electron-electron interaction, that ground state energy is given by

$$
\begin{equation*}
E_{\mathrm{He}}=\left(\frac{\hbar^{2}}{m a}-e^{2} \frac{1}{2 \pi \epsilon_{0}}\right) \frac{\langle\psi| \frac{1}{r_{1}}+\frac{1}{r_{2}}|\psi\rangle}{\langle\psi \mid \psi\rangle}-\frac{\hbar^{2}}{m a^{2}} . \tag{6.183}
\end{equation*}
$$

We'll need a couple helper integrals

$$
\begin{equation*}
4 \pi \int_{0}^{\infty} r^{2} d r e^{-2 r / a}=\pi a^{3} \tag{6.184}
\end{equation*}
$$

and

$$
\begin{equation*}
4 \pi \int_{0}^{\infty} r d r e^{-2 r / a}=\pi a^{2} \tag{6.185}
\end{equation*}
$$

To normalize the wavefunction, we need a six-fold integral over both the spatial domains. With only radial dependence that is

$$
\begin{equation*}
\langle\psi \mid \psi\rangle=(4 \pi)^{2} \int_{0}^{\infty} r_{1}^{2} d r_{1} e^{-2 r_{1} / a} \int_{0}^{\infty} r_{2}^{2} d r_{2} e^{-2 r_{2} / a}=\pi^{2} a^{6} \tag{6.186}
\end{equation*}
$$

We also need the inverse radial expectations. Calculating the expectation of $1 / r_{1}$ is sufficient, and is

$$
\begin{align*}
\langle\psi| \frac{1}{r_{1}}|\psi\rangle & =(4 \pi)^{2} \int_{0}^{\infty} r_{1} d r_{1} e^{-2 r_{1} / a} \int_{0}^{\infty} r_{2}^{2} d r_{2} e^{-2 r_{2} / a}  \tag{6.187}\\
& =\pi^{2} a^{5} .
\end{align*}
$$

So, without the electron-electron interaction, the ground state energy is

$$
\begin{align*}
E_{\mathrm{He}} & =\left(\frac{\hbar^{2}}{m a}-e^{2} \frac{1}{2 \pi \epsilon_{0}}\right) \frac{2 \pi^{2} a^{5}}{\pi^{2} a^{6}}-\frac{\hbar^{2}}{m a^{2}} \\
& =\frac{\hbar^{2}}{m a^{2}}-e^{2} \frac{1}{\pi \epsilon_{0} a} . \tag{6.188}
\end{align*}
$$

To evaluate the interaction term, a Fourier transform representation of that inverse radial distance can be employed

$$
\begin{equation*}
\frac{1}{|\mathbf{r}|}=\frac{1}{2 \pi^{2}} \int d^{3} k \frac{e^{i \mathbf{k} \cdot \mathbf{r}}}{\mathbf{k}^{2}} \tag{6.189}
\end{equation*}
$$

where this is understood to be the $\epsilon \rightarrow 0$ limit of

$$
\begin{equation*}
\frac{1}{|\mathbf{r}|} e^{-\epsilon|\mathbf{r}|}=\frac{1}{2 \pi^{2}} \int d^{3} k \frac{e^{i \mathbf{k} \cdot \mathbf{r}}}{\mathbf{k}^{2}+\epsilon^{2}} \tag{6.190}
\end{equation*}
$$

See [3] for a demonstration of this identity, and the contour used to evaluate the RHS of eq. (6.190). Employing this inverse radial representation, the $r_{1}$ and $r_{2}$ contributions to the interaction can be decoupled

$$
\begin{align*}
\frac{e^{2}}{4 \pi \epsilon_{0}}\langle\psi| & \frac{1}{\left|\mathbf{r}_{1}-\mathbf{r}_{2}\right|}|\psi\rangle \\
= & \frac{e^{2}}{4 \pi \epsilon_{0}} \int 2 \pi d r_{1} d \theta_{1} r_{1}^{2} \sin \left(\theta_{1}\right) \times \\
& \quad \int 2 \pi d r_{2} d \theta_{2} r_{2}^{2} \sin \left(\theta_{2}\right) e^{-2\left(r_{1}+r_{2}\right) / a} \frac{1}{2 \pi^{2}} \int d^{3} k \frac{e^{i \mathbf{k} \cdot\left(\mathbf{r}_{1}-\mathbf{r}_{2}\right)}}{\mathbf{k}^{2}}  \tag{6.191}\\
= & \frac{e^{2}}{4 \pi \epsilon_{0}} \frac{1}{2 \pi^{2}} \int d^{3} k \frac{1}{\mathbf{k}^{2}} \int 2 \pi d r_{1} d \theta_{1} r_{1}^{2} \sin \left(\theta_{1}\right) e^{-2 r_{1} / a+i \mathbf{k} \cdot \mathbf{r}_{1}} \times \\
& \quad \int 2 \pi d r_{2} d \theta_{2} r_{2}^{2} \sin \left(\theta_{2}\right) e^{-2 r_{2} / a-i \mathbf{k} \cdot \mathbf{r}_{2}} .
\end{align*}
$$

The spatial domain integrals can now be evaluated separately. With a coordinate system picked so that $\mathbf{k}= \pm k \hat{\mathbf{z}}$, that gives

$$
\begin{align*}
2 \pi \int d r d \theta r^{2} \sin (\theta) e^{-2 r / a+i \mathbf{k} \cdot \mathbf{r}} & =2 \pi \int_{0}^{\infty} d r r^{2} e^{-2 r / a} \int_{0}^{\pi} d \theta \frac{d}{d \theta}(-\cos (\theta)) e^{ \pm i k r \cos \theta} \\
& =2 \pi \int_{0}^{\infty} d r r^{2} e^{-2 r / a} \int_{-1}^{1} d u e^{\mp i k r u} \\
& =2 \pi \int_{0}^{\infty} d r r^{2} e^{-2 r / a} \frac{e^{\mp i k r}-e^{ \pm i k r}}{\mp i k r} \\
& =2 \pi \frac{2}{k} \int_{0}^{\infty} d r r e^{-2 r / a} \sin (k r) \\
& =2 \pi \frac{2}{k} \int_{0}^{\infty} d r r e^{-2 r / a} \sin (k r) \\
& =\frac{16 \pi a^{3}}{\left(1+a^{2} k^{2}\right)^{2}} \tag{6.192}
\end{align*}
$$

We see that the specific orientation used to evaluate the integral does not matter, so we have

$$
\begin{align*}
\frac{e^{2}}{4 \pi \epsilon_{0}}\langle\psi| \frac{1}{\left|\mathbf{r}_{1}-\mathbf{r}_{2}\right|}|\psi\rangle & =\frac{e^{2}}{4 \pi \epsilon_{0}} \frac{1}{2 \pi^{2}} \int d^{3} k \frac{1}{\mathbf{k}^{2}} \frac{\left(16 \pi a^{3}\right)^{2}}{\left(1+a^{2} k^{2}\right)^{4}} \\
& =\frac{e^{2}}{4 \pi \epsilon_{0}} \frac{1}{2 \pi^{2}}(4 \pi) 16^{2} \pi^{2} a^{6} \int d k k^{2} \frac{1}{\mathbf{k}^{2}} \frac{1}{\left(1+a^{2} k^{2}\right)^{4}} \\
& =\frac{5 \pi e^{2} a^{5}}{32 \epsilon_{0}} . \tag{6.193}
\end{align*}
$$

Rescaling with the normalization factor gives

$$
\begin{align*}
\frac{e^{2}}{4 \pi \epsilon_{0}}\langle\psi| \frac{1}{\left|\mathbf{r}_{1}-\mathbf{r}_{2}\right|}|\psi\rangle /\langle\psi \mid \psi\rangle & =\frac{5 \pi e^{2} a^{5}}{32 \epsilon_{0}} \frac{1}{\pi^{2} a^{6}} \\
& =\frac{5 e^{2}}{32 \pi \epsilon_{0} a} . \tag{6.194}
\end{align*}
$$

Adding this electron-electron interaction to the Helium ground energy calculated in eq. (6.188) gives

$$
\begin{equation*}
E_{\mathrm{He}}=\frac{\hbar^{2}}{m a^{2}}-\frac{27}{32} e^{2} \frac{1}{\pi \epsilon_{0} a} . \tag{6.195}
\end{equation*}
$$

For the minimum we want to solve

$$
\begin{align*}
0 & =\frac{\partial E}{\partial a} \\
& =-2 \frac{\hbar^{2}}{m a^{3}}+\frac{27}{32} e^{2} \frac{1}{\pi \epsilon_{0} a^{2}}, \tag{6.196}
\end{align*}
$$

which has the minimum at

$$
\begin{equation*}
a=-\frac{64 \hbar^{2} \pi \epsilon_{0}}{27 m e^{2}} . \tag{6.197}
\end{equation*}
$$

Note that $m$ should really be treated as the reduced mass of the electron, but doing so isn't numerically significant. The final result for the variational ground state energy is

$$
\begin{equation*}
E_{\mathrm{He}}=-\left(\frac{27}{16}\right)^{2} \frac{e^{2}}{4 \pi \epsilon_{0} a_{0}} \approx-77.5 \mathrm{eV} \tag{6.198}
\end{equation*}
$$

In atomic units this is

$$
\begin{equation*}
E_{\mathrm{He}}=-\left(\frac{27}{16}\right)^{2} \frac{e^{2}}{a_{0}} \approx-2.848 \frac{e^{2}}{a_{0}} \tag{6.199}
\end{equation*}
$$

In [4] the measured value is stated as $-2.90 e^{2} / a_{0}$. Table 1 of [2], which lists high precision calculations of all the energy levels, has $-2.903724 e^{2} / a_{0}$ for the 1 s energy level. The calculated value of eq. (6.198) is about $2 \%$ off the mark.

See ps7:heliumAtomGroundStateWithInteraction.nb, for a complete end to end verification of the calculations above.

Exercise 6.5 Harmonic oscillator variation. ([4] pr. 24.3)
Consider a 1D harmonic oscillator with an unnormalized trial wavefunction $\psi_{v}(x)=e^{-\beta|x|}$. Minimize the ground state energy with respect to $\beta$, thus obtaining the optimal $\beta$ as well as the variational ground state energy. Compare with the exact result. Note that you need to be careful evaluating derivatives since the wavefunction has a 'cusp' at $x=0$.

## Answer for Exercise 6.5

In order to make the derivatives of the trial function better behaved at the origin, we can treat it as a distribution, writing

$$
\begin{equation*}
\psi(x)=\Theta(x) e^{-\beta x}+\Theta(-x) e^{\beta x} \tag{6.200}
\end{equation*}
$$

This has a derivative

$$
\begin{align*}
\psi^{\prime}(x) & =\delta(x) e^{-\beta x}-\delta(-x) e^{\beta x}+\beta\left(-\Theta(x) e^{-\beta x}+\Theta(-x) e^{\beta x}\right)  \tag{6.201}\\
& =-2 \delta(x) \sinh (\beta x)+\beta\left(-\Theta(x) e^{-\beta x}+\Theta(-x) e^{\beta x}\right)
\end{align*}
$$

Using $\delta^{\prime}(x)=-\delta(x) / x$, the second derivative is

$$
\begin{align*}
\psi^{\prime \prime}(x)= & +2 \delta(x) \sinh (\beta x) / x-2 \beta \delta(x) \cosh (\beta x)+\beta\left(-\delta(x) e^{-\beta x}-\delta(-x) e^{\beta x}\right) \\
& +\beta^{2}\left(\Theta(x) e^{-\beta x}+\Theta(-x) e^{\beta x}\right) \\
= & +2 \delta(x) \beta\left(\frac{\sinh (\beta x)}{\beta x}-\cosh (\beta x)\right)-2 \beta \delta(x) \cosh (\beta x)+\beta^{2} e^{-\beta|x|} \tag{6.202}
\end{align*}
$$

Because

$$
\begin{equation*}
\int \delta(x) \cosh (\beta x) f(x) d x=f(0) \tag{6.203}
\end{equation*}
$$

and

$$
\begin{align*}
\int \delta(x)\left(\frac{\sinh (\beta x)}{\beta x}-\cosh (\beta x)\right) f(x) d x & =(1-1) f(0)  \tag{6.204}\\
& =0
\end{align*}
$$

this second derivative can be simplified to

$$
\begin{equation*}
\psi^{\prime \prime}(x)=-2 \beta \delta(x)+\beta^{2} e^{-\beta|x|} \tag{6.205}
\end{equation*}
$$

This has the $\beta^{2} \psi(x)$ value that we expect at points away from the origin. All the expectations can now be computed. The normalization is

$$
\begin{align*}
\langle\psi \mid \psi\rangle & =2 \int_{0}^{\infty} e^{-2 \beta x} d x \\
& =\left.2 \frac{e^{-2 \beta x}}{-2 \beta}\right|_{0} ^{\infty}  \tag{6.206}\\
& =\frac{1}{\beta}
\end{align*}
$$

Observe that

$$
\begin{align*}
\frac{d^{2}}{d \beta^{2}} \int_{0}^{\infty} e^{-2 \beta x} d x & =(-2)^{2} \int_{0}^{\infty} x^{2} e^{-2 \beta x} d x \\
& =\frac{d}{d \beta}\left(-\frac{1}{2 \beta^{2}}\right)  \tag{6.207}\\
& =\frac{1}{\beta^{3}}
\end{align*}
$$

so

$$
\begin{equation*}
\int_{0}^{\infty} x^{2} e^{-2 \beta x} d x=\frac{1}{4 \beta^{3}} \tag{6.208}
\end{equation*}
$$

a result we will need later. The kinetic portion of the energy expectation is

$$
\begin{align*}
\langle\psi| \frac{p^{2}}{2 m}|\psi\rangle & =-\frac{\hbar^{2}}{2 m} \int_{-\infty}^{\infty} e^{-\beta x}\left(-2 \beta \delta(x)+\beta^{2} e^{-\beta|x|}\right) \\
& =-\frac{\hbar^{2}}{2 m} \beta^{2} \frac{1}{\beta}-\frac{\hbar^{2}}{2 m}(-2 \beta)  \tag{6.209}\\
& =\frac{\hbar^{2}}{2 m} \beta
\end{align*}
$$

The potential portion of the energy expectation is

$$
\begin{align*}
\langle\psi| \frac{1}{2} m \omega^{2} x^{2}|\psi\rangle & =m \omega^{2} \int_{0}^{\infty} x^{2} e^{-2 \beta x} d x \\
& =m \omega^{2} \frac{1}{4 \beta^{3}} \tag{6.210}
\end{align*}
$$

Adding things up we have

$$
\begin{align*}
\bar{E}(\beta) & =\frac{\langle\psi| H|\psi\rangle}{\langle\psi \mid \psi\rangle} \\
& =\frac{\frac{\hbar^{2}}{2 m} \beta+\frac{m \omega^{2}}{4 \beta^{3}}}{\frac{1}{\beta}} \\
& =\frac{\hbar^{2}}{2 m} \beta^{2}+\frac{m \omega^{2}}{4 \beta^{2}}  \tag{6.211}\\
& =\frac{\hbar \omega}{2}\left(\frac{\hbar}{m \omega} \beta^{2}+\frac{m \omega}{2 \hbar \beta^{2}}\right) \\
& =\frac{\hbar \omega}{2}\left(x_{0}^{2} \beta^{2}+\frac{1}{2 x_{0}^{2} \beta^{2}}\right) .
\end{align*}
$$

Minimizing gives

$$
\begin{align*}
0 & =\frac{d}{d \beta}\left(x_{0}^{2} \beta^{2}+\frac{1}{2 x_{0}^{2} \beta^{2}}\right)  \tag{6.212}\\
& =2 x_{0}^{2} \beta-\frac{1}{x_{0}^{2} \beta^{3}}
\end{align*}
$$

or

$$
\begin{equation*}
\beta^{4}=\frac{1}{2 x_{0}^{4}} \tag{6.213}
\end{equation*}
$$

which gives

$$
\begin{equation*}
\beta=\frac{1}{2^{1 / 4} x_{0}} \tag{6.214}
\end{equation*}
$$

The energy at this value of $\beta$ is

$$
\begin{align*}
\bar{E}_{\min } & =\frac{\hbar \omega}{2}\left(x_{0}^{2} \frac{1}{\sqrt{2} x_{0}^{2}}+\frac{\sqrt{2} x_{0}^{2}}{2 x_{0}^{2}}\right)  \tag{6.215}\\
& =\frac{\hbar \omega}{2} \frac{2}{\sqrt{2}}
\end{align*}
$$

or

$$
\begin{equation*}
\bar{E}_{\min }=\frac{\hbar \omega}{2} \sqrt{2}>\hbar \omega\left(0+\frac{1}{2}\right) \tag{6.216}
\end{equation*}
$$

We find that the trial function that minimizes the average energy is

$$
\begin{equation*}
\psi(x)=2^{1 / 4} x_{0} \exp \left(-2^{-1 / 4}|x| / x_{0}\right) \tag{6.217}
\end{equation*}
$$

with an average energy that is $1.41 \times$ the actual ground state energy for the harmonic oscillator.

Exercise 6.6 Energy of absolute value potential. ([15] pr. 5.21)
Estimate the lowest eigenvalue $\lambda$ of the differential equation

$$
\begin{equation*}
\frac{d^{2}}{d x^{2}} \psi+(\lambda-|x|) \psi=0 \tag{6.218}
\end{equation*}
$$

Using $\alpha$ variation with the trial function

$$
\psi= \begin{cases}c(\alpha-|x|) & |x|<\alpha  \tag{6.219}\\ 0 & |x|>\alpha\end{cases}
$$

Answer for Exercise 6.6
First rewrite the differential equation in a Hamiltonian like fashion

$$
\begin{equation*}
H \psi=-\frac{d^{2}}{d x^{2}} \psi+|x| \psi=\lambda \psi \tag{6.220}
\end{equation*}
$$

We need the derivatives of the trial distribution. The first derivative is

$$
\begin{align*}
\frac{d}{d x} \psi & =-c \frac{d}{d x}|x| \\
& =-c \frac{d}{d x}(x \theta(x)-x \theta(-x))  \tag{6.221}\\
& =-c(\theta(x)-\theta(-x)+x \delta(x)+x \delta(-x)) \\
& =-c(\theta(x)-\theta(-x)+2 x \delta(x)) .
\end{align*}
$$

The second derivative is

$$
\begin{align*}
\frac{d^{2}}{d x^{2}} \psi & =-c \frac{d}{d x}(\theta(x)-\theta(-x)+2 x \delta(x)) \\
& =-c\left(\delta(x)+\delta(-x)+2 \delta(x)+2 x \delta^{\prime}(x)\right) \\
& =-c\left(4 \delta(x)+2 x \frac{-\delta(x)}{x}\right)  \tag{6.222}\\
& =-2 c \delta(x) .
\end{align*}
$$

This gives

$$
\begin{equation*}
H \psi=-2 c \delta(x)+|x| c(\alpha-|x|) \tag{6.223}
\end{equation*}
$$

We are now set to compute some of the inner products. The normalization is the simplest

$$
\begin{align*}
\langle\psi \mid \psi\rangle & =c^{2} \int_{-\alpha}^{\alpha}(\alpha-|x|)^{2} d x \\
& =2 c^{2} \int_{0}^{\alpha}(x-\alpha)^{2} d x \\
& =2 c^{2} \int_{-\alpha}^{0} u^{2} d u  \tag{6.224}\\
& =2 c^{2}\left(-\frac{(-\alpha)^{3}}{3}\right) \\
& =\frac{2}{3} c^{2} \alpha^{3} .
\end{align*}
$$

For the energy

$$
\begin{align*}
\langle\psi \mid H \psi\rangle & =c^{2} \int d x(\alpha-|x|)(-2 \delta(x)+|x|(\alpha-|x|)) \\
& =c^{2}\left(-2 \alpha+\int_{-\alpha}^{\alpha} d x(\alpha-|x|)^{2}|x|\right) \\
& =c^{2}\left(-2 \alpha+2 \int_{-\alpha}^{0} d u u^{2}(u+\alpha)\right) \\
& =c^{2}\left(-2 \alpha+\left.2\left(\frac{u^{4}}{4}+\alpha \frac{u^{3}}{3}\right)\right|_{-\alpha} ^{0}\right)  \tag{6.225}\\
& =c^{2}\left(-2 \alpha-2\left(\frac{\alpha^{4}}{4}-\frac{\alpha^{4}}{3}\right)\right) \\
& =c^{2}\left(-2 \alpha+\frac{1}{6} \alpha^{4}\right)
\end{align*}
$$

The energy estimate is

$$
\begin{align*}
\bar{E} & =\frac{\langle\psi \mid H \psi\rangle}{\langle\psi \mid \psi\rangle} \\
& =\frac{-2 \alpha+\frac{1}{6} \alpha^{4}}{\frac{2}{3} \alpha^{3}}  \tag{6.226}\\
& =-\frac{3}{\alpha^{2}}+\frac{1}{4} \alpha .
\end{align*}
$$

This has its minimum at

$$
\begin{equation*}
0=-\frac{6}{\alpha^{3}}+\frac{1}{4}, \tag{6.227}
\end{equation*}
$$

or

$$
\begin{equation*}
\alpha=2 \times 3^{1 / 3} . \tag{6.228}
\end{equation*}
$$

Back substitution into the energy gives

$$
\begin{align*}
\bar{E} & =-\frac{3}{4 \times 3^{2 / 3}}+\frac{1}{2} 3^{1 / 3} \\
& =\frac{3^{4 / 3}}{4} \approx 1.08 . \tag{6.229}
\end{align*}
$$

The problem says the exact answer is 1.019 , so the variation gets within $6 \%$.

## Exercise 6.7 Anharmonic oscillator. (2015 ps8 p1)

Consider a quantum particle in the ground state of a 1D anharmonic oscillator potential

$$
\begin{equation*}
V(x)=\frac{1}{2} m \omega^{2} x^{2}+\lambda x^{4}=V_{0}+\lambda V^{\prime} . \tag{6.230}
\end{equation*}
$$

Compute the first and second order energy shift of this oscillator perturbatively in $\lambda$.
Answer for Exercise 6.7
Using ps8:harmonicOscillatorRaiseAndLoweringOperators.nb the action of the potential on the ground state is

$$
\begin{align*}
V^{\prime}|0\rangle & =x^{4}|0\rangle \\
& =x_{0}^{4}\left(\frac{3}{4}|0\rangle+\frac{3}{\sqrt{2}}|2\rangle+\sqrt{\frac{3}{2}}|4\rangle\right) . \tag{6.231}
\end{align*}
$$

That allows us to compute the first order energy shift

$$
\begin{align*}
\Delta^{(1)} & =\langle 0| V^{\prime}|0\rangle \\
& =\frac{3}{4} x_{0}^{4} . \tag{6.232}
\end{align*}
$$

Writing the perturbed state as

$$
\begin{equation*}
|n\rangle=|0\rangle+\lambda|0\rangle^{\prime}+\lambda^{2}|0\rangle^{\prime \prime}+\cdots, \tag{6.233}
\end{equation*}
$$

the first order perturbation $|0\rangle^{\prime}$ of the ground state is

$$
\begin{align*}
|0\rangle^{\prime} & =\sum_{m \neq 0} \frac{|m\rangle\langle m| x^{4}|0\rangle}{\hbar \omega / 2-\hbar \omega(m+1 / 2)} \\
& =-\frac{x_{0}^{4}}{\hbar \omega} \sum_{m \neq 0} \frac{|m\rangle\langle m|}{m}\left(\frac{3}{4}|0\rangle+\frac{3}{\sqrt{2}}|2\rangle+\sqrt{\frac{3}{2}}|4\rangle\right)  \tag{6.234}\\
& =-\frac{x_{0}^{4}}{\hbar \omega}\left(\frac{1}{2} \frac{3}{\sqrt{2}}|2\rangle+\frac{1}{4} \sqrt{\frac{3}{2}}|4\rangle\right) .
\end{align*}
$$

The second order energy shift can now be calculated, and is

$$
\begin{align*}
\Delta^{(2)} & =\langle 0| V^{\prime}|0\rangle^{\prime} \\
& =-\frac{x_{0}^{8}}{\hbar \omega}\left(\frac{3}{4}\langle 0|+\frac{3}{\sqrt{2}}\langle 2|+\sqrt{\frac{3}{2}}\langle 4|\right)\left(\frac{1}{2} \frac{3}{\sqrt{2}}|2\rangle+\frac{1}{4} \sqrt{\frac{3}{2}}|4\rangle\right)  \tag{6.235}\\
& =-\frac{x_{0}^{8}}{\hbar \omega} \frac{21}{8} .
\end{align*}
$$

To second order the total energy shift is

$$
\begin{equation*}
\Delta=\frac{3}{4} \lambda x_{0}^{4}-\frac{21 x_{0}^{8} \lambda^{2}}{8 \hbar \omega} \tag{6.236}
\end{equation*}
$$

Exercise $6.8 \quad$ Quadrupolar potential. (2015 ps8 p2)
Consider a p-orbital electron of hydrogen with $|n, l=1, m\rangle$, with $m=$ $0, \pm 1$, subject to an external potential

$$
\begin{equation*}
V(x, y, z)=\lambda\left(x^{2}-y^{2}\right) \tag{6.237}
\end{equation*}
$$

with $\lambda$ being a constant. For fixed $n$, obtain the correct eigenstates which diagonalize the perturbation, without worrying about doing radial integrals explicitly. Show that the three-fold degeneracy of the p-orbital is completely broken by the perturbation to linear order in $\lambda$.
Answer for Exercise 6.8
The potential in spherical coordinates is

$$
\begin{equation*}
V=\lambda r^{2} \sin ^{2} \theta\left(\cos ^{2} \phi-\sin ^{2} \phi\right)=\lambda r^{2} \sin ^{2} \theta \cos (2 \phi) \tag{6.238}
\end{equation*}
$$

The p-orbital wave functions are

$$
\begin{equation*}
\psi_{n 1 m}(r, \theta, \phi)=R_{n}(r) Y_{1, m}(\theta, \phi) \tag{6.239}
\end{equation*}
$$

where

$$
\begin{align*}
Y_{1,1}(\theta, \phi) & =-\frac{1}{2} \sqrt{\frac{3}{2 \pi}} e^{i \phi} \sin \theta, \\
Y_{1,0}(\theta, \phi) & =\frac{1}{2} \sqrt{\frac{3}{\pi}} \cos \theta,  \tag{6.240}\\
Y_{1,-1}(\theta, \phi) & =\frac{1}{2} \sqrt{\frac{3}{2 \pi}} e^{-i \phi} \sin \theta .
\end{align*}
$$

That is enough information to construct the matrix element of the perturbing potential with respect to these states. Those are

$$
\begin{align*}
& \left\langle n^{\prime} 1 m^{\prime}\right| V|n 1 m\rangle \\
& =\int_{0}^{\infty} r^{2} d r \int_{0}^{\pi} \sin \theta d \theta \times \\
& \quad \int_{0}^{2 \pi} d \phi R_{n}(r) Y_{1, m^{\prime}}^{*}(\theta, \phi) \lambda r^{2} \sin ^{2} \theta \cos (2 \phi) R_{n}(r) Y_{1, m}(\theta, \phi) \\
& =\lambda \int_{0}^{\infty} r^{4} d r R_{n}^{2}(r) \int_{0}^{\pi} \sin ^{3} \theta \cos (2 \phi) d \theta \int_{0}^{2 \pi} d \phi Y_{1, m^{\prime}}^{*}(\theta, \phi) Y_{1, m}(\theta, \phi) \\
& =\lambda \int_{0}^{\infty} r^{4} d r R_{n}^{2}(r)\left[\begin{array}{ccc}
0 & 0 & -\frac{2}{5} \\
0 & 0 & 0 \\
-\frac{2}{5} & 0 & 0
\end{array}\right] . \tag{6.241}
\end{align*}
$$

See ps8:quadrupolarPotentialPorbitalSplitting.nb for a computation of the matrix. It has eigenvalues

$$
\begin{equation*}
\lambda \int_{0}^{\infty} r^{4} d r R_{n}^{2}(r)\left\{-\frac{2}{5}, \frac{2}{5}, 0\right\} . \tag{6.242}
\end{equation*}
$$

We see that the radial factor $R_{n}(r)$ of these wave function provides only a constant adjustment to the energy levels splitting that breaks the degeneracy. That degeneracy is completely broken by this perturbation.

The respective eigenvectors for this matrix are

$$
\left\{\frac{1}{\sqrt{2}}\left[\begin{array}{l}
1  \tag{6.243}\\
0 \\
1
\end{array}\right], \frac{1}{\sqrt{2}}\left[\begin{array}{c}
1 \\
0 \\
-1
\end{array}\right],\left[\begin{array}{l}
0 \\
1 \\
0
\end{array}\right]\right\},
$$

so the wave functions, say $\left\{\psi_{n,-1}, \psi_{n, 1}, \psi_{n, 0}\right\}$, that diagonalize this perturbation potential are

$$
\begin{align*}
\psi_{n,-1}(r, \theta, \phi) & =\frac{R_{n}(r)}{\sqrt{2}}\left(Y_{-1,1}(\theta, \phi)+Y_{1,1}(\theta, \phi)\right)=\frac{R_{n}(r)}{2 i} \sqrt{\frac{3}{\pi}} \sin \phi \sin \theta, \\
\psi_{n, 1}(r, \theta, \phi) & =\frac{R_{n}(r)}{\sqrt{2}}\left(Y_{-1,1}(\theta, \phi)-Y_{1,1}(\theta, \phi)\right)=-\frac{R_{n}(r)}{2} \sqrt{\frac{3}{\pi}} \cos \phi \sin \theta, \\
\psi_{n, 0}(r, \theta, \phi) & =R_{n}(r) Y_{1,0}(\theta, \phi)=\frac{R_{n}(r)}{2} \sqrt{\frac{3}{\pi}} \cos \theta . \tag{6.244}
\end{align*}
$$

It is natural to adjust the phases above, forming an alternate basis

$$
\begin{equation*}
\left\{-\psi_{n, 1}, i \psi_{n,-1}, \psi_{n, 0}\right\}=\frac{R_{n}(r)}{2} \sqrt{\frac{3}{\pi}} \hat{\mathbf{r}}, \tag{6.245}
\end{equation*}
$$

where $\hat{\mathbf{r}}=\{\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta\}$, the set of components of the unit vector parameterized by $\theta, \phi$. In this basis that level splitting is

$$
\lambda \int_{0}^{\infty} r^{4} d r R_{n}^{2}(r)\left\{\frac{2}{5},-\frac{2}{5}, 0\right\},
$$

respectively.
Exercise $6.9 \quad$ Harmonic oscillator. (2015 ps8 p3)
Consider a 2D harmonic oscillator with

$$
\begin{equation*}
H=\frac{p_{x}^{2}}{2 m}+\frac{p_{y}^{2}}{2 m}+\frac{1}{2} m \omega^{2}\left(x^{2}+y^{2}\right), \tag{6.246}
\end{equation*}
$$

and turn on an anharmonic perturbation

$$
\begin{equation*}
V=\lambda g_{1} \frac{m^{2} \omega^{3}}{\hbar}\left(x^{4}+y^{4}\right)+\lambda^{2} g_{2} m \omega^{2} x y . \tag{6.247}
\end{equation*}
$$

Note that the potentials have been altered from the original problem statement to have dimensions of energy with dimensionless scale factors $g_{1}, g_{2}, \lambda$.
a. Derive the equations for the energy shifts and perturbed states for a second order perturbing potential of the form above.
b. Find the perturbed eigenstate and the corresponding energy shifts up to $O\left(\lambda^{2}\right)$ for the ground state. Ignore terms of $O\left(\lambda^{3}\right)$.
c. Do the same for the first two degenerate states.

Part a. With a $\lambda^{2}$ perturbation we have to step back and revisit the derivation of the energy level and perturbed state formulas. Given

$$
\begin{equation*}
H=H_{0}+\lambda V_{1}+\lambda^{2} V_{2}, \tag{6.248}
\end{equation*}
$$

with known solution $H_{0}\left|n^{(0)}\right\rangle=E^{(0)}\left|n^{(0)}\right\rangle$, we seek the a power series representation of the perturbed ket and an energy shift $\Delta$

$$
\begin{equation*}
|n\rangle=\left|n_{0}\right\rangle+\lambda\left|n_{1}\right\rangle+\lambda^{2}\left|n_{2}\right\rangle+\cdots \tag{6.249}
\end{equation*}
$$

$$
\begin{equation*}
\Delta=\lambda \Delta^{(1)}+\lambda^{2} \Delta^{(2)}+\cdots \tag{6.250}
\end{equation*}
$$

where

$$
\begin{equation*}
H|n\rangle=\left(E^{(0)}+\Delta\right)|n\rangle . \tag{6.251}
\end{equation*}
$$

We can assume that the we have the same sort of representation of the perturbed state

$$
\begin{equation*}
|n\rangle=\left|n^{(0)}\right\rangle+\frac{\bar{P}_{n}}{E^{(0)}-H_{0}}\left(\lambda_{1} V_{1}+\lambda^{2} V_{2}-\Delta\right)|n\rangle, \tag{6.252}
\end{equation*}
$$

where

$$
\begin{equation*}
\bar{P}_{n}=1-\left|n^{(0)}\right\rangle\left\langle n^{(0)}\right|=\sum_{m \neq n}\left|m^{(0)}\right\rangle\left\langle m^{(0)}\right| . \tag{6.253}
\end{equation*}
$$

To check this, operating with $E^{(0)}-H_{0}$, we have

$$
\begin{align*}
\left(E^{(0)}-H_{0}\right)|n\rangle & =\left(E^{(0)}-H_{0}\right)\left|n^{(0)}\right\rangle+\bar{P}_{n}\left(\lambda_{1} V_{1}+\lambda^{2} V_{2}-\Delta\right)|n\rangle \\
& =\left(1-\left|n^{(0)}\right\rangle\left\langle n^{(0)}\right|\right)\left(H-H_{0}-\Delta\right)|n\rangle \tag{6.254}
\end{align*}
$$

or

$$
\begin{align*}
\left(E^{(0)}-H+\Delta\right)|n\rangle & =-\left|n^{(0)}\right\rangle\left\langle n^{(0)}\right|\left(H-H_{0}-\Delta\right)|n\rangle \\
& =-\left|n^{(0)}\right\rangle\left(\left(E^{(0)}+\Delta\right)-E^{(0)}-\Delta\right)\left\langle n^{(0)} \mid n\right\rangle  \tag{6.255}\\
& =0 .
\end{align*}
$$

The LHS is also zero as desired, showing that eq. (6.252) is the desired perturbation relationship. For the perturbed state we are looking for just
the $\lambda^{1}$ terms of eq. (6.252), which after dropping all second order and higher terms is

$$
\begin{equation*}
\left|n_{0}\right\rangle+\lambda\left|n_{1}\right\rangle=\left|n_{0}\right\rangle+\sum_{m \neq n} \frac{\left|m^{(0)}\right\rangle\left\langle m^{(0)}\right|}{E^{(0)}-E_{m}}\left(\lambda V_{1}-\lambda \Delta^{(1)}\right)\left(\left|n_{0}\right\rangle+\lambda\left|n_{1}\right\rangle\right) \tag{6.256}
\end{equation*}
$$

so the first order state perturbation is

$$
\begin{equation*}
\left|n_{1}\right\rangle=\sum_{m \neq n} \frac{\left|m^{(0)}\right\rangle\left\langle m^{(0)}\right|}{E^{(0)}-E_{m}}\left(V_{1}-\Delta^{(1)}\right)\left|n_{0}\right\rangle \tag{6.257}
\end{equation*}
$$

The $\Delta^{(1)}$ contribution drops out, leaving

$$
\begin{equation*}
\left|n_{1}\right\rangle=\sum_{m \neq n} \frac{\left|m^{(0)}\right\rangle\left\langle m^{(0)}\right|}{E^{(0)}-E_{m}} V_{1}\left|n_{0}\right\rangle \tag{6.258}
\end{equation*}
$$

just as we had for a strictly first order perturbing potential. For the energy shifts consider the braket

$$
\begin{align*}
\left\langle n^{(0)}\right| H-H_{0}|n\rangle & =\left\langle n^{(0)}\right| V_{1} \Delta^{(1)}+V_{2} \Delta^{(2)}|n\rangle \\
& =\left(\left(E^{(0)}+\Delta\right)-E^{(0)}\right)\left\langle n^{(0)} \mid n\right\rangle  \tag{6.259}\\
& =\Delta\left\langle n^{(0)} \mid n\right\rangle,
\end{align*}
$$

or

$$
\begin{equation*}
\Delta\left\langle n^{(0)} \mid n\right\rangle=\left\langle n^{(0)}\right| V_{1} \Delta^{(1)}+V_{2} \Delta^{(2)}|n\rangle \tag{6.260}
\end{equation*}
$$

Expanding both sides in powers of $\lambda$ we have

$$
\begin{equation*}
\sum_{r=1, s=0} \lambda^{r+s} \Delta^{(r)}\left\langle n^{(0)} \mid n_{s}\right\rangle=\sum_{m=0} \lambda^{m+1}\left\langle n^{(0)}\right| V_{1}\left|n_{m}\right\rangle+\lambda^{m+2}\left\langle n^{(0)}\right| V_{2}\left|n_{m}\right\rangle \tag{6.261}
\end{equation*}
$$

With $\left|n^{(0)}\right\rangle=\left|n_{0}\right\rangle$ as required in the $\lambda \rightarrow 0$ limit, the $\lambda=1$ contribution to these sums is

$$
\begin{equation*}
\Delta^{(1)}=\left\langle n_{0}\right| V_{1}\left|n_{0}\right\rangle \tag{6.262}
\end{equation*}
$$

The second order contribution is

$$
\begin{equation*}
\Delta^{(1)}\left\langle n^{(0)} \mid n_{1}\right\rangle+\Delta^{(2)}\left\langle n^{(0)} \mid n_{0}\right\rangle=\left\langle n^{(0)}\right| V_{1}\left|n_{1}\right\rangle+\left\langle n^{(0)}\right| V_{2}\left|n_{0}\right\rangle, \tag{6.263}
\end{equation*}
$$

or

$$
\begin{equation*}
\Delta^{(2)}=\left\langle n_{0}\right| V_{1}\left|n_{1}\right\rangle+\left\langle n_{0}\right| V_{2}\left|n_{0}\right\rangle-\left\langle n_{0}\right| V_{1}\left|n_{0}\right\rangle\left\langle n_{0} \mid n_{1}\right\rangle . \tag{6.264}
\end{equation*}
$$

However, from eq. (6.258) we see that $\left|n_{1}\right\rangle$ has no $\left|n_{0}\right\rangle$ component, this means the second order shift is just

$$
\begin{equation*}
\Delta^{(2)}=\left\langle n_{0}\right| V_{1}\left|n_{1}\right\rangle+\left\langle n_{0}\right| V_{2}\left|n_{0}\right\rangle \tag{6.265}
\end{equation*}
$$

Part b. In ps8:2dHarmonicOscillatorOperators.nb, for an initial state $\left|n_{0}\right\rangle=|0,0\rangle$, it is calculated that

$$
\begin{align*}
V_{1}|0,0\rangle & =g_{1} \frac{\hbar \omega}{x_{0}^{4}}\left(x^{4}+y^{4}\right)|0,0\rangle \\
& =g_{1} \hbar \omega\left(\frac{3}{2}|0,0\rangle+\frac{3}{\sqrt{2}}(|2,0\rangle+|0,2\rangle)+\sqrt{\frac{3}{2}}(|4,0\rangle+|0,4\rangle)\right) \tag{6.266}
\end{align*}
$$

The first energy shift is

$$
\begin{equation*}
\Delta^{(1)}=\langle 0,0| V_{1}|0,0\rangle=\frac{3}{2} g_{1} \hbar \omega \tag{6.267}
\end{equation*}
$$

and the first order perturbation of the state is

$$
\begin{align*}
\left|n_{1}\right\rangle= & g_{1} \hbar \omega\left(\frac{3}{\sqrt{2}} \frac{|2,0\rangle+|0,2\rangle}{\hbar \omega(1+0+0)-\hbar \omega(1+2+0)}\right. \\
& \left.+\sqrt{\frac{3}{2}} \frac{|4,0\rangle+|0,4\rangle}{\hbar \omega(1+0+0)-\hbar \omega(1+4+0)}\right) \tag{6.268}
\end{align*}
$$

or

$$
\begin{equation*}
\left|n_{1}\right\rangle=-g_{1}\left(\frac{3}{2 \sqrt{2}}(|2,0\rangle+|0,2\rangle)+\frac{1}{4} \sqrt{\frac{3}{2}}(|4,0\rangle+|0,4\rangle)\right) \tag{6.269}
\end{equation*}
$$

We can calculate

$$
\begin{align*}
\langle 0,0| V_{2}|0,0\rangle & =0 \\
\langle 0,0| V_{1}\left|n_{1}\right\rangle & =-\frac{21}{4} \hbar \omega g_{1}^{2} \tag{6.270}
\end{align*}
$$

so the second order energy shift is

$$
\begin{equation*}
\Delta^{(2)}=-\frac{21}{4} \hbar \omega g_{1}^{2} \tag{6.271}
\end{equation*}
$$

Finally, the ground state energy shift, to second order in $\lambda$, is

$$
\begin{equation*}
\hbar \omega \rightarrow \hbar \omega+\frac{3}{2} \hbar \omega g_{1} \lambda-\frac{21}{4} \hbar \omega g_{1}^{2} \lambda^{2} . \tag{6.272}
\end{equation*}
$$

For the ground state, there is no contribution from the second order potential $\lambda^{2} V_{2}$.

Part c. The next two highest states are $|1,0\rangle,|0,1\rangle$ both with unperturbed energy eigenvalues $2 \hbar \omega$. For a basis spanning the $\{|1,0\rangle,|0,1\rangle\}$ subspace, the matrix element of the perturbed Hamiltonian is

$$
\begin{align*}
H_{0}+\lambda V_{1}+\lambda^{2} V_{2} & =2 \hbar \omega I+\frac{9}{2} g_{1} \lambda \hbar \omega I+\frac{1}{2} g_{2} \lambda^{2} \hbar \omega \sigma_{1} \\
& =\hbar \omega\left[\begin{array}{cc}
2+\frac{9}{2} g_{1} \lambda & \frac{1}{2} g_{2} \lambda^{2} \\
\frac{1}{2} g_{2} \lambda^{2} & 2+\frac{9}{2} g_{1} \lambda
\end{array}\right] . \tag{6.273}
\end{align*}
$$

Since the eigenvalues of $\left[\begin{array}{ll}a & b \\ b & a\end{array}\right]$ are just $a \pm b$, the energy splitting to first order for these first two degenerate states is

$$
\begin{equation*}
2 \hbar \omega \rightarrow \hbar \omega\left(2+\frac{9}{2} g_{1} \lambda \pm \frac{g_{2} \lambda^{2}}{2}\right) \tag{6.274}
\end{equation*}
$$

While the $x y$ perturbation potential left the ground state untouched, it is responsible for the energy level splitting for the degenerate states $|1,0\rangle,|1,0\rangle$.

## Exercise $6.10 \quad$ Hyperfine levels. (2015 ps8 p4)

We can schematically model the hyperfine interaction between the electron and proton spins as $A \mathbf{S}_{\mathrm{e}} \cdot \mathbf{S}_{\mathrm{p}}$ where $A$ is the hyperfine interaction energy.
a. Consider the spin- $1 / 2$ proton interacting with a spin- $1 / 2$ electron. What are the spin eigenstates and eigenvalues?
b. Now consider applying a magnetic field which leads to an extra term

$$
\begin{equation*}
-B\left(g_{\mathrm{e}} \mu_{\mathrm{e}} S_{\mathrm{e}}^{z}+g_{\mathrm{p}} \mu_{\mathrm{p}} S_{\mathrm{p}}^{z}\right) \tag{6.275}
\end{equation*}
$$

with gyromagnetic ratios $g_{\mathrm{e}} \approx-2$ and $g_{\mathrm{p}} \approx 5.5$, with magnetic moments $\mu_{\mathrm{e}}=e / 2 m_{\mathrm{e}}$ and $\mu_{\mathrm{p}}=e / 2 m_{\mathrm{p}}$. The large nuclear mass ensures $\mu_{\mathrm{e}} / \mu_{\mathrm{p}} \sim 2000$, so let us simply set $\mu_{\mathrm{p}}=0$. For convenience, set $B g_{\mathrm{e}} \mu_{\mathrm{e}} \rightarrow B_{\text {eff }}$ so the Hamiltonian becomes

$$
\begin{equation*}
H=A \mathbf{S}_{\mathrm{e}} \cdot \mathbf{S}_{\mathrm{p}}-B_{\mathrm{eff}} S_{\mathrm{e}}^{z} \tag{6.276}
\end{equation*}
$$

so the only dimensionless parameter is $B_{\text {eff }} / A$.
Using perturbation theory (degenerate or non-degenerate as appropriate) find how the coupled hyperfine levels split for weak field $B_{\text {eff }} / A \ll 1$. Also consider the strong field limit $B_{\text {eff }} / A \gg 1$.
c. Compute the full field evolution of the levels and compare with the perturbative low field regime result and the high field regime result.

## Answer for Exercise 6.10

Part $a$. With respect to the basis $\beta=|++\rangle,|-+\rangle,|+-\rangle,|--\rangle$, where
 $\mathbf{S}_{\mathrm{p}}$ operators (not of the respective $S^{z}$ operators), we can compute the unperturbed interaction Hamiltonian. For brevity, let $S_{\mathrm{ep}}=\mathbf{S}_{\mathrm{e}} \cdot \mathbf{S}_{\mathrm{p}}$

$$
\begin{align*}
& A \mathbf{S}_{\mathrm{e}} \cdot \mathbf{S}_{\mathrm{p}} \\
& \quad=A\left[\begin{array}{llll}
\langle++| S_{\mathrm{ep}}|++\rangle & \langle++| S_{\mathrm{ep}}|-+\rangle & \langle++| S_{\mathrm{ep}}|+-\rangle & \langle++| S_{\mathrm{ep}}|--\rangle \\
\langle-+| S_{\mathrm{ep}}|++\rangle & \langle-+| S_{\mathrm{ep}}|-+\rangle & \langle-+| S_{\mathrm{ep}}|+-\rangle & \langle-+| S_{\mathrm{ep}}|--\rangle \\
\langle+-| S_{\mathrm{ep}}|++\rangle & \langle+-| S_{\mathrm{ep}}|-+\rangle & \langle+-| S_{\mathrm{ep}}|+-\rangle & \langle+-| S_{\mathrm{ep}}|--\rangle \\
\langle--| S_{\mathrm{ep}}|++\rangle & \langle--| S_{\mathrm{ep}}|-+\rangle & \langle--| S_{\mathrm{ep}}|+-\rangle & \langle--| S_{\mathrm{ep}}|--\rangle
\end{array}\right] . \tag{6.277}
\end{align*}
$$

Each of the matrix elements may be expanded in terms of the Pauli basis, for example

$$
\begin{align*}
\langle++| S_{\mathrm{ep}}|-+\rangle & =\frac{\hbar^{2}}{4}\langle+| \sigma_{\mathrm{e}}|-\rangle\langle+| \sigma_{\mathrm{p}}|+\rangle \\
& =\frac{\hbar^{2}}{4}\langle+\mid-\rangle\langle+\mid+\rangle \tag{6.278}
\end{align*}
$$

$$
\begin{align*}
& A \mathbf{S}_{\mathrm{e}} \cdot \mathbf{S}_{\mathrm{p}}=\frac{A \hbar^{2}}{4}\left[\begin{array}{llll}
\langle+\mid+\rangle\langle+\mid+\rangle & \langle+\mid-\rangle\langle+\mid+\rangle & \langle+\mid+\rangle\langle+\mid-\rangle & \langle+\mid-\rangle\langle+\mid-\rangle \\
\langle-\mid+\rangle\langle+\mid+\rangle & \langle-\mid-\rangle\langle+\mid+\rangle & \langle-\mid+\rangle\langle+\mid-\rangle & \langle-\mid-\rangle\langle+\mid-\rangle \\
\langle+\mid+\rangle\langle-\mid+\rangle & \langle+\mid-\rangle\langle-\mid+\rangle & \langle+\mid+\rangle\langle-\mid-\rangle & \langle+\mid-\rangle\langle-\mid-\rangle \\
\langle-\mid+\rangle\langle-\mid+\rangle & \langle-\mid-\rangle\langle-\mid+\rangle & \langle-\mid+\rangle\langle-\mid-\rangle & \langle-\mid-\rangle\langle-\mid-\rangle
\end{array}\right] \\
& =\frac{A \hbar^{2}}{4}\left[\begin{array}{cccc}
(1)(1) & (0)(1) & (1)(0) & (0)(0) \\
(0)(1) & (-1)(1) & (0)(0) & (-1)(0) \\
(1)(0) & (0)(0) & (1)(-1) & (0)(-1) \\
(0)(0) & (-1)(0) & (0)(-1) & (-1)(-1)
\end{array}\right] \\
& =\frac{A \hbar^{2}}{4}\left[\begin{array}{cc}
\sigma_{3} & 0 \\
0 & -\sigma_{3}
\end{array}\right] \text {. } \tag{6.279}
\end{align*}
$$

The spin eigenstates are the basis elements of $\beta$ above, with respective eigenvalues

$$
\begin{equation*}
\left\{A \hbar^{2} / 4,-A \hbar^{2} / 4,-A \hbar^{2} / 4, A \hbar^{2} / 4\right\} \tag{6.280}
\end{equation*}
$$

Part b. The matrix representation of the perturbation potential is

$$
\begin{align*}
& -B_{\mathrm{eff}} S_{\mathrm{e}}^{z}=-\frac{B_{\mathrm{eff}} \hbar}{2} \times \\
& \left.\qquad \begin{array}{llll}
\langle+| \sigma_{\mathrm{e}}^{z}|+\rangle\langle+\mid+\rangle & \langle+| \sigma_{\mathrm{e}}^{z}|-\rangle\langle+\mid+\rangle & \langle+| \sigma_{\mathrm{e}}^{z}|+\rangle\langle+\mid-\rangle & \langle+| \sigma_{\mathrm{e}}^{z}|-\rangle\langle+\mid-\rangle \\
\langle-| \sigma_{\mathrm{e}}^{z}|+\rangle\langle+\mid+\rangle & \langle-| \sigma_{\mathrm{e}}^{z}|-\rangle\langle+\mid+\rangle & \langle-| \sigma_{\mathrm{e}}^{z}|+\rangle\langle+\mid-\rangle & \langle-| \sigma_{\mathrm{e}}^{z}|-\rangle\langle+\mid-\rangle \\
\langle+| \sigma_{\mathrm{e}}^{z}|+\rangle\langle-\mid+\rangle & \langle+| \sigma_{\mathrm{e}}^{z}|-\rangle\langle-\mid+\rangle & \langle+| \sigma_{\mathrm{e}}^{z}|+\rangle\langle-\mid-\rangle & \langle+| \sigma_{\mathrm{e}}^{z}|-\rangle\langle-\mid-\rangle \\
\langle-| \sigma_{\mathrm{e}}^{z}|+\rangle\langle-\mid+\rangle & \langle-| \sigma_{\mathrm{e}}^{z}|-\rangle\langle-\mid+\rangle & \langle-| \sigma_{\mathrm{e}}^{z}|+\rangle\langle-\mid-\rangle & \langle-| \sigma_{\mathrm{e}}^{z}|-\rangle\langle-\mid-\rangle
\end{array}\right] \\
& \quad=-\frac{B_{\mathrm{eff}} \hbar}{2}\left[\begin{array}{cc}
\sigma_{\mathrm{e}}^{z} & 0 \\
0 & \sigma_{\mathrm{e}}^{z}
\end{array}\right], \tag{6.281}
\end{align*}
$$

Assuming the $\mathbf{S}_{\mathrm{e}}$ operator is directed along $\hat{\mathbf{n}}=(\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta)$ with eigenkets
the representation of the $\sigma_{\mathrm{e}}^{z}$ operator is

$$
\begin{align*}
\sigma_{\mathrm{e}}^{z} & =\left[\begin{array}{cc}
\cos \theta & -\sin \theta \\
-\sin \theta & -\cos \theta
\end{array}\right]  \tag{6.284}\\
& =U\left[\begin{array}{cc}
1 & 0 \\
0 & -1
\end{array}\right] U^{-1},
\end{align*}
$$

where

$$
U=\left[\begin{array}{cc}
-\cos (\theta / 2) & \sin (\theta / 2)  \tag{6.285}\\
\sin (\theta / 2) & \cos (\theta / 2)
\end{array}\right] .
$$

This is demonstrated in ps8:PauliMatrixSpinOperators.nb. The full Hamiltonian can now be written in block matrix form

$$
H=\frac{A \hbar^{2}}{4}\left[\begin{array}{cc}
\sigma_{z} & 0  \tag{6.286}\\
0 & -\sigma_{z}
\end{array}\right]-\frac{B_{\mathrm{eff}} \hbar}{2}\left[\begin{array}{cc}
U \sigma_{z} U^{-1} & 0 \\
0 & U \sigma_{z} U^{-1}
\end{array}\right] .
$$

Transforming the Hamiltonian to the $S_{\mathrm{e}}^{z}$ basis we have

$$
H^{\prime}=\frac{A \hbar^{2}}{4}\left[\begin{array}{cc}
U^{-1} \sigma_{z} U & 0  \tag{6.287}\\
0 & -U^{-1} \sigma_{z} U
\end{array}\right]-\frac{B_{\mathrm{eff}} \hbar}{2}\left[\begin{array}{cc}
\sigma_{z} & 0 \\
0 & \sigma_{z}
\end{array}\right] .
$$

For $B_{\text {eff }} \ll A$, the first order energy splitting can be read off by inspection

$$
\begin{align*}
\frac{\hbar^{2} A}{4} & \rightarrow \frac{\hbar^{2} A}{4}-\frac{B_{\mathrm{eff}} \hbar}{2}, \\
-\frac{\hbar^{2} A}{4} & \rightarrow-\frac{\hbar^{2} A}{4}+\frac{B_{\mathrm{eff}} \hbar}{2}, \\
-\frac{\hbar^{2} A}{4} & \rightarrow-\frac{\hbar^{2} A}{4}-\frac{B_{\mathrm{eff}} \hbar}{2},  \tag{6.288}\\
\frac{\hbar^{2} A}{4} & \rightarrow \frac{\hbar^{2} A}{4}+\frac{B_{\mathrm{eff}} \hbar}{2} .
\end{align*}
$$

For the strong field limit, we can flip the problem, and consider $A \mathbf{S}_{\mathrm{e}} \cdot \mathbf{S}_{\mathrm{p}}$ to be a perturbation of an initial Hamiltonian $H_{0}=-B_{\mathrm{eff}} S_{\mathrm{e}}^{z}$. The diagonalization of that perturbation is just eq. (6.286) so the first order energy shifts are

$$
\begin{align*}
-\frac{\hbar B_{\mathrm{eff}}}{2} & \rightarrow-\frac{B_{\mathrm{eff}} \hbar}{2}+\frac{\hbar^{2} A}{4}, \\
\frac{\hbar B_{\mathrm{eff}}}{2} & \rightarrow \frac{B_{\mathrm{eff}} \hbar}{2}-\frac{\hbar^{2} A}{4}  \tag{6.289}\\
-\frac{\hbar B_{\mathrm{eff}}}{2} & \rightarrow-\frac{B_{\mathrm{eff}} \hbar}{2}-\frac{\hbar^{2} A}{4}, \\
\frac{\hbar B_{\mathrm{eff}}}{2} & \rightarrow \frac{B_{\mathrm{eff}} \hbar}{2}+\frac{\hbar^{2} A}{4}
\end{align*}
$$

The splitting is the same to first order, but the starting energies are different.

Part a. See ps8:PauliMatrixSpinOperators.nb for the full field solutions, which are

$$
\begin{align*}
& -\frac{\hbar}{4} \sqrt{4 B_{\mathrm{eff}}^{2}-4 A \hbar \cos \theta B_{\mathrm{eff}}+A^{2} \hbar^{2}} \\
& \frac{\hbar}{4} \sqrt{4 B_{\mathrm{eff}}^{2}-4 A \hbar \cos \theta B_{\mathrm{eff}}+A^{2} \hbar^{2}}  \tag{6.290}\\
& -\frac{\hbar}{4} \sqrt{4 B_{\mathrm{eff}}^{2}+4 A \hbar \cos \theta B_{\mathrm{eff}}+A^{2} \hbar^{2}} \\
& \frac{\hbar}{4} \sqrt{4 B_{\mathrm{eff}}^{2}+4 A \hbar \cos \theta B_{\mathrm{eff}}+A^{2} \hbar^{2}}
\end{align*}
$$

For the weak field $B_{\text {eff }} \ll A$ the respective approximations of these energies are

$$
\begin{align*}
& -\frac{A \hbar^{2}}{4}-\frac{B_{\mathrm{eff}} \hbar \cos \theta}{2} \\
& \frac{A \hbar^{2}}{4}-\frac{B_{\mathrm{eff}} \hbar \cos \theta}{2} \\
& -\frac{A \hbar^{2}}{4}+\frac{B_{\mathrm{eff}} \hbar \cos \theta}{2}  \tag{6.291}\\
& \frac{A \hbar^{2}}{4}+\frac{B_{\mathrm{eff}} \hbar \cos \theta}{2}
\end{align*}
$$

whereas for the strong field $B_{\text {eff }} \gg A$ the respective approximations of these energies are

$$
\begin{gather*}
-\frac{\hbar B_{\mathrm{eff}}}{2}-\frac{A \hbar^{2} \cos \theta}{4} \\
\frac{\hbar B_{\mathrm{eff}}}{2}-\frac{A \hbar^{2} \cos \theta}{4} \\
-\frac{\hbar B_{\mathrm{eff}}}{2}+\frac{A \hbar^{2} \cos \theta}{4},  \tag{6.292}\\
\frac{\hbar B_{\mathrm{eff}}}{2}+\frac{A \hbar^{2} \cos \theta}{4}
\end{gather*}
$$

The full field solution has an orientation specific coupling that the first order perturbative solution does not find, so the perturbation is most accurate when the electron spin orientation is close to the z-axis $(\hat{\mathbf{n}} \cdot \hat{\mathbf{z}}=$ $\cos \theta \approx 1$ ).

## Exercise 6.11 2D SHO xy perturbation. ([15] pr. 5.4)

Given a 2D SHO with Hamiltonian

$$
\begin{equation*}
H_{0}=\frac{1}{2 m}\left(p_{x}^{2}+p_{y}^{2}\right)+\frac{m \omega^{2}}{2}\left(x^{2}+y^{2}\right) \tag{6.293}
\end{equation*}
$$

a. What are the energies and degeneracies of the three lowest states?
b. With perturbation

$$
\begin{equation*}
V=m \omega^{2} x y \tag{6.294}
\end{equation*}
$$

calculate the first order energy perturbations and the zeroth order perturbed states.
c. Solve the $H_{0}+\delta V$ problem exactly, and compare.

## Answer for Exercise 6.11

Part a. Recall that we have

$$
\begin{equation*}
H\left|n_{1}, n_{2}\right\rangle=\hbar \omega\left(n_{1}+n_{2}+1\right)\left|n_{1}, n_{2}\right\rangle \tag{6.295}
\end{equation*}
$$

So the three lowest energy states are $|0,0\rangle,|1,0\rangle,|0,1\rangle$ with energies

$$
\hbar \omega, 2 \hbar \omega, 2 \hbar \omega
$$

respectively (with a two fold degeneracy for the second two energy eigenkets.)

Part b. Consider the action of $x y$ on the $\beta=\{|0,0\rangle,|1,0\rangle,|0,1\rangle\}$ subspace. Those are

$$
\begin{align*}
x y|0,0\rangle & =\frac{x_{0}^{2}}{2}\left(a+a^{\dagger}\right)\left(b+b^{\dagger}\right)|0,0\rangle \\
& =\frac{x_{0}^{2}}{2}\left(b+b^{\dagger}\right)|1,0\rangle  \tag{6.296}\\
& =\frac{x_{0}^{2}}{2}|1,1\rangle
\end{align*}
$$

$$
\begin{align*}
x y|1,0\rangle & =\frac{x_{0}^{2}}{2}\left(a+a^{\dagger}\right)\left(b+b^{\dagger}\right)|1,0\rangle \\
& =\frac{x_{0}^{2}}{2}\left(a+a^{\dagger}\right)|1,1\rangle  \tag{6.297}\\
& =\frac{x_{0}^{2}}{2}(|0,1\rangle+\sqrt{2}|2,1\rangle) . \\
x y|0,1\rangle & =\frac{x_{0}^{2}}{2}\left(a+a^{\dagger}\right)\left(b+b^{\dagger}\right)|0,1\rangle \\
& =\frac{x_{0}^{2}}{2}\left(b+b^{\dagger}\right)|1,1\rangle  \tag{6.298}\\
& =\frac{x_{0}^{2}}{2}(|1,0\rangle+\sqrt{2}|1,2\rangle) .
\end{align*}
$$

The matrix representation of $m \omega^{2} x y$ with respect to the subspace spanned by basis $\beta$ above is

$$
x y \sim \frac{1}{2} \hbar \omega\left[\begin{array}{lll}
0 & 0 & 0  \tag{6.299}\\
0 & 0 & 1 \\
0 & 1 & 0
\end{array}\right]
$$

This diagonalizes with

$$
\begin{align*}
& U=\left[\begin{array}{cc}
1 & 0 \\
0 & \tilde{U}
\end{array}\right],  \tag{6.300a}\\
& \tilde{U}=\frac{1}{\sqrt{2}}\left[\begin{array}{cc}
1 & 1 \\
1 & -1
\end{array}\right],  \tag{6.300b}\\
& D=\frac{1}{2} \hbar \omega\left[\begin{array}{ccc}
0 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & -1
\end{array}\right],  \tag{6.300c}\\
& x y=U D U^{\dagger}=U D U . \tag{6.300d}
\end{align*}
$$

The unperturbed Hamiltonian in the original basis is

$$
H_{0}=\hbar \omega\left[\begin{array}{cc}
1 & 0  \tag{6.301}\\
0 & 2 I
\end{array}\right]
$$

so the transformation to the diagonal $x y$ basis leaves the initial Hamiltonian unaltered

$$
\begin{align*}
H_{0}^{\prime} & =U^{\dagger} H_{0} U \\
& =\hbar \omega\left[\begin{array}{cc}
1 & 0 \\
0 & \tilde{U} 2 I \tilde{U}
\end{array}\right]  \tag{6.302}\\
& =\hbar \omega\left[\begin{array}{cc}
1 & 0 \\
0 & 2 I
\end{array}\right] .
\end{align*}
$$

Now we can compute the first order energy shifts almost by inspection. Writing the new basis as $\beta^{\prime}=\{|0\rangle,|1\rangle,|2\rangle\}$ those energy shifts are just the diagonal elements from the $x y$ operators matrix representation

$$
\begin{align*}
& E_{0}^{(1)}=\langle 0| V|0\rangle=0, \\
& E_{1}^{(1)}=\langle 1| V|1\rangle=\frac{1}{2} \hbar \omega,  \tag{6.303}\\
& E_{2}^{(1)}=\langle 2| V|2\rangle=-\frac{1}{2} \hbar \omega .
\end{align*}
$$

The new energies are

$$
\begin{align*}
& E_{0} \rightarrow \hbar \omega, \\
& E_{1} \rightarrow \hbar \omega(2+\delta / 2),  \tag{6.304}\\
& E_{2} \rightarrow \hbar \omega(2-\delta / 2) .
\end{align*}
$$

Part c. For the exact solution, it's possible to rotate the coordinate system in a way that kills the explicit $x y$ term of the perturbation. That we could do this for $x, y$ operators wasn't obvious to me, but after doing so (and rotating the momentum operators the same way) the new operators still have the required commutators. Let

$$
\begin{align*}
{\left[\begin{array}{l}
u \\
v
\end{array}\right] } & =\left[\begin{array}{cc}
\cos \theta & \sin \theta \\
-\sin \theta & \cos \theta
\end{array}\right]\left[\begin{array}{l}
x \\
y
\end{array}\right] \\
& =\left[\begin{array}{c}
x \cos \theta+y \sin \theta \\
-x \sin \theta+y \cos \theta
\end{array}\right] . \tag{6.305}
\end{align*}
$$

Similarly, for the momentum operators, let

$$
\begin{align*}
{\left[\begin{array}{c}
p_{u} \\
p_{v}
\end{array}\right] } & =\left[\begin{array}{cc}
\cos \theta & \sin \theta \\
-\sin \theta & \cos \theta
\end{array}\right]\left[\begin{array}{c}
p_{x} \\
p_{y}
\end{array}\right]  \tag{6.306}\\
& =\left[\begin{array}{c}
p_{x} \cos \theta+p_{y} \sin \theta \\
-p_{x} \sin \theta+p_{y} \cos \theta
\end{array}\right]
\end{align*}
$$

For the commutators of the new operators we have

$$
\begin{align*}
{\left[u, p_{u}\right] } & =\left[x \cos \theta+y \sin \theta, p_{x} \cos \theta+p_{y} \sin \theta\right] \\
& =\left[x, p_{x}\right] \cos ^{2} \theta+\left[y, p_{y}\right] \sin ^{2} \theta \\
& =i \hbar\left(\cos ^{2} \theta+\sin ^{2} \theta\right)  \tag{6.307}\\
& =i \hbar
\end{align*}
$$

$$
\begin{align*}
{\left[v, p_{v}\right] } & =\left[-x \sin \theta+y \cos \theta,-p_{x} \sin \theta+p_{y} \cos \theta\right] \\
& =\left[x, p_{x}\right] \sin ^{2} \theta+\left[y, p_{y}\right] \cos ^{2} \theta  \tag{6.308}\\
& =i \hbar
\end{align*}
$$

$$
\left[u, p_{v}\right]=\left[x \cos \theta+y \sin \theta,-p_{x} \sin \theta+p_{y} \cos \theta\right]
$$

$$
\begin{equation*}
=\cos \theta \sin \theta\left(-\left[x, p_{x}\right]+\left[y, p_{p}\right]\right) \tag{6.309}
\end{equation*}
$$

$$
=0
$$

We see that the new operators are canonical conjugate as required. For this problem, we just want a 45 degree rotation, with

$$
\begin{align*}
& x=\frac{1}{\sqrt{2}}(u+v)  \tag{6.311}\\
& y=\frac{1}{\sqrt{2}}(u-v)
\end{align*}
$$

We have

$$
\begin{align*}
x^{2}+y^{2} & =\frac{1}{2}\left((u+v)^{2}+(u-v)^{2}\right) \\
& =\frac{1}{2}\left(2 u^{2}+2 v^{2}+2 u v-2 u v\right)  \tag{6.312}\\
& =u^{2}+v^{2}
\end{align*}
$$

$$
\begin{align*}
p_{x}^{2}+p_{y}^{2} & =\frac{1}{2}\left(\left(p_{u}+p_{v}\right)^{2}+\left(p_{u}-p_{v}\right)^{2}\right) \\
& =\frac{1}{2}\left(2 p_{u}^{2}+2 p_{v}^{2}+2 p_{u} p_{v}-2 p_{u} p_{v}\right)  \tag{6.313}\\
& =p_{u}^{2}+p_{v}^{2}
\end{align*}
$$

and

$$
\begin{align*}
x y & =\frac{1}{2}((u+v)(u-v)) \\
& =\frac{1}{2}\left(u^{2}-v^{2}\right) \tag{6.314}
\end{align*}
$$

The perturbed Hamiltonian is

$$
\begin{align*}
H_{0}+\delta V & =\frac{1}{2 m}\left(p_{u}^{2}+p_{v}^{2}\right)+\frac{1}{2} m \omega^{2}\left(u^{2}+v^{2}+\delta u^{2}-\delta v^{2}\right) \\
& =\frac{1}{2 m}\left(p_{u}^{2}+p_{v}^{2}\right)+\frac{1}{2} m \omega^{2}\left(u^{2}(1+\delta)+v^{2}(1-\delta)\right) \tag{6.315}
\end{align*}
$$

In this coordinate system, the corresponding eigensystem is

$$
\begin{equation*}
H\left|n_{1}, n_{2}\right\rangle=\hbar \omega\left(1+n_{1} \sqrt{1+\delta}+n_{2} \sqrt{1-\delta}\right)\left|n_{1}, n_{2}\right\rangle \tag{6.316}
\end{equation*}
$$

For small $\delta$

$$
\begin{equation*}
n_{1} \sqrt{1+\delta}+n_{2} \sqrt{1-\delta} \approx n_{1}+n_{2}+\frac{1}{2} n_{1} \delta-\frac{1}{2} n_{2} \delta \tag{6.317}
\end{equation*}
$$

so

$$
\begin{equation*}
H\left|n_{1}, n_{2}\right\rangle \approx \hbar \omega\left(1+n_{1}+n_{2}+\frac{1}{2} n_{1} \delta-\frac{1}{2} n_{2} \delta\right)\left|n_{1}, n_{2}\right\rangle \tag{6.318}
\end{equation*}
$$

The lowest order perturbed energy levels are

$$
\begin{align*}
& |0,0\rangle \rightarrow \hbar \omega  \tag{6.319}\\
& |1,0\rangle \rightarrow \hbar \omega\left(2+\frac{1}{2} \delta\right),  \tag{6.320}\\
& |0,1\rangle \rightarrow \hbar \omega\left(2-\frac{1}{2} \delta\right) . \tag{6.321}
\end{align*}
$$

The degeneracy of the $|0,1\rangle,|1,0\rangle$ states has been split, and to first order match the zeroth order perturbation result.

## Exercise 6.12 Perturbation of two state Hamiltonian. ([15] pr. 5.11)

Given a two-state system

$$
\begin{align*}
H & =H_{0}+\lambda V \\
& =\left[\begin{array}{ll}
E_{1} & \lambda \Delta \\
\lambda \Delta & E_{2}
\end{array}\right] . \tag{6.322}
\end{align*}
$$

a. Solve the system exactly.
b. Find the first order perturbed states and second order energy shifts, and compare to the exact solution.
c. Solve the degenerate case for $E_{1}=E_{2}$, and compare to the exact solution.
Answer for Exercise 6.12

Part $a$. The energy eigenvalues $\epsilon$ are given by

$$
\begin{equation*}
0=\left(E_{1}-\epsilon\right)\left(E_{2}-\epsilon\right)-(\lambda \Delta)^{2}, \tag{6.323}
\end{equation*}
$$

or

$$
\begin{equation*}
\epsilon^{2}-\epsilon\left(E_{1}+E_{2}\right)+E_{1} E_{2}=(\lambda \Delta)^{2} \tag{6.324}
\end{equation*}
$$

After rearranging this is

$$
\begin{equation*}
\epsilon=\frac{E_{1}+E_{2}}{2} \pm \sqrt{\left(\frac{E_{1}-E_{2}}{2}\right)^{2}+(\lambda \Delta)^{2}} \tag{6.325}
\end{equation*}
$$

Notice that for $E_{2}=E_{1}$ we have

$$
\begin{equation*}
\epsilon=E_{1} \pm \lambda \Delta \tag{6.326}
\end{equation*}
$$

Since a change of basis can always put the problem in a form so that $E_{1}>$ $E_{2}$, let's assume that to make an approximation of the energy eigenvalues for $|\lambda \Delta| \ll\left(E_{1}-E_{2}\right) / 2$

$$
\begin{align*}
\epsilon & =\frac{E_{1}+E_{2}}{2} \pm \frac{E_{1}-E_{2}}{2} \sqrt{1+\frac{(2 \lambda \Delta)^{2}}{\left(E_{1}-E_{2}\right)^{2}}} \\
& \approx \frac{E_{1}+E_{2}}{2} \pm \frac{E_{1}-E_{2}}{2}\left(1+2 \frac{(\lambda \Delta)^{2}}{\left(E_{1}-E_{2}\right)^{2}}\right)  \tag{6.327}\\
& =\frac{E_{1}+E_{2}}{2} \pm \frac{E_{1}-E_{2}}{2} \pm \frac{(\lambda \Delta)^{2}}{E_{1}-E_{2}} \\
& =E_{1}+\frac{(\lambda \Delta)^{2}}{E_{1}-E_{2}}, E_{2}+\frac{(\lambda \Delta)^{2}}{E_{2}-E_{1}} .
\end{align*}
$$

For the perturbed states, starting with the plus case, if
we must have

$$
\begin{align*}
0 & =\left(E_{1}-\left(E_{1}+\frac{(\lambda \Delta)^{2}}{E_{1}-E_{2}}\right)\right) a+\lambda \Delta b \\
& =\left(-\frac{(\lambda \Delta)^{2}}{E_{1}-E_{2}}\right) a+\lambda \Delta b, \tag{6.329}
\end{align*}
$$

so

Similarly for the minus case we must have

$$
\begin{align*}
0 & =\lambda \Delta a+\left(E_{2}-\left(E_{2}+\frac{(\lambda \Delta)^{2}}{E_{2}-E_{1}}\right)\right) b \\
& =\lambda \Delta b+\left(-\frac{(\lambda \Delta)^{2}}{E_{2}-E_{1}}\right) b, \tag{6.331}
\end{align*}
$$

for

Part b. For the perturbation the first energy shift for perturbation of the $1+\rangle$ state is

The first order energy shift for the perturbation of the $|-\rangle$ state is also zero. The perturbed $|+\rangle$ state is

The numerator matrix element is

$$
\begin{align*}
\langle-| V|+\rangle & =\left[\begin{array}{ll}
0 & 1
\end{array}\right]\left[\begin{array}{ll}
0 & \Delta \\
\Delta & 0
\end{array}\right]\left[\begin{array}{l}
1 \\
0
\end{array}\right] \\
& =\left[\begin{array}{ll}
0 & 1
\end{array}\right]\left[\begin{array}{l}
0 \\
\Delta
\end{array}\right]  \tag{6.335}\\
& =\Delta
\end{align*}
$$

so

Observe that this matches the first order series expansion of the exact value above. For the perturbation of $|-\rangle$ we need the matrix element

$$
\begin{align*}
\langle+| V|-\rangle & =\left[\begin{array}{ll}
1 & 0
\end{array}\right]\left[\begin{array}{ll}
0 & \Delta \\
\Delta & 0
\end{array}\right]\left[\begin{array}{l}
0 \\
1
\end{array}\right] \\
& =\left[\begin{array}{ll}
1 & 0
\end{array}\right]\left[\begin{array}{l}
\Delta \\
0
\end{array}\right]  \tag{6.337}\\
& =\Delta,
\end{align*}
$$

so it's clear that the perturbed ket is
also matching the approximation found from the exact computation. The second order energy shifts can now be calculated

$$
\begin{align*}
\langle+| V|+\rangle^{\prime} & =\left[\begin{array}{ll}
1 & 0
\end{array}\right]\left[\begin{array}{ll}
0 & \Delta \\
\Delta & 0
\end{array}\right]\left[\begin{array}{c}
1 \\
\frac{\Delta}{E_{1}-E_{2}}
\end{array}\right] \\
& =\left[\begin{array}{ll}
1 & 0
\end{array}\right]\left[\begin{array}{c}
\frac{\Delta^{2}}{E_{1}-E_{2}} \\
\Delta
\end{array}\right]  \tag{6.339}\\
& =\frac{\Delta^{2}}{E_{1}-E_{2}}
\end{align*}
$$

and

$$
\begin{align*}
\langle-| V|-\rangle^{\prime} & =\left[\begin{array}{ll}
0 & 1
\end{array}\right]\left[\begin{array}{cc}
0 & \Delta \\
\Delta & 0
\end{array}\right]\left[\begin{array}{c}
\frac{\Delta}{E_{2}-E_{1}} \\
1
\end{array}\right] \\
& =\left[\begin{array}{ll}
0 & 1
\end{array}\right]\left[\begin{array}{c}
\Delta \\
\frac{\Delta^{2}}{E_{2}-E_{1}}
\end{array}\right]  \tag{6.340}\\
& =\frac{\Delta^{2}}{E_{2}-E_{1}}
\end{align*}
$$

The energy perturbations are therefore

$$
\begin{align*}
E_{1} & \rightarrow E_{1}+\frac{(\lambda \Delta)^{2}}{E_{1}-E_{2}}  \tag{6.341}\\
E_{2} & \rightarrow E_{2}+\frac{(\lambda \Delta)^{2}}{E_{2}-E_{1}}
\end{align*}
$$

This is what we had by approximating the exact case.

Part c. For the $E_{2}=E_{1}$ case, we'll have to diagonalize the perturbation potential. That is

$$
\begin{align*}
V & =U \bigwedge U^{\dagger} \\
\bigwedge & =\left[\begin{array}{cc}
\Delta & 0 \\
0 & -\Delta
\end{array}\right]  \tag{6.342}\\
U & =U^{\dagger}=\frac{1}{\sqrt{2}}\left[\begin{array}{cc}
1 & 1 \\
1 & -1
\end{array}\right] .
\end{align*}
$$

A change of basis for the Hamiltonian is

$$
\begin{align*}
H^{\prime} & =U^{\dagger} H U \\
& =U^{\dagger} H_{0} U+\lambda U^{\dagger} V U \\
& =E_{1} U^{\dagger}+\lambda U^{\dagger} V U  \tag{6.343}\\
& =H_{0}+\lambda \bigwedge .
\end{align*}
$$

We can now calculate the perturbation energy with respect to the new basis, say $\{|1\rangle,|2\rangle\}$. Those energy shifts are

$$
\begin{align*}
& \Delta^{(1)}=\langle 1| V|1\rangle=\Delta \\
& \Delta^{(2)}=\langle 2| V|2\rangle=-\Delta . \tag{6.344}
\end{align*}
$$

The perturbed energies are therefore

$$
\begin{align*}
& E_{1} \rightarrow E_{1}+\lambda \Delta \\
& E_{2} \rightarrow E_{2}-\lambda \Delta, \tag{6.345}
\end{align*}
$$

which matches eq. (6.326), the exact result.
Exercise 6.13 Spherically symmetric 3D potential. ([15] pr. 5.16)
a. For a particle in a spherically symmetric potential $V(r)$ show that

$$
\begin{equation*}
|\psi(0)|^{2}=\frac{m}{2 \pi \hbar^{2}}\left\langle\frac{d V}{d r}\right\rangle \tag{6.346}
\end{equation*}
$$

for all s-states, ground or excited.
b. Show this is the case for the 3D SHO and hydrogen wave functions.
Answer for Exercise 6.13

Part a. The text works a problem that looks similar to this by considering the commutator of an operator $A$, later set to $A=p_{r}=-i \hbar \partial / \partial r$ the radial momentum operator. First it is noted that

$$
\begin{equation*}
0=\langle n l m|[H, A]|n l m\rangle, \tag{6.347}
\end{equation*}
$$

since $H$ operating to either the right or the left is the energy eigenvalue $E_{n}$. Next it appears the author uses an angular momentum factoring of the squared momentum operator. Looking earlier in the text that factoring is found to be

$$
\begin{equation*}
\frac{\mathbf{p}^{2}}{2 m}=\frac{1}{2 m r^{2}} \mathbf{L}^{2}-\frac{\hbar^{2}}{2 m}\left(\frac{\partial^{2}}{\partial r^{2}}+\frac{2}{r} \frac{\partial}{\partial r}\right) \tag{6.348}
\end{equation*}
$$

With

$$
\begin{equation*}
R=-\frac{\hbar^{2}}{2 m}\left(\frac{\partial^{2}}{\partial r^{2}}+\frac{2}{r} \frac{\partial}{\partial r}\right) \tag{6.349}
\end{equation*}
$$

we have

$$
\begin{align*}
0 & =\langle n l m|\left[H, p_{r}\right]|n l m\rangle \\
& =\langle n l m|\left[\frac{\mathbf{p}^{2}}{2 m}+V(r), p_{r}\right]|n l m\rangle \\
& =\langle n l m|\left[\frac{1}{2 m r^{2}} \mathbf{L}^{2}+R+V(r), p_{r}\right]|n l m\rangle  \tag{6.350}\\
& =\langle n l m|\left[\frac{-\hbar^{2} l(l+1)}{2 m r^{2}}+R+V(r), p_{r}\right]|n l m\rangle .
\end{align*}
$$

Let's consider the commutator of each term separately. First

$$
\begin{align*}
{\left[V, p_{r}\right] \psi } & =V p_{r} \psi-p_{r} V \psi \\
& =V p_{r} \psi-\left(p_{r} V\right) \psi-V p_{r} \psi \\
& =-\left(p_{r} V\right) \psi  \tag{6.351}\\
& =i \hbar \frac{\partial V}{\partial r} \psi
\end{align*}
$$

Setting $V(r)=1 / r^{2}$, we also have

$$
\begin{equation*}
\left[\frac{1}{r^{2}}, p_{r}\right] \psi=-\frac{2 i \hbar}{r^{3}} \psi \tag{6.352}
\end{equation*}
$$

Finally

$$
\begin{align*}
{\left[\frac{\partial^{2}}{\partial r^{2}}+\frac{2}{r} \frac{\partial}{\partial r}, \frac{\partial}{\partial r}\right] } & =\left(\partial_{r r}+\frac{2}{r} \partial_{r}\right) \partial_{r}-\partial_{r}\left(\partial_{r r}+\frac{2}{r} \partial_{r}\right) \\
& =\partial_{r r r}+\frac{2}{r} \partial_{r r}-\left(\partial_{r r r}-\frac{2}{r^{2}} \partial_{r}+\frac{2}{r} \partial_{r r}\right)  \tag{6.353}\\
& =-\frac{2}{r^{2}} \partial_{r},
\end{align*}
$$

so

$$
\begin{align*}
{\left[R, p_{r}\right] } & =-\frac{2}{r^{2}} \frac{-\hbar^{2}}{2 m} p_{r}  \tag{6.354}\\
& =\frac{\hbar^{2}}{m r^{2}} p_{r}
\end{align*}
$$

Putting all the pieces back together, we've got

$$
\begin{align*}
0 & =\langle n l m|\left[\frac{-\hbar^{2} l(l+1)}{2 m r^{2}}+R+V(r), p_{r}\right]|n l m\rangle  \tag{6.355}\\
& =i \hbar\langle n l m|\left(\frac{\hbar^{2} l(l+1)}{m r^{3}}-\frac{i \hbar}{m r^{2}} p_{r}+\frac{\partial V}{\partial r}\right)|n l m\rangle .
\end{align*}
$$

Since s-states are those for which $l=0$, this means

$$
\begin{aligned}
\left\langle\frac{\partial V}{\partial r}\right\rangle & =\frac{i \hbar}{m}\left\langle\frac{1}{r^{2}} p_{r}\right\rangle \\
& =\frac{\hbar^{2}}{m}\left\langle\frac{1}{r^{2}} \frac{\partial}{\partial r}\right\rangle \\
& =\frac{\hbar^{2}}{m} \int_{0}^{\infty} d r \int_{0}^{\pi} d \theta \int_{0}^{2 \pi} d \phi r^{2} \sin \theta \psi^{*}(r, \theta, \phi) \frac{1}{r^{2}} \frac{\partial \psi(r, \theta, \phi)}{\partial r}
\end{aligned}
$$

Since s-states are spherically symmetric, this is

$$
\begin{equation*}
\left\langle\frac{\partial V}{\partial r}\right\rangle=\frac{4 \pi \hbar^{2}}{m} \int_{0}^{\infty} d r \psi^{*} \frac{\partial \psi}{\partial r} \tag{6.357}
\end{equation*}
$$

That integral is

$$
\begin{equation*}
\int_{0}^{\infty} d r \psi^{*} \frac{\partial \psi}{\partial r}=\left.|\psi|^{2}\right|_{0} ^{\infty}-\int_{0}^{\infty} d r \frac{\partial \psi^{*}}{\partial r} \psi \tag{6.358}
\end{equation*}
$$

With the hydrogen atom, our radial wave functions are real valued. It's reasonable to assume that we can do the same for other real-valued spherical potentials. If that is the case, we have

$$
\begin{equation*}
2 \int_{0}^{\infty} d r \psi^{*} \frac{\partial \psi}{\partial r}=|\psi(0)|^{2} \tag{6.359}
\end{equation*}
$$

and

$$
\begin{equation*}
\left\langle\frac{\partial V}{\partial r}\right\rangle=\frac{2 \pi \hbar^{2}}{m}|\psi(0)|^{2} \tag{6.360}
\end{equation*}
$$

which completes this part of the problem.

Part b. For a hydrogen like atom, in atomic units, we have

$$
\begin{align*}
\left\langle\frac{\partial V}{\partial r}\right\rangle & =\left\langle\frac{\partial}{\partial r}\left(-\frac{Z e^{2}}{r}\right)\right\rangle \\
& =Z e^{2}\left\langle\frac{1}{r^{2}}\right\rangle \\
& =Z e^{2} \frac{Z^{2}}{n^{3} a_{0}^{2}(l+1 / 2)}  \tag{6.361}\\
& =\frac{\hbar^{2}}{m a_{0}} \frac{2 Z^{3}}{n^{3} a_{0}^{2}} \\
& =\frac{2 \hbar^{2} Z^{3}}{m n^{3} a_{0}^{3}}
\end{align*}
$$

On the other hand for $n=1$, we have

$$
\begin{align*}
\frac{2 \pi \hbar^{2}}{m}\left|R_{10}(0)\right|^{2}\left|Y_{00}\right|^{2} & =\frac{2 \pi \hbar^{2}}{m} \frac{Z^{3}}{a_{0}^{3}} 4 \frac{1}{4 \pi} \\
& =\frac{2 \hbar^{2} Z^{3}}{m a_{0}^{3}} \tag{6.362}
\end{align*}
$$

and for $n=2$, we have

$$
\begin{align*}
\frac{2 \pi \hbar^{2}}{m}\left|R_{20}(0)\right|^{2}\left|Y_{00}\right|^{2} & =\frac{2 \pi \hbar^{2}}{m} \frac{Z^{3}}{8 a_{0}^{3}} 4 \frac{1}{4 \pi}  \tag{6.363}\\
& =\frac{\hbar^{2} Z^{3}}{4 m a_{0}^{3}}
\end{align*}
$$

These both match the potential derivative expectation when evaluated for the s-orbital $(l=0)$.

In sakuraiProblem5.16bSHO.nb is a verification for the 3D SHO ground state. There it was found that

$$
\begin{align*}
\left\langle\frac{\partial V}{\partial r}\right\rangle & =\frac{2 \pi \hbar^{2}}{m}|\psi(0)|^{2}  \tag{6.364}\\
& =2 \sqrt{\frac{m \omega^{3} \hbar}{\pi}}
\end{align*}
$$

Exercise $6.14 \quad L_{y}$ perturbation. ([15] pr. 5.17(a))
Find the first non-zero energy shift for the perturbed Hamiltonian

$$
\begin{equation*}
H=A \mathbf{L}^{2}+B L_{z}+C L_{y}=H_{0}+V \tag{6.365}
\end{equation*}
$$

Answer for Exercise 6.14
The energy eigenvalues for state $|l, m\rangle$ prior to perturbation are

$$
\begin{equation*}
A \hbar^{2} l(l+1)+B \hbar m \tag{6.366}
\end{equation*}
$$

The first order energy shift is zero

$$
\begin{align*}
\Delta^{1} & =\langle l, m| C L_{y}|l, m\rangle \\
& =\frac{C}{2 i}\langle l, m|\left(L_{+}-L_{-}\right)|l, m\rangle  \tag{6.367}\\
& =0
\end{align*}
$$

so we need the second order shift. Assuming no degeneracy to start, the perturbed state is

$$
\begin{equation*}
|l, m\rangle^{\prime}=\sum^{\prime} \frac{\left|l^{\prime}, m^{\prime}\right\rangle\left\langle l^{\prime}, m^{\prime}\right|}{E_{l, m}-E_{l^{\prime}, m^{\prime}}} V|l, m\rangle \tag{6.368}
\end{equation*}
$$

and the next order energy shift is

$$
\begin{align*}
\Delta^{2} & =\langle l m| V \sum^{\prime} \frac{\left|l^{\prime}, m^{\prime}\right\rangle\left\langle l^{\prime}, m^{\prime}\right|}{E_{l, m}-E_{l^{\prime}, m^{\prime}}} V|l, m\rangle \\
& =\sum^{\prime} \frac{\langle l, m| V\left|l^{\prime}, m^{\prime}\right\rangle\left\langle l^{\prime}, m^{\prime}\right|}{E_{l, m}-E_{l^{\prime}, m^{\prime}}} V|l, m\rangle \\
& =\sum \frac{\left.\left|\left\langle l^{\prime}, m^{\prime}\right| V\right| l, m\right\rangle\left.\right|^{2}}{E_{l, m}-E_{l^{\prime}, m^{\prime}}} \\
& =\sum_{m^{\prime} \neq m} \frac{\left.\left|\left\langle l, m^{\prime}\right| V\right| l, m\right\rangle\left.\right|^{2}}{E_{l, m}-E_{l, m^{\prime}}}  \tag{6.369}\\
& =\sum_{m^{\prime} \neq m} \frac{\left.\left|\left\langle l, m^{\prime}\right| V\right| l, m\right\rangle\left.\right|^{2}}{\left(A \hbar^{2} l(l+1)+B \hbar m\right)-\left(A \hbar^{2} l(l+1)+B \hbar m^{\prime}\right)} \\
& =\frac{1}{B \hbar} \sum_{m^{\prime} \neq m} \frac{\left.\left|\left\langle l, m^{\prime}\right| V\right| l, m\right\rangle\left.\right|^{2}}{m-m^{\prime}} .
\end{align*}
$$

The sum over $l^{\prime}$ was eliminated because $V$ only changes the $m$ of any state $|l, m\rangle$, so the matrix element $\left\langle l^{\prime}, m^{\prime}\right| V|l, m\rangle$ must includes a $\delta_{l^{\prime}, l}$ factor. Since we are now summing over $m^{\prime} \neq m$, some of the matrix elements in the numerator should now be non-zero, unlike the case when the zero first order energy shift was calculated in eq. (6.367).

$$
\begin{align*}
\left\langle l, m^{\prime}\right| & C L_{y}|l, m\rangle \\
= & \frac{C}{2 i}\left\langle l, m^{\prime}\right|\left(L_{+}-L_{-}\right)|l, m\rangle \\
= & \frac{C}{2 i}\left\langle l, m^{\prime}\right|\left(L_{+}|l, m\rangle-L_{-}|l, m\rangle\right) \\
= & \frac{C \hbar}{2 i}\left\langle l, m^{\prime}\right|(\sqrt{(l-m)(l+m+1)}|l, m+1\rangle \\
& \quad-\sqrt{(l+m)(l-m+1)}|l, m-1\rangle) \\
& =\frac{C \hbar}{2 i}\left(\sqrt{(l-m)(l+m+1)} \delta_{m^{\prime}, m+1}-\sqrt{(l+m)(l-m+1)} \delta_{m^{\prime}, m-1}\right) \tag{6.370}
\end{align*}
$$

After squaring and summing, the cross terms will be zero since they involve products of delta functions with different indices. That leaves

$$
\begin{align*}
\Delta^{2} & =\frac{C^{2} \hbar}{4 B} \sum_{m^{\prime} \neq m} \frac{(l-m)(l+m+1) \delta_{m^{\prime}, m+1}-(l+m)(l-m+1) \delta_{m^{\prime}, m-1}}{m-m^{\prime}} \\
& =\frac{C^{2} \hbar}{4 B}\left(\frac{(l-m)(l+m+1)}{m-(m+1)}-\frac{(l+m)(l-m+1)}{m-(m-1)}\right) \\
& =\frac{C^{2} \hbar}{4 B}\left(-\left(l^{2}-m^{2}+l-m\right)-\left(l^{2}-m^{2}+l+m\right)\right) \\
& =-\frac{C^{2} \hbar}{2 B}\left(l^{2}-m^{2}+l\right) \tag{6.371}
\end{align*}
$$

so to first order the energy shift is

$$
A \hbar^{2} l(l+1)+B \hbar m \rightarrow \hbar l(l+1)\left(A \hbar-\frac{C^{2}}{2 B}\right)+B \hbar m+\frac{C^{2} m^{2} \hbar}{2 B}
$$

Exact perturbation equation If we wanted to solve the Hamiltonian exactly, we've have to diagonalize the $2 m+1$ dimensional Hamiltonian

$$
\begin{align*}
& \left\langle l, m^{\prime}\right| H|l, m\rangle \\
& =\left(A \hbar^{2} l(l+1)+B \hbar m\right) \delta_{m^{\prime}, m} \\
& \quad+\frac{C \hbar}{2 i}\left(\sqrt{(l-m)(l+m+1)} \delta_{m^{\prime}, m+1}-\sqrt{(l+m)(l-m+1)} \delta_{m^{\prime}, m-1}\right) \tag{6.373}
\end{align*}
$$

This Hamiltonian matrix has a very regular structure

$$
\begin{align*}
H & =\left(A l(l+1) \hbar^{2}-B \hbar(l+1)\right) I \\
& +B \hbar\left[\begin{array}{cccc}
1 & & & \\
& 2 & & \\
& & 3 & \\
& & \ddots & \\
& & & 2 l+1
\end{array}\right] \\
& +\frac{C \hbar}{i}\left[\begin{array}{ccccc}
0 & -c_{2 l-1,1} & \\
c_{2 l-1,1} & 0 & -c_{2 l-2,2} \\
& c_{2 l-2,2} & & \\
& & & \ddots & \\
& & & & c_{1,2 l-1}
\end{array}\right] \tag{6.374}
\end{align*}
$$

where $c_{a, b}=\sqrt{a b}$.
Solving for the eigenvalues of this Hamiltonian for increasing $l$ in Mathematica (sakuraiProblem5.17a.nb), it appears that the eigenvalues are

$$
\begin{equation*}
\lambda_{m}=A \hbar^{2}(l)(l+1)+\hbar m B \sqrt{1+\frac{4 C^{2}}{B^{2}}} \tag{6.375}
\end{equation*}
$$

so to first order in $C^{2}$, these are

$$
\begin{equation*}
\lambda_{m}=A \hbar^{2}(l)(l+1)+\hbar m B\left(1+\frac{2 C^{2}}{B^{2}}\right) \tag{6.376}
\end{equation*}
$$

We have a $C^{2} \hbar / B$ term in both the perturbative energy shift eq. (6.371), and the first order expansion of the exact solution eq. (6.375). Comparing this for the $l=5$ case, the coefficients of $C^{2} \hbar / B$ in eq. (6.371) are all negative

$$
\begin{equation*}
-17.5,-17 .,-16.5,-16 .,-15.5,-15 .,-14.5,-14 .,-13.5,-13 .,-12.5 \tag{6.377}
\end{equation*}
$$

whereas the coefficient of $C^{2} \hbar / B$ in the first order expansion of the exact solution eq. (6.375) are $2 m$, ranging from $[-10,10]$.

## Exercise 6.15 Quadratic Zeeman effect. ([15] pr. 5.18)

Work out the quadratic Zeeman effect for the ground state hydrogen atom due to the usually neglected $e^{2} \mathbf{A}^{2} / 2 m_{e} c^{2}$ term in the Hamiltonian. Answer for Exercise 6.15

For a z-oriented magnetic field we can use

$$
\begin{equation*}
\mathbf{A}=\frac{B}{2}(-y, x, 0), \tag{6.378}
\end{equation*}
$$

so the perturbation potential is

$$
\begin{align*}
V & =\frac{e^{2} \mathbf{A}^{2}}{2 m_{e} c^{2}} \\
& =\frac{e^{2} \mathbf{B}^{2}\left(x^{2}+y^{2}\right)}{8 m_{e} c^{2}}  \tag{6.379}\\
& =\frac{e^{2} \mathbf{B}^{2} r^{2} \sin ^{2} \theta}{8 m_{e} c^{2}}
\end{align*}
$$

The ground state wave function is

$$
\begin{align*}
\psi_{0} & =\langle\mathbf{x} \mid 0\rangle \\
& =\frac{1}{\sqrt{\pi a_{0}^{3}}} e^{-r / a_{0}}, \tag{6.380}
\end{align*}
$$

so the energy shift is

$$
\begin{align*}
\Delta & =\langle 0| V|0\rangle \\
& =\frac{1}{\pi a_{0}^{3}} 2 \pi \frac{e^{2} \mathbf{B}^{2}}{8 m_{e} c^{2}} \int_{0}^{\infty} r^{2} \sin \theta e^{-2 r / a_{0}} r^{2} \sin ^{2} \theta d r d \theta \\
& =\frac{e^{2} \mathbf{B}^{2}}{4 a_{0}^{3} m_{e} c^{2}} \int_{0}^{\infty} r^{4} e^{-2 r / a_{0}} d r \int_{0}^{\pi} \sin ^{3} \theta d \theta  \tag{6.381}\\
& =-\left.\frac{e^{2} \mathbf{B}^{2}}{4 a_{0}^{3} m_{e} c^{2}} \frac{4!}{\left(2 / a_{0}\right)^{4+1}}\left(u-\frac{u^{3}}{3}\right)\right|_{1} ^{-1} \\
& =\frac{e^{2} a_{0}^{2} \mathbf{B}^{2}}{4 m_{e} c^{2}}
\end{align*}
$$

If this energy shift is written in terms of a diamagnetic susceptibility $\chi$ defined by

$$
\begin{equation*}
\Delta=-\frac{1}{2} \chi \mathbf{B}^{2} \tag{6.382}
\end{equation*}
$$

the diamagnetic susceptibility is

$$
\begin{equation*}
\chi=-\frac{e^{2} a_{0}^{2}}{2 m_{e} c^{2}} \tag{6.383}
\end{equation*}
$$

## Exercise $6.16 \quad$ Electron and Boson hyperfine spin interaction. (2015 final)

This problem is a variation of problem set 8 problem 4 , but instead of the spin interaction of a Fermionic nucleus with an electron, this problem was to look at the spin interaction of an electron with a Bosonic nucleus.

The interaction Hamiltonian in this problem includes a Zeeman field

$$
\begin{equation*}
H=J \mathbf{S}_{e} \cdot \mathbf{S}_{n}+B S_{e}^{z} \tag{6.384}
\end{equation*}
$$

a. Solve the problem exactly.
b. Do a first order perturbative solution.

Answer for Exercise 6.16

Part $a$. With two states for the electron spin and three for the nuclear spin, the state space for the spin system is six dimensional. Without the Zeeman contribution the action of Hamiltonian is

$$
\begin{align*}
H|\hat{\mathbf{n}} ;+\rangle|1\rangle & =\frac{J \hbar^{2}}{2}|\hat{\mathbf{n}} ;+\rangle|1\rangle, \\
H|\hat{\mathbf{n}} ;-\rangle|1\rangle & =-\frac{J \hbar^{2}}{2}|\hat{\mathbf{n}} ;-\rangle|1\rangle, \\
H|\hat{\mathbf{n}} ;+\rangle|0\rangle & =0|\hat{\mathbf{n}} ;+\rangle|0\rangle  \tag{6.385}\\
H|\hat{\mathbf{n}} ;-\rangle|0\rangle & =0|\hat{\mathbf{n}} ;-\rangle|0\rangle \\
H|\hat{\mathbf{n}} ;+\rangle|-1\rangle & =-\frac{J \hbar^{2}}{2}|\hat{\mathbf{n}} ;+\rangle|-1\rangle, \\
H|\hat{\mathbf{n}} ;-\rangle|-1\rangle & =\frac{J \hbar^{2}}{2}|\hat{\mathbf{n}} ;-\rangle|-1\rangle
\end{align*}
$$

This can be put into block matrix form

$$
J \mathbf{S}_{e} \cdot \mathbf{S}_{n}=\frac{J \hbar^{2}}{2}\left[\begin{array}{ccc}
\sigma_{z} & 0 & 0  \tag{6.386}\\
0 & 0 & 0 \\
0 & 0 & -\sigma_{z}
\end{array}\right]
$$

The Zeeman field in the basis above is

$$
B S_{e}^{z}=\frac{B \hbar}{2}\left[\begin{array}{lll}
R & 0 & 0  \tag{6.387}\\
0 & R & 0 \\
0 & 0 & R
\end{array}\right]
$$

where $R$ is the representation of $\sigma_{z}$ in a $|\hat{\mathbf{n}} ;+\rangle,|\hat{\mathbf{n}} ;-\rangle$ basis. Representing the electron spin states as

$$
\begin{align*}
& |\hat{\mathbf{n}} ;+\rangle=\left[\begin{array}{c}
\cos (\theta / 2) e^{-i \phi} \\
\sin (\theta / 2)
\end{array}\right]=\cos (\theta / 2) e^{-i \phi}|\hat{\mathbf{z}} ;+\rangle+\sin (\theta / 2)|\hat{\mathbf{z}} ;-\rangle  \tag{6.388}\\
& |\hat{\mathbf{n}} ;-\rangle=\left[\begin{array}{c}
-\sin (\theta / 2) e^{-i \phi} \\
\cos (\theta / 2)
\end{array}\right]=-\sin (\theta / 2) e^{-i \phi}|\hat{\mathbf{z}} ;+\rangle+\cos (\theta / 2)|\hat{\mathbf{z}} ;-\rangle
\end{align*}
$$

so the $S_{e}^{z}$ action on the spin states is

$$
\begin{align*}
S_{e}^{z}|\hat{\mathbf{n}} ;+\rangle & =\frac{\hbar}{2}\left(\cos (\theta / 2) e^{-i \phi}|\hat{\mathbf{z}} ;+\rangle-\sin (\theta / 2)|\hat{\mathbf{z}} ;-\rangle\right) \\
& =\frac{\hbar}{2}\left[\begin{array}{c}
\cos (\theta / 2) e^{-i \phi} \\
-\sin (\theta / 2)
\end{array}\right] \tag{6.389}
\end{align*}
$$

and

$$
\begin{align*}
S_{e}^{z}|\hat{\mathbf{n}} ;-\rangle & =\frac{\hbar}{2}\left(-\sin (\theta / 2) e^{-i \phi}|\hat{\mathbf{z}} ;+\rangle-\cos (\theta / 2)|\hat{\mathbf{z}} ;-\rangle\right) \\
& =\frac{\hbar}{2}\left[\begin{array}{c}
-\sin (\theta / 2) e^{-i \phi} \\
-\cos (\theta / 2)
\end{array}\right] \tag{6.390}
\end{align*}
$$

The matrix elements for $S_{e}^{z}$ are

$$
\begin{align*}
&\langle\hat{\mathbf{n}} ;+| S_{e}^{z}|\hat{\mathbf{n}} ;+\rangle=\frac{\hbar}{2}\left[\begin{array}{ll}
\cos (\theta / 2) e^{i \phi} & \sin (\theta / 2)
\end{array}\right]\left[\begin{array}{c}
\cos (\theta / 2) e^{-i \phi} \\
-\sin (\theta / 2)
\end{array}\right]  \tag{6.391}\\
&=\frac{\hbar}{2} \cos \theta, \\
&\langle\hat{\mathbf{n}} ;+| S_{e}^{z}|\hat{\mathbf{n}} ;-\rangle=\frac{\hbar}{2}\left[\cos (\theta / 2) e^{i \phi}\right.  \tag{6.392}\\
&\sin (\theta / 2)]\left[\begin{array}{c}
-\sin (\theta / 2) e^{-i \phi} \\
-\cos (\theta / 2)
\end{array}\right] \\
&=-\frac{\hbar}{2} \sin \theta,
\end{align*}
$$

$$
\left.\begin{array}{rl}
\langle\hat{\mathbf{n}} ;-| S_{e}^{z}|\hat{\mathbf{n}} ;+\rangle & =\frac{\hbar}{2}\left[-\sin (\theta / 2) e^{i \phi}\right.  \tag{6.393}\\
\cos (\theta / 2)
\end{array}\right]\left[\begin{array}{c}
\cos (\theta / 2) e^{-i \phi} \\
-\sin (\theta / 2)
\end{array}\right]
$$

$$
\left.\begin{array}{rl}
\langle\hat{\mathbf{n}} ;-| S_{e}^{z}|\hat{\mathbf{n}} ;-\rangle & =\frac{\hbar}{2}\left[-\sin (\theta / 2) e^{i \phi}\right.  \tag{6.394}\\
\cos (\theta / 2)
\end{array}\right]\left[\begin{array}{c}
-\sin (\theta / 2) e^{-i \phi} \\
-\cos (\theta / 2)
\end{array}\right]
$$

or

$$
R=\left[\begin{array}{cc}
\cos \theta & -\sin \theta  \tag{6.395}\\
-\sin \theta & -\cos \theta
\end{array}\right]
$$

Part b. In finalExamProblem6HyperfineInteraction.nb the exact solution to the system is found to be

$$
\begin{equation*}
E \in\left\{ \pm \frac{B \hbar}{2}, \pm \frac{\hbar}{2} \sqrt{B^{2} \pm 2 B J \hbar \cos \theta+J^{2} \hbar^{2}}\right\} \tag{6.396}
\end{equation*}
$$

or with $m \in\{-1,0,1\}$

$$
\begin{equation*}
E \in \pm \frac{\hbar}{2} \sqrt{B^{2}+2 B J m \hbar \cos \theta+m^{2} J^{2} \hbar^{2}} \tag{6.397}
\end{equation*}
$$

In the strong field case where $B \gg \hbar J$, we have

$$
\begin{equation*}
E \approx \pm \frac{\hbar B}{2}\left(1+\frac{J}{B} m \hbar \cos \theta+\frac{1}{2} m^{2} \frac{J^{2}}{B^{2}} \hbar^{2}\right) \tag{6.398}
\end{equation*}
$$

whereas in the weak field case where $B \ll \hbar J$, we have

$$
E \approx \begin{cases} \pm \frac{\hbar B}{2} & \text { if } m=0  \tag{6.399}\\ \pm \frac{\hbar^{2} J}{2}\left(1+\frac{B}{\hbar J} m \cos \theta+\frac{1}{2} \frac{B^{2}}{J^{2} \hbar^{2}}\right) & \text { if } m \neq 0\end{cases}
$$

Note that the result above is exact for the $m=0$ case.

Perturbative solutions The degenerate perturbation requires diagonalizing $B S_{e}^{z}$. Since $R$ is a representation of $S_{e}^{z}$, the eigenvalues of $\frac{B \hbar R}{2}$ are $\pm B \hbar / 2$. This means that the first order shifts to the energy eigenvalues are

$$
\begin{align*}
\frac{J \hbar^{2}}{2} & \rightarrow \frac{J \hbar^{2}}{2}+\frac{B \hbar}{2}, \\
-\frac{J \hbar^{2}}{2} & \rightarrow-\frac{J \hbar^{2}}{2}-\frac{B \hbar}{2}, \\
0 & \rightarrow+\frac{B \hbar}{2},  \tag{6.400}\\
0 & \rightarrow-\frac{B \hbar}{2}, \\
-\frac{J \hbar^{2}}{2} & \rightarrow-\frac{J \hbar^{2}}{2}+\frac{B \hbar}{2}, \\
\frac{J \hbar^{2}}{2} & \rightarrow \frac{J \hbar^{2}}{2}-\frac{B \hbar}{2} .
\end{align*}
$$

Trig

$$
\begin{align*}
& 1+\cos x=2 \cos ^{2} \frac{x}{2} \\
& 1-\cos x=2 \sin ^{2} \frac{x}{2}  \tag{A.2}\\
& \sin x=2 \sin \frac{x}{2} \cos \frac{x}{2} \tag{A.3}
\end{align*}
$$

$2 \cos a \cos b=\cos (a+b)+\cos (a-b)$
$2 \sin a \sin b=\cos (a-b)-\cos (a-b)$
$2 \cos a \sin b=\sin (a+b)-\sin (a-b)$
$2 \sin a \cos b=\sin (a-b)+\sin (a+b)$
$\cos (a \pm b)=\cos a \cos b \mp \sin a \sin b$
$\sin (a \pm b)=\sin a \cos b \pm \cos a \sin b$

Basics
$\langle x \mid p\rangle \propto e^{i p x / \hbar}$

$$
\begin{equation*}
|\Psi(t)\rangle=U|\Psi(0)\rangle \tag{A.11}
\end{equation*}
$$

$U=e^{-i H t / \hbar}, \quad$ for time independent $H$

$$
\begin{equation*}
p=-i \hbar \frac{\partial}{\partial x} \tag{A.13}
\end{equation*}
$$

$$
\begin{equation*}
x=i \hbar \frac{\partial}{\partial p} \tag{A.14}
\end{equation*}
$$

$$
\begin{equation*}
H=i \hbar \frac{\partial}{\partial t} \tag{A.15}
\end{equation*}
$$

$$
\begin{equation*}
A \sim\left(\left\langle a_{\mathrm{row}}\right| A\left|a_{\mathrm{column}}\right\rangle\right) \tag{A.16}
\end{equation*}
$$

$$
\begin{align*}
& \rho=\psi^{*} \psi \\
& \mathbf{j}=\frac{\hbar}{m} \operatorname{Im}\left(\psi^{*} \boldsymbol{\nabla} \psi\right)=-\frac{i \hbar}{2 m}\left(\psi^{*} \boldsymbol{\nabla} \psi-\psi \boldsymbol{\nabla} \psi^{*}\right)  \tag{A.17}\\
& \frac{\partial \rho}{\partial t}+\boldsymbol{\nabla} \cdot \mathbf{j}=0 \tag{A.18}
\end{align*}
$$

Commutators

$$
\begin{align*}
& {[x, p]=i \hbar}  \tag{A.19}\\
& {\left[x_{j}, F(\mathbf{p})\right]=i \hbar \frac{\partial F}{\partial p_{j}}} \\
& {\left[p_{j}, G(\mathbf{x})\right]=-i \hbar \frac{\partial G}{\partial x_{j}}} \tag{A.20}
\end{align*}
$$

$$
\begin{equation*}
e^{A \mu} B e^{-A \mu}=B+\mu[A, B]+\frac{\mu^{2}}{2!}[A,[A, B]]+\cdots \tag{A.21}
\end{equation*}
$$

$$
\begin{equation*}
[A, B]_{\text {classical }}=\frac{\partial A}{\partial q} \frac{\partial B}{\partial p}-\frac{\partial A}{\partial p} \frac{\partial B}{\partial q} \tag{A.22}
\end{equation*}
$$

Heisenberg picture

$$
\begin{align*}
& A_{\mathrm{H}}=U^{\dagger} A U  \tag{A.23}\\
& \frac{d A_{\mathrm{H}}}{d t}=\frac{1}{i \hbar}\left[A_{\mathrm{H}}, H\right]  \tag{A.24}\\
& \frac{d}{d t}\langle\mathbf{x} \cdot \mathbf{p}\rangle=\left\langle\frac{\mathbf{p}^{2}}{m}\right\rangle-\langle\mathbf{x} \cdot \nabla V\rangle \tag{A.25}
\end{align*}
$$

Density operator

$$
\begin{align*}
& \rho=\sum_{i} w_{i}\left|\alpha^{i}\right\rangle\left\langle\alpha^{i}\right|, \quad \sum_{i} w_{i}=1  \tag{A.26}\\
& {[A]=\sum_{i} w_{i}\left\langle\alpha^{i}\right| A\left|\alpha^{i}\right\rangle=\operatorname{tr}(\rho A)} \tag{A.27}
\end{align*}
$$

$S=-\operatorname{tr}(\rho \ln \rho)=-\operatorname{tr}\left(\rho_{k k} \ln \rho_{k k}\right)$

$$
\begin{align*}
\left|\psi_{m n}\right\rangle & =\left|a_{m}\right\rangle_{1} \otimes\left|a_{n}\right\rangle \\
|\psi\rangle & =\sum_{m n} c_{m n}\left|\psi_{m n}\right\rangle \\
\rho & =|\psi\rangle\langle\psi|  \tag{A.29}\\
\operatorname{tr}_{2}(\rho) & =\rho_{2}=\sum_{a}{ }_{2}\langle a| \rho|a\rangle_{2}
\end{align*}
$$

$$
\sum_{a}{ }_{1}\langle a| \operatorname{tr}_{2}(\rho)|a\rangle_{1}=\sum_{n}\left|c_{m n}\right|^{2}
$$

## Pauli matrices

$$
\begin{align*}
& \sigma_{x}=\left[\begin{array}{ll}
0 & 1 \\
1 & 0
\end{array}\right] \\
& \sigma_{y}=\left[\begin{array}{cc}
0 & -i \\
i & 0
\end{array}\right]  \tag{A.30}\\
& \sigma_{z}=\left[\begin{array}{cc}
1 & 0 \\
0 & -1
\end{array}\right] \\
& {\left[\sigma_{a}, \sigma_{b}\right]=2 i \epsilon_{a b c} \sigma_{c}}  \tag{A.31}\\
& \left|S_{x} ; \pm\right\rangle=\frac{1}{\sqrt{2}}\left[\begin{array}{c}
1 \\
\pm 1
\end{array}\right]  \tag{A.32a}\\
& \left|S_{y} ; \pm\right\rangle=\frac{1}{\sqrt{2}}\left[\begin{array}{c}
1 \\
\pm i
\end{array}\right]  \tag{A.32b}\\
& \left|S_{z} ; \pm\right\rangle=\left[\begin{array}{l}
1 \\
0
\end{array}\right],\left[\begin{array}{l}
0 \\
1
\end{array}\right] \tag{A.32c}
\end{align*}
$$

For $\hat{\mathbf{n}}=(\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta)$, the eigenkets of $\mathbf{S} \cdot \hat{\mathbf{n}}$ are

$$
\begin{align*}
& |\hat{\mathbf{n}} ;+\rangle=\left[\begin{array}{c}
\cos (\theta / 2) e^{-i \phi / 2} \\
\sin (\theta / 2) e^{i \phi / 2}
\end{array}\right]=e^{-i \sigma_{z} \phi / 2} e^{-i \sigma_{y} \theta / 2}|\hat{\mathbf{z}} ;+\rangle  \tag{A.33}\\
& |\hat{\mathbf{n}} ;-\rangle=\left[\begin{array}{c}
-\sin (\theta / 2) e^{-i \phi / 2} \\
\cos (\theta / 2) e^{i \phi / 2}
\end{array}\right]=e^{-i \sigma_{z} \phi / 2} e^{-i \sigma_{y}(\theta+\pi) / 2}|\hat{\mathbf{z}} ;+\rangle  \tag{A.34}\\
& H=-\frac{e}{m c} \mathbf{S} \cdot \mathbf{B}=-\frac{e B}{m c} S_{z} \tag{A.35}
\end{align*}
$$

Harmonic oscillator

$$
\begin{align*}
& x_{0}^{2}=\frac{\hbar}{m \omega} \\
& p_{0}^{2}=m \omega \hbar  \tag{A.37}\\
& x(t)=\frac{x_{0}}{\sqrt{2}}\left(a e^{-i \omega t}+a^{\dagger} e^{i \omega t}\right)  \tag{A.38}\\
& p(t)=\frac{i \hbar}{\sqrt{2} x_{0}}\left(a^{\dagger} e^{i \omega t}-a e^{-i \omega t}\right)  \tag{A.39}\\
& a, a^{\dagger}=\frac{1}{\sqrt{2} x_{0}}\left(x \mp x_{0}^{2} \frac{d}{d x}\right) \tag{A.40}
\end{align*}
$$

$$
\begin{equation*}
a|n\rangle=\sqrt{n}|n-1\rangle \tag{A.41}
\end{equation*}
$$

$$
\begin{equation*}
a^{\dagger}|n\rangle=\sqrt{n+1}|n+1\rangle \tag{A.42}
\end{equation*}
$$

$x(t)=x(0) \cos \omega t+\frac{p(0)}{m \omega} \sin \omega t$
$p(t)=p(0) \cos \omega t-m \omega x(0) \sin \omega t$
$x(t)^{2}=\frac{\hbar \omega}{2}\left(a e^{-i \omega t}+a^{\dagger} e^{i \omega t}\right)^{2}$
$p(t)^{2}=\frac{\hbar \omega}{2}\left(a e^{-i \omega t}-a^{\dagger} e^{i \omega t}\right)\left(a^{\dagger} e^{i \omega t}-a e^{-i \omega t}\right)$

$$
\begin{equation*}
N=a^{\dagger} a \tag{A.47}
\end{equation*}
$$

$$
\begin{equation*}
N|n\rangle=n|n\rangle \tag{A.48}
\end{equation*}
$$

$$
\begin{equation*}
[N, a]=-a \tag{A.49}
\end{equation*}
$$

$$
\begin{equation*}
\left[N, a^{\dagger}\right]=a^{\dagger} \tag{A.50}
\end{equation*}
$$

$$
\begin{equation*}
H=\hbar \omega\left(N+\frac{1}{2}\right) \tag{A.51}
\end{equation*}
$$

$$
\begin{align*}
& {\left[a, a^{\dagger}\right]=1}  \tag{A.52}\\
& a=\frac{1}{x_{0} \sqrt{2}}\left(x+\frac{i p}{m \omega}\right) \tag{A.53}
\end{align*}
$$

$$
\begin{equation*}
\langle x \mid n\rangle=\frac{1}{\pi^{1 / 4} \sqrt{2^{n} n!}} x_{0}^{-(n+1 / 2)}\left(x-x_{0}^{2} \frac{d}{d x}\right)^{n} e^{-\left(x / x_{0}\right)^{2} / 2} \tag{A.54}
\end{equation*}
$$

$$
\begin{equation*}
|n\rangle=\frac{\left(a^{\dagger}\right)^{n}}{\sqrt{n!}}|0\rangle \tag{A.55}
\end{equation*}
$$

Coherent states

$$
\begin{align*}
& a|z\rangle=z|z\rangle  \tag{A.56}\\
& |z\rangle=e^{-|z|^{2} / 2+z a^{\dagger}}|0\rangle \tag{A.57}
\end{align*}
$$

$$
\begin{align*}
& \langle z| a a^{\dagger}|z\rangle=\langle z| 1+a^{\dagger} a|z\rangle  \tag{A.58}\\
& \langle x\rangle(0)=x_{0} \equiv \sqrt{\frac{2 \hbar}{m \omega}} \operatorname{Re} z=\sqrt{\frac{\hbar}{2 m \omega}}\left(z+z^{*}\right)  \tag{A.59}\\
& \langle p\rangle(0)=p_{0} \equiv \sqrt{2 m \hbar \omega} \operatorname{Im} z=-i \sqrt{\frac{m \hbar \omega}{2}}\left(z-z^{*}\right)
\end{align*}
$$

Electromagnetism

$$
\begin{align*}
& \left(\mathbf{A}^{\prime}, \phi^{\prime}\right)=\left(\mathbf{A}+\boldsymbol{\nabla} \chi, \phi-\partial_{t} \chi\right)  \tag{A.60}\\
& \mathbf{E}=-\frac{\partial \mathbf{A}}{\partial t}-\boldsymbol{\nabla} \phi  \tag{A.61}\\
& \mathbf{B}=\boldsymbol{\nabla} \times \mathbf{A}
\end{align*}
$$

$$
\mathbf{A}=\left\{\begin{array}{cl}
\frac{B \rho_{a}^{2}}{2 \rho} \hat{\boldsymbol{\phi}} & \text { if } \rho \geq \rho_{a} \Longrightarrow \mathbf{B}  \tag{A.62}\\
& =0 \\
\frac{B \rho}{2} \hat{\boldsymbol{\phi}} & \text { if } \rho<\rho_{a} \Longrightarrow \mathbf{B} \\
& =B \hat{\mathbf{z}}
\end{array}\right.
$$

$$
\begin{equation*}
\Phi \equiv \oint \mathbf{A} \cdot d \mathbf{l} \tag{A.63}
\end{equation*}
$$

Aharonov-Bohm and Magnetic fields

$$
\begin{align*}
& H=\frac{1}{2 m}\left(\mathbf{p}-\frac{q}{c} \mathbf{A}\right)+q \phi  \tag{A.64}\\
& \Pi=\mathbf{p}-\frac{e}{c} \mathbf{A} \\
& {\left[\Pi_{x}, \Pi_{y}\right] }=\frac{i e \hbar}{c} B_{z} \\
& H=\frac{1}{2 m} \Pi^{2}=\hbar \omega\left(b^{\dagger} b+\frac{1}{2}\right)  \tag{A.65}\\
& \omega=\frac{e B_{0}}{m c} \\
& b=\frac{1}{\sqrt{2 m \omega \hbar}}\left(\Pi_{x}+i \Pi_{y}\right)
\end{align*}
$$

$$
\begin{equation*}
\mathbf{A}=\frac{B_{0}}{2}(-y, x, 0), \mathbf{A}=B_{0}(0, x, 0) \Longrightarrow \mathbf{B}=B_{0} \hat{\mathbf{z}} \tag{A.66}
\end{equation*}
$$

Dirac equation

Generators

$$
\begin{equation*}
T_{\mathbf{a}}=e^{-i \mathbf{p} \cdot \mathbf{a} / \hbar} \tag{A.71}
\end{equation*}
$$

$$
\begin{equation*}
T_{\tilde{\mathbf{p}}}=e^{i \tilde{\mathbf{p}} \cdot \mathbf{x} / \hbar} \tag{A.72}
\end{equation*}
$$

$$
\begin{equation*}
\mathscr{D}(\hat{\mathbf{n}}, \phi)=\exp (-i \mathbf{J} \cdot \hat{\mathbf{n}} \phi / \hbar) \tag{A.73}
\end{equation*}
$$

## Calculus

$$
\begin{align*}
& \boldsymbol{\nabla}^{2} \psi=\frac{1}{\rho} \partial_{\rho}\left(\rho \partial_{\rho} \psi\right)+\frac{1}{\rho^{2}} \partial_{\phi \phi} \psi+\partial_{z z} \psi  \tag{A.74}\\
& \boldsymbol{\nabla}^{2} \psi=\frac{1}{r^{2}} \partial_{r}\left(r^{2} \partial_{r} \psi\right)+\frac{1}{r^{2} \sin \theta} \partial_{\theta}\left(\sin \theta \partial_{\theta} \psi\right)+\frac{1}{r^{2} \sin ^{2} \theta} \partial_{\phi \phi}  \tag{A.75}\\
& \int_{-\infty}^{\infty} \exp \left(a x^{2}\right) d x=\sqrt{\frac{-\pi}{a}}  \tag{A.76}\\
& \Gamma(t)=\int_{0}^{\infty} x^{t-1} e^{-x} d x=(t-1)!  \tag{A.77}\\
& \int_{0}^{\infty} e^{-\alpha r} r^{n} d r=\frac{n!}{\alpha^{n+1}}  \tag{A.78}\\
& \lim _{N \rightarrow \infty}\left(1+\frac{x}{N}\right)^{N}=e^{x}
\end{align*}
$$

Symmetries

$$
\begin{align*}
& \pi^{\dagger}=\pi=\pi^{-1}  \tag{A.80}\\
& \pi^{\dagger} \mathbf{x} \pi=-\mathbf{x} \\
& \pi^{\dagger} \mathbf{p} \pi=-\mathbf{p}  \tag{A.82}\\
& \pi|\mathbf{x}\rangle=|-\mathbf{x}\rangle \\
& \langle\mathbf{x}| \pi|\psi\rangle=\langle-\mathbf{x} \mid \psi\rangle=\psi(-\mathbf{x}) \tag{A.84}
\end{align*}
$$

$$
\begin{equation*}
[\pi, \mathbf{J}]=0 \tag{A.85}
\end{equation*}
$$

$$
\begin{equation*}
\Theta H=H \Theta \tag{A.86}
\end{equation*}
$$

$$
\begin{equation*}
\Theta \mathbf{p} \Theta^{-1}=-\mathbf{p} \tag{A.87}
\end{equation*}
$$

$$
\begin{equation*}
\Theta \mathbf{J} \Theta^{-1}=-\mathbf{J} \tag{A.88}
\end{equation*}
$$

$\Theta \mathbf{x} \Theta^{-1}=\mathbf{x}$
$\langle\mathbf{x}| \Theta|\alpha\rangle=\langle\alpha \mid \mathbf{x}\rangle$
$\Theta|j, m\rangle=i^{2 m}|j,-m\rangle$
$\Theta^{2}=(-1)^{2 j}$
$\Theta=-i \sigma_{y} \eta K, \quad \eta=i$
$K i=-i$
$\Theta|\hat{\mathbf{n}} ;+\rangle=\eta|\hat{\mathbf{n}} ;-\rangle$
$\Theta|\hat{\mathbf{n}} ;-\rangle=-\eta|\hat{\mathbf{n}} ;+\rangle$

Spin

$$
\begin{align*}
& {\left[J_{x}, J_{y}\right]=i \hbar J_{z}}  \tag{A.94}\\
& {\left[J_{r}, J_{s}\right]=i \hbar \epsilon_{r s t} J_{t}}  \tag{A.95}\\
& {\left[J^{2}, J_{ \pm}\right]=0}  \tag{A.96}\\
& {\left[J_{z}, J_{ \pm}\right]= \pm \hbar J_{ \pm}}  \tag{A.97}\\
& {\left[J_{+}, J_{-}\right]=2 \hbar J_{z}} \tag{A.98}
\end{align*}
$$

$$
\begin{equation*}
\mathbf{J}=\mathbf{L} \otimes 1+1 \otimes \mathbf{S}=\mathbf{L}+\mathbf{S} \tag{A.99}
\end{equation*}
$$

$$
\begin{equation*}
J_{z}|j, m\rangle=m \hbar|j, m\rangle \tag{A.100}
\end{equation*}
$$

$$
\begin{equation*}
\mathbf{J}^{2}|j, m\rangle=j(j+1) \hbar^{2}|j, m\rangle \tag{A.101}
\end{equation*}
$$

$J_{ \pm}|j, m\rangle=\hbar \sqrt{(j \mp m)(j \pm m+1)}|j, m\rangle$
$\mathscr{D}(R)=\exp (-i \mathbf{L} \cdot \hat{\mathbf{n}} \phi / \hbar) \otimes \exp (-i \mathbf{S} \cdot \hat{\mathbf{n}} \phi / \hbar)$

## Spin one representation

$$
\begin{align*}
& J_{x}=\frac{\hbar}{\sqrt{2}}\left[\begin{array}{lll}
0 & 1 & 0 \\
1 & 0 & 1 \\
0 & 1 & 0
\end{array}\right]  \tag{A.104}\\
& J_{y}=\frac{\hbar i}{\sqrt{2}}\left[\begin{array}{ccc}
0 & -1 & 0 \\
1 & 0 & -1 \\
0 & 1 & 0
\end{array}\right]  \tag{A.105}\\
& J_{z}=\hbar\left[\begin{array}{ccc}
1 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & -1
\end{array}\right] \tag{A.106}
\end{align*}
$$

Angular momentum

$$
\begin{align*}
\mathbf{L}^{2} & =L_{z}^{2}+\frac{1}{2}\left(L_{+} L_{-}+L_{-} L_{+}\right) \\
& =-\hbar^{2}\left(\frac{1}{\sin ^{2} \theta} \partial_{\phi \phi}+\frac{1}{\sin \theta} \partial_{\theta}\left(\sin \theta \partial_{\theta}\right)\right)  \tag{A.107}\\
& =\mathbf{x}^{2} \mathbf{p}^{2}-(\mathbf{x} \cdot \mathbf{p})^{2}+i \hbar \mathbf{x} \cdot \mathbf{p} \\
& =r^{2} \mathbf{p}^{2}+\hbar^{2}\left(r^{2} \partial_{r r}+2 r \partial_{r}\right)
\end{align*}
$$

$$
\begin{align*}
L_{x} & =-i \hbar\left(-\sin \phi \partial_{\theta}-\cot \theta \cos \phi \partial_{\phi}\right) \\
L_{y} & =-i \hbar\left(\cos \phi \partial_{\theta}-\cot \theta \sin \phi \partial_{\phi}\right)  \tag{A.108}\\
L_{z} & =-i \hbar \partial_{\phi}
\end{align*}
$$

$$
\begin{equation*}
\left[L_{x}, L_{y}\right]=i \hbar L_{z} \tag{A.109}
\end{equation*}
$$

$$
\begin{equation*}
\left\{L_{+}, L_{-}\right\}=2\left(L_{x}^{2}+L_{y}^{2}\right)=2\left(\mathbf{L}^{2}-L_{z}^{2}\right) \tag{A.110}
\end{equation*}
$$

$$
\begin{equation*}
L_{ \pm}=-i \hbar e^{ \pm i \phi}\left( \pm i \partial_{\theta}-\cot \theta \partial_{\phi}\right)=L_{x} \pm i L_{y} \tag{A.111}
\end{equation*}
$$

$$
\begin{equation*}
L_{ \pm}|l, m\rangle=\hbar \sqrt{(l \mp m)(l \pm m+1)}|l, m \pm 1\rangle \tag{A.112}
\end{equation*}
$$

$$
\begin{equation*}
\mathbf{L}^{2}|l, m\rangle=l(l+1) \hbar^{2}|l, m\rangle \tag{A.113}
\end{equation*}
$$

$$
\begin{equation*}
L_{z}|l, m\rangle=m \hbar|l, m\rangle \tag{A.114}
\end{equation*}
$$

## Spherical harmonics

$$
\begin{align*}
& Y_{00}=\frac{1}{2 \sqrt{\pi}} \\
& Y_{10}=\frac{1}{2} \sqrt{\frac{3}{\pi}} \cos (\theta) \\
& Y_{20}=\frac{1}{4} \sqrt{\frac{5}{\pi}}\left(3 \cos ^{2}(\theta)-1\right) \\
& Y_{30}=\frac{1}{4} \sqrt{\frac{7}{\pi}}\left(5 \cos ^{3}(\theta)-3 \cos (\theta)\right) \\
& Y_{40}=\frac{3\left(\left(35 \cos ^{4}(\theta)-30 \cos ^{2}(\theta)+3\right)\right.}{16 \sqrt{\pi}} \\
& Y_{50}=\frac{1}{16} \sqrt{\frac{11}{\pi}}\left(63 \cos ^{5}(\theta)-70 \cos ^{3}(\theta)+15 \cos (\theta)\right) \\
& Y_{60}=\frac{1}{32} \sqrt{\frac{13}{\pi}}\left(231 \cos ^{6}(\theta)-315 \cos ^{4}(\theta)+105 \cos ^{2}(\theta)-5\right) \\
& Y_{70}=\frac{1}{32} \sqrt{\frac{15}{\pi}}\left(429 \cos ^{7}(\theta)-693 \cos ^{5}(\theta)+315 \cos ^{3}(\theta)-35 \cos (\theta)\right) \tag{A.115}
\end{align*}
$$

$$
\begin{equation*}
Y_{l,-m}=(-1)^{m} Y_{l m}^{*} \tag{A.116}
\end{equation*}
$$

Hydrogen wavefunctions

$$
\begin{equation*}
\psi_{1 s}=\psi_{100}=\frac{1}{\sqrt{\pi}}\left(\frac{Z}{a_{0}}\right)^{3 / 2} e^{-Z r / a_{0}} \tag{A.117a}
\end{equation*}
$$

$$
\begin{align*}
& \psi_{2 s}=\psi_{200}=\frac{1}{4 \sqrt{2 \pi}}\left(\frac{Z}{a_{0}}\right)^{3 / 2}\left(2-\frac{r Z}{a_{0}}\right) e^{-Z r / 2 a_{0}} \\
& \psi_{2 p_{x}}=\frac{1}{\sqrt{2}}\left(\psi_{2,1,-1}-\psi_{2,1,1}\right)=\frac{1}{4 \sqrt{2 \pi}}\left(\frac{Z}{a_{0}}\right)^{3 / 2} \frac{r Z}{a_{0}} e^{-r Z / 2 a_{0}} \sin \theta \cos \phi  \tag{A.117c}\\
& \psi_{2 p_{y}}=\frac{i}{\sqrt{2}}\left(\psi_{2,1,-1}+\psi_{2,1,1}\right)=\frac{1}{4 \sqrt{2 \pi}}\left(\frac{Z}{a_{0}}\right)^{3 / 2} \frac{r Z}{a_{0}} e^{-r Z / 2 a_{0}} \sin \theta \sin \phi  \tag{A.117d}\\
& \text { (A.117c) }  \tag{A.117e}\\
& \psi_{2 p_{z}}=\psi_{210}=\frac{1}{4 \sqrt{2 \pi}}\left(\frac{Z}{a_{0}}\right)^{3 / 2} \frac{r Z}{a_{0}} e^{-r Z / 2 a_{0}} \cos \theta
\end{align*}
$$

I looked to [10] to see where to add in the $a_{0}$ factors.
Energy levels are $n$ dependent only

$$
\begin{align*}
& E_{n}=-\frac{Z^{2} e^{2}}{2 n^{2} a_{0}}  \tag{A.118}\\
& a_{0}=\frac{4 \pi \epsilon_{0} \hbar^{2}}{m e^{2}} \tag{A.119}
\end{align*}
$$

Perturbation

$$
\begin{equation*}
H=H_{0}+\lambda V \tag{A.120a}
\end{equation*}
$$

$$
\begin{equation*}
|n\rangle=\sum_{j=0} \lambda^{j}\left|n_{j}\right\rangle \tag{A.120b}
\end{equation*}
$$

$$
\begin{equation*}
\Delta_{n}=\sum_{j=1} \lambda^{j} \Delta_{n_{j}} \tag{A.120c}
\end{equation*}
$$

$$
\begin{equation*}
\bar{P}_{n}=\sum_{m \neq n}|m\rangle\langle m| \tag{A.120d}
\end{equation*}
$$

$$
\begin{align*}
& \left|n_{0}\right\rangle=\left|n^{(0)}\right\rangle \\
& \left|n_{1}\right\rangle=\frac{\bar{P}_{n}}{E_{n}^{(0)}-H_{0}} V\left|n_{0}\right\rangle \\
& \left|n_{2}\right\rangle=\frac{\bar{P}_{n}}{E_{n}^{(0)}-H_{0}}\left(V-\Delta_{n_{1}}\right)\left|n_{1}\right\rangle  \tag{A.121}\\
& \left|n_{3}\right\rangle=\frac{\bar{P}_{n}}{E_{n}^{(0)}-H_{0}}\left(V\left|n_{2}\right\rangle-\Delta_{n_{1}}\left|n_{2}\right\rangle-\Delta_{n_{2}}\left|n_{1}\right\rangle\right) \\
& \left|n_{j}\right\rangle=\frac{\bar{P}_{n}}{E_{n}^{(0)}-H_{0}}\left(V\left|n_{j-1}\right\rangle-\sum_{k=1}^{j-1} \Delta_{n_{k}}\left|n_{j-k}\right\rangle\right)
\end{align*}
$$

$$
\begin{equation*}
\Delta_{n_{j}}=\left\langle n_{0}\right| V\left|n_{j-1}\right\rangle \tag{A.122}
\end{equation*}
$$

$$
\begin{equation*}
f(|x|)=\Theta(x) f(x)+\Theta(-x) f(-x) \tag{A.123}
\end{equation*}
$$

$$
\begin{equation*}
\frac{d}{d x} \delta(x)=-\frac{\delta(x)}{x} \tag{A.124}
\end{equation*}
$$

Clebsch Ex. $l_{1}=2, l_{2}=1$

$$
\begin{align*}
& |3,3\rangle=|2,2\rangle \otimes|1,1\rangle \\
& |3,2\rangle=\frac{1}{\langle 3,2| L_{-}|3,3\rangle}\left(\left(L_{-}|2,2\rangle\right) \otimes|1,1\rangle+|2,2\rangle \otimes L_{-}|1,1\rangle\right)  \tag{A.125}\\
& |2,2\rangle: a|1\rangle+b|2\rangle \rightarrow-b|1\rangle+a|2\rangle \\
& |1,1\rangle=|3,1\rangle \times|2,1\rangle
\end{align*}
$$

ODDS AND ENDS.

## B

## B. 1 SCHWARTZ INEQUALITY IN BRA-KET NOTATION.

Motivation In [15] the Schwartz inequality

$$
\begin{equation*}
\langle a \mid a\rangle\langle b \mid b\rangle \geq|\langle a \mid b\rangle|^{2}, \tag{B.1}
\end{equation*}
$$

is used in the derivation of the uncertainty relation. The proof of the Schwartz inequality uses a sneaky substitution that doesn't seem obvious, and is even less obvious since there is a typo in the value to be substituted. Let's understand where that sneakiness is coming from.

Without being sneaky My ancient first year linear algebra text [11] contains a non-sneaky proof, but it only works for real vector spaces. Recast in bra-ket notation, this method examines the bounds of the norms of sums and differences of unit state vectors (i.e. $\langle a \mid a\rangle=\langle b \mid b\rangle=1$.)

$$
\begin{align*}
\langle a-b \mid a-b\rangle & =\langle a \mid a\rangle+\langle b \mid b\rangle-\langle a \mid b\rangle-\langle b \mid a\rangle \\
& =2-2 \operatorname{Re}\langle a \mid b\rangle  \tag{B.2}\\
& \geq 0,
\end{align*}
$$

so

$$
\begin{equation*}
1 \geq \operatorname{Re}\langle a \mid b\rangle . \tag{B.3}
\end{equation*}
$$

Similarly

$$
\begin{align*}
\langle a+b \mid a+b\rangle & =\langle a \mid a\rangle+\langle b \mid b\rangle+\langle a \mid b\rangle+\langle b \mid a\rangle \\
& =2+2 \operatorname{Re}\langle a \mid b\rangle  \tag{B.4}\\
& \geq 0,
\end{align*}
$$

SO

$$
\begin{equation*}
\operatorname{Re}\langle a \mid b\rangle \geq-1 . \tag{B.5}
\end{equation*}
$$

This means that for normalized state vectors

$$
\begin{equation*}
-1 \leq \operatorname{Re}\langle a \mid b\rangle \leq 1, \tag{B.6}
\end{equation*}
$$

or

$$
\begin{equation*}
|\operatorname{Re}\langle a \mid b\rangle| \leq 1 . \tag{B.7}
\end{equation*}
$$

Writing out the unit vectors explicitly, that last inequality is

$$
\begin{equation*}
\left|\operatorname{Re}\left\langle\left.\frac{a}{\sqrt{\langle a \mid a\rangle}} \right\rvert\, \frac{b}{\sqrt{\langle b \mid b\rangle}}\right\rangle\right| \leq 1, \tag{B.8}
\end{equation*}
$$

squaring and rearranging gives

$$
\begin{equation*}
|\operatorname{Re}\langle a \mid b\rangle|^{2} \leq\langle a \mid a\rangle\langle b \mid b\rangle . \tag{B.9}
\end{equation*}
$$

This is similar to, but not identical to the Schwartz inequality. The Schwartz inequality cannot be demonstrated with this argument, since $|\operatorname{Re}\langle a \mid b\rangle| \leq$ $|\langle a \mid b\rangle|$. This first year algebra method works nicely for demonstrating the inequality for real vector spaces, so a different argument is required for a complex vector space (i.e. quantum mechanics state space.)

Arguing with projected and rejected components Notice that the equality condition holds when the vectors are colinear, and the largest inequality $(0 \leq 1)$ holds when the vectors are normal to each other. Given those geometrical observations, it seems reasonable to examine the norms of projected or rejected components of a vector. To do so in bra-ket notation, the correct form of a projection operation must be determined. Care is required to get the ordering of the bra-kets right when expressing such a projection (or rejection)

Suppose we wish to calculation the rejection of $|a\rangle$ from $|b\rangle$, that is $|b-\alpha a\rangle$, such that

$$
\begin{align*}
0 & =\langle a \mid b-\alpha a\rangle  \tag{B.10}\\
& =\langle a \mid b\rangle-\alpha\langle a \mid a\rangle,
\end{align*}
$$

or

$$
\begin{equation*}
\alpha=\frac{\langle a \mid b\rangle}{\langle a \mid a\rangle} . \tag{B.11}
\end{equation*}
$$

Therefore, the projection of $|b\rangle$ on $|a\rangle$ is

$$
\begin{equation*}
\operatorname{Proj}_{|a\rangle}|b\rangle=\frac{\langle a \mid b\rangle}{\langle a \mid a\rangle}|a\rangle=\frac{\langle b \mid a\rangle^{*}}{\langle a \mid a\rangle}|a\rangle . \tag{B.12}
\end{equation*}
$$

The conventional way to write this in QM is in the operator form

$$
\begin{equation*}
\operatorname{Proj}_{|a\rangle}|b\rangle=\frac{|a\rangle\langle a|}{\langle a \mid a\rangle}|b\rangle . \tag{B.13}
\end{equation*}
$$

In this form the rejection of $|a\rangle$ from $|b\rangle$ can be expressed as

$$
\begin{equation*}
\operatorname{Rej}_{|a\rangle}|b\rangle=|b\rangle-\frac{|a\rangle\langle a|}{\langle a \mid a\rangle}|b\rangle \tag{B.14}
\end{equation*}
$$

This state vector is normal to $|a\rangle$ as desired

$$
\begin{align*}
\left\langle a \left\lvert\, b-\frac{\langle a \mid b\rangle}{\langle a \mid a\rangle} a\right.\right\rangle & =\langle a \mid b\rangle-\frac{\langle a \mid b\rangle}{\langle a \mid a\rangle}\langle a \mid a\rangle  \tag{B.15}\\
& =0 .
\end{align*}
$$

How about it's length? That is

$$
\begin{align*}
\left\langle\left. b-\frac{\langle a \mid b\rangle}{\langle a \mid a\rangle} a \right\rvert\, b-\frac{\langle a \mid b\rangle}{\langle a \mid a\rangle} a\right\rangle & =\langle b \mid b\rangle-2 \frac{|\langle a \mid b\rangle|^{2}}{\langle a \mid a\rangle}+\frac{|\langle a \mid b\rangle|^{2}}{\langle a \mid a\rangle^{2}}\langle a \mid a\rangle  \tag{B.16}\\
& =\langle b \mid b\rangle-\frac{|\langle a \mid b\rangle|^{2}}{\langle a \mid a\rangle}
\end{align*}
$$

Observe that this must be greater to or equal to zero, so

$$
\begin{equation*}
\langle b \mid b\rangle \geq \frac{|\langle a \mid b\rangle|^{2}}{\langle a \mid a\rangle} . \tag{B.17}
\end{equation*}
$$

Rearranging this gives eq. (B.1) as desired. The Schwartz proof in [15] obscures the geometry involved and starts with

$$
\begin{equation*}
\langle b+\lambda a \mid b+\lambda a\rangle \geq 0, \tag{B.18}
\end{equation*}
$$

where the "proof" is nothing more than a statement that one can "pick" $\lambda=-\langle b \mid a\rangle /\langle a \mid a\rangle$. The Pythagorean context of the Schwartz inequality is not mentioned, and without thinking about it, one is left wondering what sort of magic hat that $\lambda$ selection came from.

## B. 2 AN OBSERVATION ABOUT THE GEOMETRY OF PAULI X,Y MATRICES.

Motivation The conventional form for the Pauli matrices is

$$
\begin{align*}
& \sigma_{x}=\left[\begin{array}{ll}
0 & 1 \\
1 & 0
\end{array}\right], \\
& \sigma_{y}=\left[\begin{array}{cc}
0 & -i \\
i & 0
\end{array}\right],  \tag{B.19}\\
& \sigma_{z}=\left[\begin{array}{cc}
1 & 0 \\
0 & -1
\end{array}\right] .
\end{align*}
$$

In [4] these forms are derived based on the commutation relations

$$
\begin{equation*}
\left[\sigma_{r}, \sigma_{s}\right]=2 i \epsilon_{r s t} \sigma_{t}, \tag{B.20}
\end{equation*}
$$

by defining raising and lowering operators $\sigma_{ \pm}=\sigma_{x} \pm i \sigma_{y}$ and figuring out what form the matrix must take. I noticed an interesting geometrical relation hiding in that derivation if $\sigma_{+}$is not assumed to be real.

Derivation For completeness, I'll repeat the argument of [4], which builds on the commutation relations of the raising and lowering operators. Those are

$$
\begin{align*}
{\left[\sigma_{z}, \sigma_{ \pm}\right] } & =\sigma_{z}\left(\sigma_{x} \pm i \sigma_{y}\right)-\left(\sigma_{x} \pm i \sigma_{y}\right) \sigma_{z} \\
& =\left[\sigma_{z}, \sigma_{x}\right] \pm i\left[\sigma_{z}, \sigma_{y}\right] \\
& =2 i \sigma_{y} \pm i(-2 i) \sigma_{x}  \tag{B.21}\\
& = \pm 2\left(\sigma_{x} \pm i \sigma_{y}\right) \\
& = \pm 2 \sigma_{ \pm},
\end{align*}
$$

and

$$
\begin{align*}
{\left[\sigma_{+}, \sigma_{-}\right] } & =\left(\sigma_{x}+i \sigma_{y}\right)\left(\sigma_{x}-i \sigma_{y}\right)-\left(\sigma_{x}-i \sigma_{y}\right)\left(\sigma_{x}+i \sigma_{y}\right) \\
& =-i \sigma_{x} \sigma_{y}+i \sigma_{y} \sigma_{x}-i \sigma_{x} \sigma_{y}+i \sigma_{y} \sigma_{x} \\
& =2 i\left[\sigma_{y}, \sigma_{x}\right]  \tag{B.22}\\
& =2 i(-2 i) \sigma_{z} \\
& =4 \sigma_{z}
\end{align*}
$$

From these a matrix representation containing unknown values can be assumed. Let

$$
\sigma_{+}=\left[\begin{array}{ll}
a & b  \tag{B.23}\\
c & d
\end{array}\right]
$$

The commutator with $\sigma_{z}$ can be computed

$$
\begin{align*}
{\left[\sigma_{z}, \sigma_{+}\right] } & =\left[\begin{array}{cc}
1 & 0 \\
0 & -1
\end{array}\right]\left[\begin{array}{ll}
a & b \\
c & d
\end{array}\right]-\left[\begin{array}{ll}
a & b \\
c & d
\end{array}\right]\left[\begin{array}{cc}
1 & 0 \\
0 & -1
\end{array}\right] \\
& =\left[\begin{array}{cc}
a & b \\
-c & -d
\end{array}\right]-\left[\begin{array}{cc}
a & -b \\
c & -d
\end{array}\right]  \tag{B.24}\\
& =2\left[\begin{array}{cc}
0 & b \\
-c & 0
\end{array}\right]
\end{align*}
$$

Now compare this with eq. (B.21)

$$
\begin{align*}
2\left[\begin{array}{cc}
0 & b \\
-c & 0
\end{array}\right] & =2 \sigma_{+} \\
& =2\left[\begin{array}{ll}
a & b \\
d & d
\end{array}\right] \tag{B.25}
\end{align*}
$$

This shows that $a=0$, and $d=0$. Similarly the $\sigma_{z}$ commutator with the lowering operator is

$$
\begin{align*}
{\left[\sigma_{z}, \sigma_{-}\right] } & =\left[\begin{array}{cc}
1 & 0 \\
0 & -1
\end{array}\right]\left[\begin{array}{cc}
0 & -c^{*} \\
b^{*} & 0
\end{array}\right]-\left[\begin{array}{cc}
0 & -c^{*} \\
b^{*} & 0
\end{array}\right]\left[\begin{array}{cc}
1 & 0 \\
0 & -1
\end{array}\right] \\
& =\left[\begin{array}{cc}
0 & -c^{*} \\
-b^{*} & 0
\end{array}\right]-\left[\begin{array}{cc}
0 & c^{*} \\
b^{*} & 0
\end{array}\right]  \tag{B.26}\\
& =-2\left[\begin{array}{cc}
0 & c^{*} \\
b^{*} & 0
\end{array}\right] .
\end{align*}
$$

Again comparing to eq. (B.21), we have

$$
\begin{align*}
-2\left[\begin{array}{cc}
0 & c^{*} \\
b^{*} & 0
\end{array}\right] & =-2 \sigma_{-}  \tag{B.27}\\
& =-2\left[\begin{array}{cc}
0 & -c^{*} \\
b^{*} & 0
\end{array}\right]
\end{align*}
$$

so $c=0$. Computing the commutator of the raising and lowering operators fixes $b$

$$
\begin{align*}
{\left[\sigma_{+}, \sigma_{-}\right] } & =\left[\begin{array}{ll}
0 & b \\
0 & 0
\end{array}\right]\left[\begin{array}{cc}
0 & 0 \\
b^{*} & 0
\end{array}\right]-\left[\begin{array}{cc}
0 & 0 \\
b^{*} & 0
\end{array}\right]\left[\begin{array}{ll}
0 & b \\
0 & 0
\end{array}\right] \\
& =\left[\begin{array}{cc}
|b|^{2} & 0 \\
0 & 0
\end{array}\right]-\left[\begin{array}{lll}
0 & 00 & \left.-|b|^{2}\right] \\
& =|b|^{2}\left[\begin{array}{cc}
1 & 0 \\
0 & -1
\end{array}\right] \\
& =|b|^{2} \sigma_{z} .
\end{array}\right. \text {. } \tag{B.28}
\end{align*}
$$

From eq. (B.22) it must be that $|b|^{2}=4$, so the most general form of the raising operator is

$$
\sigma_{+}=2\left[\begin{array}{cc}
0 & e^{i \phi}  \tag{B.29}\\
0 & 0
\end{array}\right]
$$

Observation The conventional choice is to set $\phi=0$, but I found it interesting to see the form of $\sigma_{x}, \sigma_{y}$ without that choice. That is

$$
\begin{align*}
\sigma_{x} & =\frac{1}{2}\left(\sigma_{+}+\sigma_{-}\right) \\
& =\left[\begin{array}{cc}
0 & e^{i \phi} \\
e^{-i \phi} & 0
\end{array}\right],  \tag{B.30}\\
\sigma_{y} & =\frac{1}{2 i}\left(\sigma_{+}-\sigma_{-}\right) \\
& =\left[\begin{array}{cc}
0 & -i e^{i \phi} \\
-i e^{-i \phi} & 0
\end{array}\right]  \tag{B.31}\\
& =\left[\begin{array}{cc}
0 & e^{i(\phi-\pi / 2)} \\
e^{-i(\phi-\pi / 2)} & 0
\end{array}\right] .
\end{align*}
$$

Notice that the Pauli matrices $\sigma_{x}$ and $\sigma_{y}$ actually both have the same form as $\sigma_{x}$, but the phase of the complex argument of each differs by $90^{\circ}$. That $90^{\circ}$ separation isn't obvious in the standard form eq. (B.19).

It's a small detail, but I thought it was kind of cool that the orthogonality of these matrix unit vector representations is built directly into the structure of their matrix representations.

## B. 3 OPERATOR MATRIX ELEMENT.

Weird dreams I woke up today having a dream still in my head from the night, but it was a strange one. I was expanding out the Dirac notation representation of an operator in matrix form, but the symbols in the kets were elaborate pictures of Disney princesses that I was drawing with forestry scenery in the background, including little bears. At the point that I woke up from the dream, I noticed that I'd gotten the proportion of the bears wrong in one of the pictures, and they looked like they were ready to eat one of the princess characters.

Guts As a side effect of this weird dream I actually started thinking about matrix element representation of operators.

When forming the matrix element of an operator using Dirac notation the elements are of the form $\langle\mathrm{row}| A \mid$ column $\rangle$. I've gotten that mixed up a couple of times, so I thought it would be helpful to write this out explicitly for a $2 \times 2$ operator representation for clarity.

To start, consider a change of basis for a single matrix element from basis $\{|q\rangle,|r\rangle\}$, to basis $\{|a\rangle,|b\rangle\}$

$$
\begin{align*}
\langle q| A|r\rangle & =\langle q \mid a\rangle\langle a| A|r\rangle+\langle q \mid b\rangle\langle b| A|r\rangle \\
& =\langle q \mid a\rangle\langle a| A|a\rangle\langle a \mid r\rangle+\langle q \mid a\rangle\langle a| A|b\rangle\langle b \mid r\rangle \\
& +\langle q \mid b\rangle\langle b| A|a\rangle\langle a \mid r\rangle+\langle q \mid b\rangle\langle b| A|b\rangle\langle b \mid r\rangle \\
& =\langle q \mid a\rangle\left[\begin{array}{cc}
\langle a| A|a\rangle & \langle a| A|b\rangle
\end{array}\right]\left[\begin{array}{c}
\langle a \mid r\rangle \\
\langle b \mid r\rangle
\end{array}\right]+\langle q \mid b\rangle\left[\begin{array}{ll}
\langle b| A|a\rangle & \langle b| A|b\rangle
\end{array}\right]\left[\begin{array}{c}
\langle a \mid r\rangle \\
\langle b \mid r\rangle
\end{array}\right] \\
& =\left[\begin{array}{ll}
\langle q \mid a\rangle & \langle q \mid b\rangle
\end{array}\right]\left[\begin{array}{cc}
\langle a| A|a\rangle & \langle a| A|b\rangle \\
\langle b| A|a\rangle & \langle b| A|b\rangle
\end{array}\right]\left[\begin{array}{c}
\langle a \mid r\rangle \\
\langle b \mid r\rangle
\end{array}\right] . \tag{B.32}
\end{align*}
$$

Suppose the matrix representation of $|q\rangle,|r\rangle$ are respectively

$$
\begin{align*}
& |q\rangle \sim\left[\begin{array}{l}
\langle a \mid q\rangle \\
\langle b \mid q\rangle
\end{array}\right],  \tag{B.33}\\
& |r\rangle \sim\left[\begin{array}{l}
\langle a \mid r\rangle \\
\langle b \mid r\rangle
\end{array}\right],
\end{align*}
$$

then

$$
\langle q| \sim\left[\begin{array}{c}
\langle a \mid q\rangle  \tag{B.34}\\
\langle b \mid q\rangle
\end{array}\right]^{\dagger}=\left[\begin{array}{ll}
\langle q \mid a\rangle & \langle q \mid b\rangle
\end{array}\right] .
$$

The matrix element is then

$$
\langle q| A|r\rangle \sim\langle q|\left[\begin{array}{ll}
\langle a| A|a\rangle & \langle a| A|b\rangle  \tag{B.35}\\
\langle b| A|a\rangle & \langle b| A|b\rangle
\end{array}\right]|r\rangle
$$

and the corresponding matrix representation of the operator is

$$
A \sim\left[\begin{array}{ll}
\langle a| A|a\rangle & \langle a| A|b\rangle  \tag{B.36}\\
\langle b| A|a\rangle & \langle b| A|b\rangle
\end{array}\right]
$$

## B. 4 GENERALIZED GAUSSIAN INTEGRALS.

Both [15] and [19] use Gaussian integrals with both (negative) real, and imaginary arguments, which give the impression that the following is true:

$$
\begin{equation*}
\int_{-\infty}^{\infty} \exp \left(a x^{2}\right) d x=\sqrt{\frac{-\pi}{a}} \tag{B.37}
\end{equation*}
$$

even when $a$ is not a real negative constant, and in particular, with values $a= \pm i$. Clearly this doesn't follow by just making a substitution $x \rightarrow$ $x / \sqrt{a}$, since that moves the integration range onto a rotated path in the complex plane when $a$ is $\pm i$. However, with some care, it can be shown that eq. (B.37) holds provided $\operatorname{Re} a \leq 0$.

To show this, this integral will be considered for the pure real case, purely imaginary, and finally the complex case with non-zero real and imaginary parts for $a$.

Real (negative) case. The first special case is $\int_{-\infty}^{\infty} \exp \left(-x^{2}\right) d x=\sqrt{\pi}$ which is easy to derive using the usual square it and integrate in circular coordinates trick.

Purely imaginary cases. Let's handle the $a= \pm i$ cases next. These can be evaluated by considering integrals over the contours of fig. B.1, where the upper plane contour is used for $a=i$ and the lower plane contour for


Figure B.1: Contours for $a= \pm i$.
$a=-i$. Since there are no poles, the integral over either such contour is zero. Credit for figuring out how to tackle that integral and what contour to use goes to Dr MV, on stackexchange [9]. For the upper plane contour we have

$$
\begin{align*}
0 & =\oint \exp \left(i z^{2}\right) d z \\
& =\int_{0}^{R} \exp \left(i x^{2}\right) d x+\int_{0}^{\pi / 4} \exp \left(i R^{2} e^{2 i \theta}\right) R i e^{i \theta} d \theta+\int_{R}^{0} \exp \left(i^{2} t^{2}\right) e^{i \pi / 4} d t \tag{B.38}
\end{align*}
$$

Observe that $i e^{2 i \theta}=i \cos (2 \theta)-\sin (2 \theta)$ which has a negative real part for all values of $\theta \neq 0$. Provided the contour is slightly deformed from the axis, that second integral has a term of the form $\sim R e^{-R^{2}}$ which tends to zero as $R \rightarrow \infty$. So in the limit, this is

$$
\begin{equation*}
\int_{0}^{\infty} \exp \left(i x^{2}\right) d x=\sqrt{\pi} e^{i \pi / 4} / 2 \tag{B.39}
\end{equation*}
$$

or

$$
\begin{equation*}
\int_{-\infty}^{\infty} \exp \left(i x^{2}\right) d x=\sqrt{i \pi} \tag{B.40}
\end{equation*}
$$

a special case of eq. (B.37) as desired. For $a=-i$ integrating around the lower plane contour, we have

$$
\begin{align*}
0= & \oint \exp \left(-i z^{2}\right) d z \\
= & \int_{0}^{R} \exp \left(i x^{2}\right) d x+\int_{0}^{-\pi / 4} \exp \left(-i R^{2} e^{2 i \theta}\right) R i e^{i \theta} d \theta  \tag{B.41}\\
& +\int_{R}^{0} \exp \left(-i(-i) t^{2}\right) e^{-i \pi / 4} d t
\end{align*}
$$

This time, in the second integral we also have $-i R^{2} e^{2 i \theta}=i R^{2} \cos (2 \theta)+$ $\sin (2 \theta)$, which also has a negative real part for $\theta \in(0, \pi / 4]$. Again the contour needs to be infinitesimally deformed ${ }^{1}$, placed just lower than the axis. This time we find

$$
\begin{equation*}
\int_{-\infty}^{\infty} \exp \left(-i x^{2}\right) d x=\sqrt{-i \pi} \tag{B.42}
\end{equation*}
$$

another special case of eq. (B.37).
Completely complex case. A similar trick can be used to evaluate the more general cases, but a bit of thought is required to figure out the contours required. More precisely, while these contours will still have a wedge of pie shape, as sketched in fig. B.2, we have to figure out the angle subtended by the edge of this piece of pie. To evaluate the integral consider

$$
\begin{align*}
0= & \oint \exp \left(e^{i \phi} z^{2}\right) d z \\
= & \int_{0}^{R} \exp \left(e^{i \phi} x^{2}\right) d x+\int_{0}^{\theta} \exp \left(e^{i \phi} R^{2} e^{2 i \mu}\right) R i e^{i \mu} d \mu  \tag{B.43}\\
& +\int_{R}^{0} \exp \left(e^{i \phi} e^{2 i \theta} t^{2}\right) e^{i \theta} d t
\end{align*}
$$

where $\phi \in(\pi / 2, \pi) \cup(\pi, 3 \pi / 2)$. We have a hope of evaluating this last integral if $\phi+2 \theta=\pi$, or

$$
\begin{equation*}
\theta=(\pi-\phi) / 2, \tag{B.44}
\end{equation*}
$$

1 Distorting the contour in this fashion seems somewhat like hand-waving. A better approach would probably follow [8] where Jordan's lemma is covered. It doesn't look like Jordan's lemma applies as is to this case, but the arguments look like they could be adapted appropriately.


Figure B.2: Contours for complex $a$.
giving

$$
\begin{align*}
\int_{0}^{R} \exp \left(e^{i \phi} x^{2}\right) d x= & e^{i(\pi-\phi) / 2} \int_{0}^{R} \exp \left(-t^{2}\right) d t \\
& -\int_{0}^{\theta} \exp \left(R^{2}(\cos (\phi+2 \mu)+i \sin (\phi+2 \mu))\right) R i e^{i \mu} d \mu \tag{B.45}
\end{align*}
$$

If the cosine is always negative on the chosen contours, then that integral will vanish in the $R \rightarrow \infty$ limit. This turns out to be the case, which can be confirmed by considering each of the contours in sequence. If the upper plane contour is used to evaluate eq. (B.43) for the $\phi \in(\pi / 2, \pi)$ case, we have

$$
\begin{equation*}
\theta \in(0, \pi / 4) \tag{B.46}
\end{equation*}
$$

Since $\phi+2 \theta=\pi$, we have

$$
\begin{equation*}
\phi+2 \mu \in(\pi / 2, \pi) \tag{B.47}
\end{equation*}
$$

and find that the cosine is strictly negative on that contour for that range of $\phi$. Picking the lower plane contour for the $\phi \in(\pi, 3 \pi / 2)$ range, we have

$$
\begin{equation*}
\theta \in(-\pi / 4,0) \tag{B.48}
\end{equation*}
$$

and so

$$
\begin{equation*}
\phi+2 \mu \in(\pi / 2,3 \pi / 2) \tag{B.49}
\end{equation*}
$$

For this range of $\phi$ the cosine on the lower plane contour is again negative as desired, so in the infinite $R$ limit we have

$$
\begin{equation*}
\int_{0}^{\infty} \exp \left(e^{i \phi} x^{2}\right) d x=\frac{1}{2} \sqrt{-\pi e^{-i \phi}} \tag{B.50}
\end{equation*}
$$

The points at $\phi=\pi / 2, \pi, 3 \pi / 2$ were omitted, but we've found the same result at those points, completing the verification of eq. (B.37) for all $\operatorname{Re} a \leq 0$.

## B. 5 A CURIOUS PROOF OF THE BAKER-CAMPBELL-HAUSDORFF FORMULA.

Equation (39) of [6] states the Baker-Campbell-Hausdorff formula for two operators $a, b$ that commute with their commutator $[a, b]$

$$
\begin{equation*}
e^{a} e^{b}=e^{a+b+[a, b] / 2} \tag{B.51}
\end{equation*}
$$

and provides the outline of an interesting method of proof. That method is to consider the derivative of

$$
\begin{equation*}
f(\lambda)=e^{\lambda a} e^{\lambda b} e^{-\lambda(a+b)} \tag{B.52}
\end{equation*}
$$

That derivative is

$$
\begin{align*}
\frac{d f}{d \lambda} & =e^{\lambda a} a e^{\lambda b} e^{-\lambda(a+b)}+e^{\lambda a} b e^{\lambda b} e^{-\lambda(a+b)}-e^{\lambda a} b e^{\lambda b}(a+b) e^{-\lambda(a+b)} \\
& =e^{\lambda a}\left(a e^{\lambda b}+b e^{\lambda b}-e^{\lambda b}(a+b)\right) e^{-\lambda(a+b)}  \tag{B.53}\\
& =e^{\lambda a}\left(\left[a, e^{\lambda b}\right]+\left[b, e^{\lambda b}\right]\right) e^{-\lambda(a+b)} \\
& =e^{\lambda a}\left[a, e^{\lambda b}\right] e^{-\lambda(a+b)}
\end{align*}
$$

The commutator above is proportional to $[a, b]$

$$
\begin{align*}
{\left[a, e^{\lambda b}\right] } & =\sum_{k=0}^{\infty} \frac{\lambda^{k}}{k!}\left[a, b^{k}\right] \\
& =\sum_{k=0}^{\infty} \frac{\lambda^{k}}{k!} k b^{k-1}[a, b]  \tag{B.54}\\
& =\lambda \sum_{k=1}^{\infty} \frac{\lambda^{k-1}}{(k-1)!} b^{k-1}[a, b] \\
& =\lambda e^{\lambda b}[a, b]
\end{align*}
$$

$$
\begin{equation*}
\frac{d f}{d \lambda}=\lambda[a, b] f \tag{B.55}
\end{equation*}
$$

To get the above, we should also do the induction demonstration for $\left[a, b^{k}\right]=$ $k b^{k-1}[a, b]$. This clearly holds for $k=0,1$. For any other $k$ we have

$$
\begin{align*}
{\left[a, b^{k+1}\right] } & =a b^{k+1}-b^{k+1} a \\
& =\left(\left[a, b^{k}\right]+b^{k} a\right) b-b^{k+1} a \\
& =k b^{k-1}[a, b] b+b^{k}([a, b]+b a)-b^{k+1} a  \tag{B.56}\\
& =k b^{k}[a, b]+b^{k}[a, b] \\
& =(k+1) b^{k}[a, b] .
\end{align*}
$$

Observe that eq. (B.55) is solved by

$$
\begin{equation*}
f=e^{\lambda^{2}[a, b] / 2} \tag{B.57}
\end{equation*}
$$

which gives

$$
\begin{equation*}
e^{\lambda^{2}[a, b] / 2}=e^{\lambda a} e^{\lambda b} e^{-\lambda(a+b)} \tag{B.58}
\end{equation*}
$$

Right multiplication by $e^{\lambda(a+b)}$ which commutes with $e^{\lambda^{2}[a, b] / 2}$ and setting $\lambda=1$ recovers eq. (B.51) as desired.

What I wonder looking at this, is what thought process led to trying this in the first place? This is not what I would consider an obvious approach to demonstrating this identity.

## B. 6 POSITION OPERATOR IN MOMENTUM SPACE REPRESENTATION.

A derivation of the position space representation of the momentum operator $-i \hbar \partial_{x}$ is made in [4], starting with the position-momentum commutator. Here I'll repeat that argument for the momentum space representation of the position operator.

What we want to do is expand the matrix element of the commutator. First using the definition of the commutator

$$
\begin{align*}
\left\langle p^{\prime}\right| X P-P X\left|p^{\prime \prime}\right\rangle & =i \hbar\left\langle p^{\prime} \mid p^{\prime \prime}\right\rangle  \tag{B.59}\\
& =i \hbar \delta p^{\prime}-p^{\prime \prime}
\end{align*}
$$

and then by inserting an identity operation in a momentum space basis

$$
\begin{align*}
& \left\langle p^{\prime}\right| X P-P X\left|p^{\prime \prime}\right\rangle \\
& =\int d p\left\langle p^{\prime}\right| X|p\rangle\langle p| P\left|p^{\prime \prime}\right\rangle-\int d p\left\langle p^{\prime}\right| P|p\rangle\langle p| X\left|p^{\prime \prime}\right\rangle \\
& =\int d p\left\langle p^{\prime}\right| X|p\rangle p \delta\left(p-p^{\prime \prime}\right)-\int d p p \delta\left(p^{\prime}-p\right)\langle p| X\left|p^{\prime \prime}\right\rangle  \tag{B.60}\\
& =\left\langle p^{\prime}\right| X\left|p^{\prime \prime}\right\rangle p^{\prime \prime}-p^{\prime}\left\langle p^{\prime}\right| X\left|p^{\prime \prime}\right\rangle .
\end{align*}
$$

So we have

$$
\begin{equation*}
\left\langle p^{\prime}\right| X\left|p^{\prime \prime}\right\rangle p^{\prime \prime}-p^{\prime}\left\langle p^{\prime}\right| X\left|p^{\prime \prime}\right\rangle=i \hbar \delta p^{\prime}-p^{\prime \prime} \tag{B.61}
\end{equation*}
$$

Because the RHS is zero whenever $p^{\prime} \neq p^{\prime \prime}$, the matrix element $\left\langle p^{\prime}\right| X\left|p^{\prime \prime}\right\rangle$ must also include a delta function. Let

$$
\begin{equation*}
\left\langle p^{\prime}\right| X\left|p^{\prime \prime}\right\rangle=\delta\left(p^{\prime}-p^{\prime \prime}\right) X\left(p^{\prime \prime}\right) \tag{B.62}
\end{equation*}
$$

Because eq. (B.61) is an operator equation that really only takes on meaning when applied to a wave function and integrated, we do that

$$
\begin{align*}
& \int d p^{\prime \prime} \delta\left(p^{\prime}-p^{\prime \prime}\right) X\left(p^{\prime \prime}\right) p^{\prime \prime} \psi\left(p^{\prime \prime}\right) \\
& \quad-\int d p^{\prime \prime} p^{\prime} \delta\left(p^{\prime}-p^{\prime \prime}\right) X\left(p^{\prime \prime}\right) \psi\left(p^{\prime \prime}\right)=\int d p^{\prime \prime} i \hbar \delta p^{\prime}-p^{\prime \prime} \psi\left(p^{\prime \prime}\right) \tag{B.63}
\end{align*}
$$

or

$$
\begin{equation*}
i \hbar \psi\left(p^{\prime}\right)=X\left(p^{\prime}\right) p^{\prime} \psi\left(p^{\prime}\right)-p^{\prime} X\left(p^{\prime}\right) \psi\left(p^{\prime}\right) \tag{B.64}
\end{equation*}
$$

Provided $X\left(p^{\prime}\right)$ operates on everything to its right, this equation is solved by setting

$$
\begin{equation*}
X\left(p^{\prime}\right)=i \hbar \frac{\partial}{\partial p^{\prime}} \tag{B.65}
\end{equation*}
$$

## b. 7 expansion of the squared angular momentum operator.

In [15] eq. 6.16 is

$$
\begin{equation*}
\mathbf{L}^{2}=\mathbf{x}^{2} \mathbf{p}^{2}-(\mathbf{x} \cdot \mathbf{p})^{2}+i \hbar \mathbf{x} \cdot \mathbf{p} \tag{B.66}
\end{equation*}
$$

and a derivation of the same. The derivation is clear right until the end, where the details for the last steps are left off. When I attempted those last steps I got a different sign for the $\mathbf{x} \cdot \mathbf{p}$ term. I also get that same difference in sign if I do this expansion myself:

$$
\begin{align*}
\mathbf{L}^{2} & =\epsilon_{i j k} x_{i} p_{j} \epsilon_{r s k} x_{r} p_{s} \\
& =\delta_{i j}^{[r s]} x_{i} p_{j} x_{r} p_{s} \\
& =x_{i} p_{j} x_{i} p_{j}-x_{i} p_{j} x_{j} p_{i} \\
& =x_{i}\left(x_{i} p_{j}-i \hbar \delta_{i j}\right) p_{j}-x_{i} p_{j}\left(p_{i} x_{j}+i \hbar \delta_{i j}\right)  \tag{B.67}\\
& =\mathbf{x}^{2} \mathbf{p}^{2}-i \hbar \mathbf{x} \cdot \mathbf{p}-x_{i} p_{j} p_{i} x_{j}-i \hbar \mathbf{x} \cdot \mathbf{p} \\
& =\mathbf{x}^{2} \mathbf{p}^{2}-2 i \hbar \mathbf{x} \cdot \mathbf{p}-x_{i} p_{i} p_{j} x_{j} \\
& =\mathbf{x}^{2} \mathbf{p}^{2}-2 i \hbar \mathbf{x} \cdot \mathbf{p}-(\mathbf{x} \cdot \mathbf{p})\left(x_{j} p_{j}-i \hbar\right) \\
& =\mathbf{x}^{2} \mathbf{p}^{2}-i \hbar \mathbf{x} \cdot \mathbf{p}-(\mathbf{x} \cdot \mathbf{p})^{2} .
\end{align*}
$$

I couldn't spot an error anywhere above, but Prof Paramekanti spotted it when I asked. The error is in the second last step above, since $p_{j} x_{j}=$ $p_{1} x_{1}+p_{2} x_{2}+p_{3} x_{3}$ and flipping the order must be done to each.

$$
\begin{align*}
\mathbf{L}^{2} & =\mathbf{x}^{2} \mathbf{p}^{2}-2 i \hbar \mathbf{x} \cdot \mathbf{p}-(\mathbf{x} \cdot \mathbf{p})\left(x_{j} p_{j}-3 i \hbar\right)  \tag{B.68}\\
& =\mathbf{x}^{2} \mathbf{p}^{2}+i \hbar \mathbf{x} \cdot \mathbf{p}-(\mathbf{x} \cdot \mathbf{p})^{2} .
\end{align*}
$$

The text is correct.

## JULIA NOTEBOOKS.

C

These Julia notebooks, can be found in
https://github.com/peeterjoot/julia.
These notebooks are text files. The julia program, available freely at www.julialang.org, is required to execute them. Some Julia code can also be evaluated with Matlab.

- Sep 24, 2015 phy1520/vectorSolenoid.jl

Plot of solenoid potential for constant interior magnetic field

- Nov 15, 2015 phy1520/rodbalancing.jl

Plug in some numbers for uncertainty time rod balancing calculation.

These Mathematica notebooks, some just trivial ones used to generate figures, others more elaborate, and perhaps some even polished, can be found in
https://github.com/peeterjoot/mathematica/tree/master/phy1520/.
The free Wolfram CDF player, is capable of read-only viewing these notebooks to some extent.

- Sep 28, 2015 anticommutingWithZeroEigenvalue.nb

Construct and verify an example of a pair of anticommuting matrices each having a zero eigenvalue, and a shared eigenvector. This was for Sakurai p.1.17, and includes some numeric commutator expansions to verify the example constructions done with that problem.

- Oct 18, 2015 ps3/ps3Fig2.nb

Plot some probability densities for ps3 p2.

- Oct 25, 2015 energyVsMomentumForDiracStepPotentialWaveFunctions.nb

A Manipulate to explore the energy vs momentum curves for a stepped potential barrier and the Dirac Hamiltonian.

- Nov 1, 2015 spinOneHalfSymbolicManipulation.nb

For a spin one-half system Sakurai leaves it to the reader (and also to problem 3.10) to verify that knowledge of the three ensemble averages $\left\langle S_{x}\right\rangle,\left\langle S_{y}\right\rangle,\left\langle S_{z}\right\rangle$ is sufficient to reconstruct the density operator. Showing that is algebraically messy, and well suited to do in Mathematica.

- Nov 2, 2015 eulerAngleVsNormalRotationPauliMatrices.nb

Comparing a unimodular transformation to an arbitrary rotation matrix was trivial. To do the same for an Euler angle rotation matrix and arbitrary matrix is less so because of sign differences. Here is a
dumb symbol expansion of the full rotation on a general vector in Pauli matrix form to have a look at the two results and see if there is anything striking. It turns out there is not anything striking. Both expressions are messy even after FullSimplify.

- Nov 16, 2015 spinOneOperatorRepresentation.nb

Spin one operators. Sakurai only seems to list the $S_{y}$ spin one matrix representation. See these in Desai, but need to verify that all my corrections to Desai eq. (27.117) are correct.

- Nov 19, 2015 diracDeltaFunctionIntegrals.nb

Double check the integral done in the Dirac delta function problem, and perform the first Fourier transformation.

- Nov 22, 2015 ps6/clebschGordan.nb

First pass of a Clebsch-Gordon computation engine, used to compute coefficients for spin one plus spin half, checking against what was done by hand for in class example.

- Nov 22, 2015 ps6/clebschGordan2.nb

Generalize and improve code to compute Clebsch-Gorgon coefficients, and apply to spin two plus spin one.

- Nov 24, 2015 spinOneDensityOperatorRepresentationEnsembleAverages.nb

Solution for Sakurai problem 3.12: What must we know in addition to the ensemble averages of the spin operators to completely characterize the density matrix of a spin 1 system? I guessed that we also needed the ensemble averages of at least the squares, but that and the unit trace, was not enough, since those relations do not provide nine equations, for the nine unknowns. Omitting the trace requirement, but also introducing variables for the other second order products was enough to solve the problem. Printing out all those products is helpful to show why this is the case: you can see visually that these products appear to span the space of $3 \times 3$ matrices.

- Nov 24, 2015 squareOfAngularMomentumOperatorSphericalCoordinates.nb

Expand out $\mathbf{L}$ in spherical coordinates in terms of $L_{x}, L_{y}, L_{z}$ operators and compare to stated result (Sakurai (6.15)).

- Nov 26, 2015 lecture 19integralsAndPlots.nb

Confirm some of the lecture 19 (variational method) integrals, and plot the energy distribution.

- Nov 28, 2015 ps7/ps7p3VariationalAbsoluteExponential.nb An attempt to do ps7 pr. 3 with Mathematica. It was hard to figure out how to make Mathematica cope with the delta functions, so I gave up and did it manually.
- Nov 28, 2015 ps7/doubleWellPotential.nb

Double well potential. ps 7 p1, part I.

- Nov 30, 2015 ps7/hydrogenAtomGroundState.nb

Verify all the hand calculations for ps7 p2, Hydrogen atom ground state. I found a better way to do the Hydrogen vs Helium ground state comparison, and rewrote all the content that this notebook applied to. This is, however, a nice standalone computation of the Hydrogen ground state energy.

- Nov 30, 2015 ps7/heliumAtomGroundStateWithInteraction.nb

Verify all the hand calculations for ps7 p2. Started with the Helium atom ground state calculation, without electron-electron interaction. This was followed up with a verification of the electron-electron interaction integrals and end results too.

- Dec 1, 2015 ps7/hydrogenComparedToHeliumGroundStateEnergy.nb

Recompute the Hydrogen vs. Helium ground state energy entirely in Mathematica to see if I am still getting the factor of 8 difference that I computed by hand. That factor of 8 turns out to be correct.

- Dec 2, 2015 visualizationOfEigenvaluesOfTwoByTwoHermitianMatrix.nb

Dynamic visualization of eigenvalues of $2 \times 2$ Hermitian matrix, as sketched in lecture 20, plus some plots with specific values.

- Dec 3, 2015 lecture21someSphericalHarmonicsAndTheirIntegrals.nb

Lecture 21. Print out some spherical harmonic functions and their integrals, to look at the conditions for the integral of $Y_{l m} z$ to be zero. Also evaluate the $z$ and $z^{2}$ brakets mentioned in the lecture.

- Dec 5, 2015 ps8/harmonicOscillatorRaiseAndLoweringOperators.nb Implement 1D SHO lowering and raising operator functions, and use to compute x operator powers on the ground state.
- Dec 6, 2015 harmonicOscillatorQuadradicPertubation.nb

Uses ps8/harmonicOscillatorRaiseAndLoweringOperators.nb to verify text computations of a small quadratic perturbation.

- Dec 6, 2015 ps8/2dHarmonicOscillatorOperators.nb

Implement 2D SHO lowering and raising operator functions, and use them to calculate $x^{4}+y^{4}$ and $x y$ operator actions on some two particle states, and some specific ps8 p3 related calculations.

- Dec 6, 2015 sakuraiProblem5.13c.nb

Sakurai. Problem 5.11 (c). Verify hand calculations (diagonalization).

- Dec 8, 2015 ps8/PauliMatrixSpinOperators.nb

Systematically compute the basis for spin-half $\boldsymbol{\sigma} \cdot \mathbf{n}$, and the matrix representation of $S_{z}$ in that basis. This was for ps 8 p 4 , hyperfine interaction.

- Dec 8, 2015 vanDerWalls.nb

Do a second order expansion of the van der Walls potential to compare to value stated. Did not figure out how to get Mathematica to simplify this gracefully and did it by hand manually instead.

- Dec 10, 2015 ps8/quadrupolarPotentialPorbitalSplitting.nb ps8 p2. Compute the matrix element for the p-orbital $(l=1)$ subset of $V=\lambda\left(x^{2}-y^{2}\right)$, diagonalize it, and find the diagonalizing basis states.
- Dec 10, 2015 lecture22Integrals.nb

Integrals from lecture 22 (van der Walls potential).

- Dec 11, 2015 sakuraiProblem3.17.nb
sakuraiProblem3.17
- Dec 13, 2015 sakuraiProblem5.17a.nb

Sakurai pr. 5.17 (a): Find energy eigenvalues for $H=A L^{2}+B L_{z}+$ $C L_{y}$.

- Dec 14, 2015 commutatorOfRadialPortionOfMomentumSquaredOperator.nb
Compute the commutator $\left[-\frac{\hbar^{2}}{2 m r^{2}} \partial_{r}\left(r^{2} \partial_{r}\right),-i \hbar \partial_{r}\right]=-\frac{\hbar^{2}}{2 m r^{2}} p_{r}$. The first operator is a component of $\mathbf{p}^{2} / 2 m$ with the $\mathbf{L}^{2}$ contribution removed.
- Dec 14, 2015 sakuraiProblem5.16bSHO.nb

Verify the relation from Sakurai pr. 5.16(a) using the ground state of the 3D SHO. Verification of the potential derivative expectations for higher values of n do not complete in reasonable times.

- Dec 15, 2015 sakuraiProblem3.33.nb

Spin three halves operator question (Sakurai 3.33). Implement operators for $S_{+}, S_{-}, S_{z}$ that act on kets $| \pm 3 / 2\rangle,| \pm 1 / 2\rangle$. These operators act on only a single basis element, but are used to construct the matrix representations for these operators (which are more general). Use those matrices to compute the representation of the Hamiltonian for the problem and compute its energy eigenvalues. Display the end result and the representations for all of $S_{+}, S_{-}, S_{x}, S_{y}, S_{z}, H=$ $A\left(3 S_{z}^{2}-\mathbf{S}^{2}\right)+B\left(S_{+}^{2}+S_{-}^{2}\right)$.

- Dec 15, 2015 spinMatrices.nb

Compute and display $S_{+}, S_{-}, S_{x}, S_{y}, S_{z}, S^{2}$ for a given spin (with $\hbar=1$ ). Use this to display the matrix representations of these operators for each of: spin $1 / 2$, spin 1 , spin $3 / 2$, spin 2 , spin $5 / 2$. Use this to solve the (spin one) eigensystem in Sakurai pr. 4.12.

- Dec 20, 2015 finalExamProblem6HyperfineInteraction.nb

Spin matrix and $S_{z}$ matrix relative to a specific normal vector $\hat{\mathbf{n}}$. This is based on PauliMatrixSpinOperators.nb from ps8 p4, hyperfine interaction, but switched up for the Fermion/Boson interaction final exam question p6.

- Dec 22, 2015 icePick.nb

Semi-classical treatment of ice-pick question from Sakurai.

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