PEETER JOOT

ADVANCED CLASSICAL OPTICS

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Notes and problems from UofT PHY485H1F 2012

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DOCUMENT VERSION

Version V0.1.12-5

Sources for this notes compilation can be found in the github repository

https://github.com/peeterjoot/phy485-optics

The last commit (Aug/22/2023), associated with this pdf was

566e6a159e1a5c9c340e0b68c390b78f2f75d30f

Should you wish to actively contribute typo fixes (or even more significant changes) to this book, you can do so by contacting me, or by forking your own copy of the associated git repositories and building the book pdf from source, and submitting a subsequent merge request.

```
#!/bin/bash

git clone git@github.com:peeterjoot/latex-notes-compilations.
    git peeterjoot
cd peeterjoot

submods="figures/phy485-optics phy485-optics mathematica latex
    "
for i in $submods ; do
   git submodule update --init $i
   (cd $i && git checkout master)
done

export PATH=`pwd`/latex/bin:$PATH

cd phy485-optics
make
```

I reserve the right to impose dictatorial control over any editing and content decisions, and may not accept merge requests as-is, or at all. That said, I will probably not refuse reasonable suggestions or merge requests.

Dedicated to:
Mom. You are a force of nature. You raised me by example, illustrating that any goal was possible once the decision to achieve it had been made, irrespective of any obstacles. The path to that goal might be a rough slog, but I knew that perseverance and hard work would get me there in the end.

PREFACE

This book is based on my lecture notes for the Fall 2012, University of Toronto Advanced Classical Optics course (PHY485H1F), taught by Prof. Joseph H. Thywissen.

My thanks to Professor Thywissen for teaching this course. He knows his subject well, and I learned a lot.

Official course description:

"This course builds on a student's knowledge of basic electromagnetic theory by focusing attention on light including elementary aspects of the propagation of optical beams and their interaction with matter. We examine light polarization, coherence, interference and diffraction as we move towards a description of lasers within a semiclassical picture in which the fields are treated classically and matter is treated quantum mechanically. In between we discuss Gaussian beam modes and their relation to optical resonators as well as fibre and slab waveguides."

In the course of the lectures the books [8], [5], and [2] are referred to, but none were required texts for the course.

This book contains a few things

- Plain old lecture notes.
- Personal notes exploring concepts from the lectures or texts.
- Assigned problems.

- Some worked problems attempted as course prep, for fun, or for exam preparation, or post exam review.
- Links to Mathematica workbooks associated with course content or these notes.

Peeter Joot peeterjoot@pm.me

CONTENTS

LIST OF FIGURES

1 MATRIX METHODS IN GEOMETRIC OPTICS.

1.1 MISSING CONTENT.

I was late. Glancing at somebody else's notes very quickly, it looked like I missed

- A derivation of Snell's law from Fermat's principle.
- The focal point formula

$$\frac{1}{f} = \frac{1}{s} + \frac{1}{s'}. \tag{1.1}$$

- The paraxial approximation was defined (small angles only, looking only near the center of a lens).

I was going to scrounge for notes to scan from somebody else so I can fill in missing content, but instead think I'll fill in with some self assigned problems.

Suggested reading for this lecture is §5 from [8].

1.2 MATRIX METHODS.

Referring to fig. 1.1 we can define a transfer out matrix, or $ABCD$

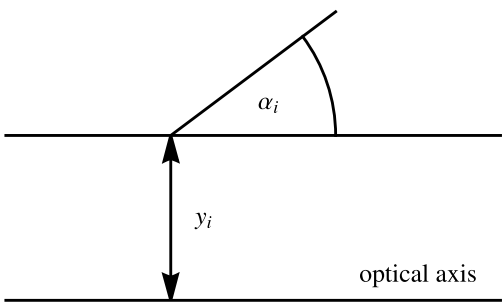

Figure 1.1: Matrix method.

matrix taking pairs of coordinates describing rays

$$\begin{bmatrix} y \\ \alpha \end{bmatrix} \tag{1.2}$$

so that the transition of the ray through the interface is described as

$$\begin{bmatrix} y_f \\ \alpha_f \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} y_i \\ \alpha_i \end{bmatrix} \tag{1.3}$$

1.2.1 *Free propagation.*

Referring to fig. 1.2 we see that

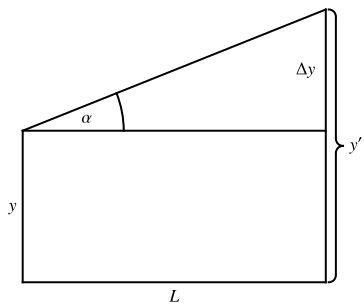

Figure 1.2: Free propagation.

$$\tan\alpha = \frac{\Delta y}{L} \approx \alpha, \tag{1.4}$$

so that

$$y' = y + \Delta y \approx y + L\alpha, \tag{1.5}$$

so that our matrix describing free propagation is

$$\begin{bmatrix} y_f \\ \alpha_f \end{bmatrix} = \begin{bmatrix} 1 & L \\ 0 & 1 \end{bmatrix} \begin{bmatrix} y_i \\ \alpha_i \end{bmatrix} \tag{1.6}$$

1.2.2 *Refraction off of a flat lens.*

Referring to fig. 1.3 where

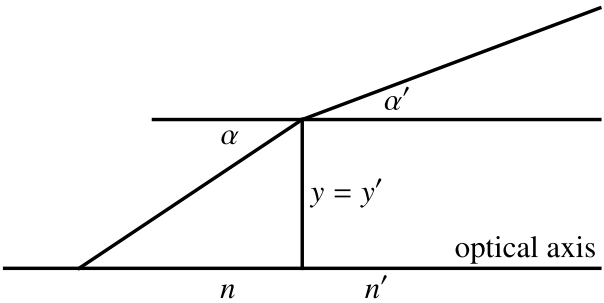

Figure 1.3: Refraction at flat surface.

$$n \sin \alpha = n' \sin \alpha'. \tag{1.7}$$

We employ the paraxial approximation

$$n\alpha \sim n'\alpha', \tag{1.8}$$

or

$$\alpha' = \frac{n'}{n}\alpha, \tag{1.9}$$

allowing for the description of refraction off of a flat lens by

$$\begin{bmatrix} y_f \\ \alpha_f \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & \frac{n'}{n} \end{bmatrix} \begin{bmatrix} y_i \\ \alpha_i \end{bmatrix} \tag{1.10}$$

1.2.3 *Refraction of a curved surface.*

1.2.3.1 *Convex refraction.*

Referring to fig. 1.4 we see that

$$\phi \approx \frac{y}{R}, \tag{1.11}$$

and can also employ Snell's law in the approximation

$$n\theta = n'\theta'. \tag{1.12}$$

From the figure we see that

$$\theta = \alpha + \phi. \tag{1.13a}$$

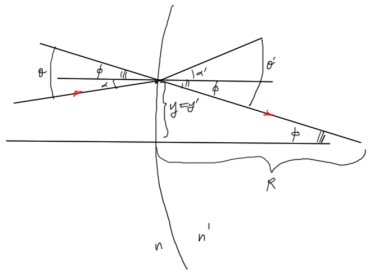

Figure 1.4: Refraction of convex surface.

$$\theta' = \alpha + \phi'. \tag{1.13b}$$

or

$$n(\alpha + \phi) = n'(\alpha' + \phi). \tag{1.14}$$

Using eq. (1.11) this is

$$n\left(\alpha + \frac{y}{R}\right) = n'\left(\alpha' + \frac{y}{R}\right), \tag{1.15}$$

which we can rearrange to find

$$\alpha' = \frac{y}{R}\left(\frac{n}{n'} - 1\right) + \frac{n}{n'}\alpha. \tag{1.16}$$

This can now be put into matrix form, yielding

$$\begin{bmatrix} y_f \\ \alpha_f \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ \frac{1}{R}\left(\frac{n}{n'} - 1\right) & \frac{n}{n'} \end{bmatrix} \begin{bmatrix} y_i \\ \alpha_i \end{bmatrix} \tag{1.17}$$

Observe that for $R \to \infty$ we have the same result as a flat surface.

1.2.3.2 *Concave refraction.*

Now, let's consider an input ray against a concave surface. We have just to flip around some of the labels as in fig. 1.5. From the figure the only difference is that the coordinate of the focus is now at $(-R, 0)$ whereas it was $(R, 0)$ before. So if we write $R' = -R$, with R being positive we find for a ray incident on a concave lens

$$\begin{bmatrix} y_f \\ \alpha_f \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ \frac{1}{R'}\left(\frac{n}{n'} - 1\right) & \frac{n}{n'} \end{bmatrix} \begin{bmatrix} y_i \\ \alpha_i \end{bmatrix} \tag{1.18}$$

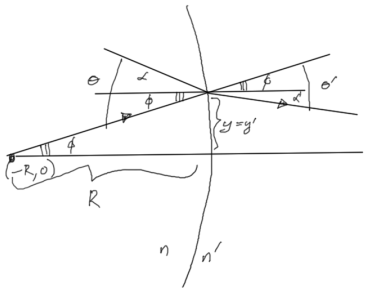

Figure 1.5: Refraction against concave surface.

Alternatively, and this is what we do, is that we allow for R to be signed, in which case this formula is true for both $R > 0$ (convex) and $R < 0$ (concave). We employ the following sign convention fig. 1.6 where $-R$ is used for concave, and $+R$ is used for convex.

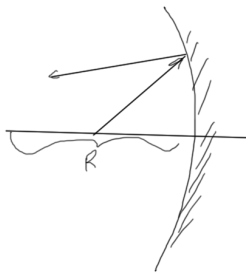

Figure 1.6: Concave curvature.

The sign conventions are illustrated in fig. 1.7 and fig. 1.8.

1.2.4 *ABCD matrix for a lens.*

Consider the lens illustrated in fig. 1.9 where once again we use the paraxial approximation, assuming we are considering only close enough to the middle that we can neglect any variation in thickness. In this approximation we have a geometry of the form fig. 1.10 where $y \ll R_1, R_2$.

Our complete transfer matrix is then given by

$$\begin{bmatrix} y_f \\ \alpha_f \end{bmatrix} = M_3 M_2 M_1 \begin{bmatrix} y_i \\ \alpha_i \end{bmatrix} = M \begin{bmatrix} y_i \\ \alpha_i \end{bmatrix}. \tag{1.19}$$

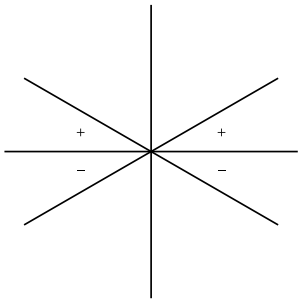

Figure 1.7: Sign convention.

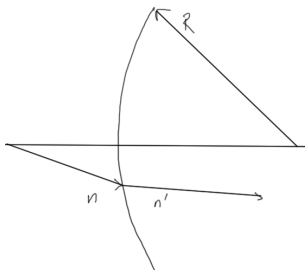

Figure 1.8: Convex curvature.

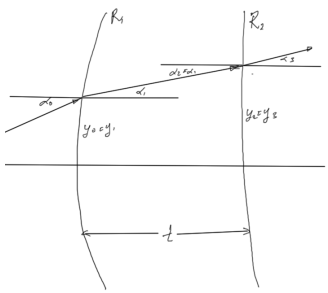

Figure 1.9: matrix for lens.

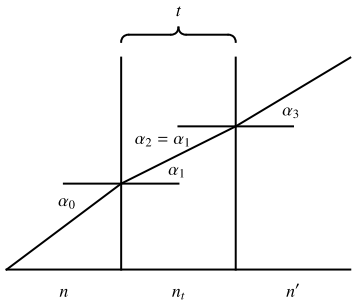

Figure 1.10: Lens paraxial approximation.

We've described M_1, M_2, M_3 individually above, and they are

$$M_1 = \begin{bmatrix} 1 & 0 \\ \frac{1}{R_1}\left(\frac{n}{n'} - 1\right) & \frac{n}{n'} \end{bmatrix} \tag{1.20a}$$

$$M_2 = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} \tag{1.20b}$$

$$M_3 = \begin{bmatrix} 1 & 0 \\ \frac{1}{R_2}\left(\frac{n'}{n} - 1\right) & \frac{n'}{n} \end{bmatrix} \tag{1.20c}$$

The matrix M is the mess

$$M = \begin{bmatrix} 1 & 0 \\ \frac{1}{R_2}\left(\frac{n'}{n} - 1\right) & \frac{n'}{n} \end{bmatrix} \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ \frac{1}{R_1}\left(\frac{n}{n'} - 1\right) & \frac{n}{n'} \end{bmatrix} \tag{1.21}$$

With only the ratio n'/n showing up, let's make the substitution

$$\frac{n'}{n} \to n, \tag{1.22}$$

effectively working with $n = 1$ outside of the lens. This gives us

$$\begin{aligned}
M &= \begin{bmatrix} 1 & 0 \\ \frac{n-1}{R_2} & n \end{bmatrix} \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ \frac{\frac{1}{n}-1}{R_1} & \frac{1}{n} \end{bmatrix} \\
&= \begin{bmatrix} 1 & t \\ \frac{n-1}{R_2} & \frac{t(n-1)}{R_2} + n \end{bmatrix} \begin{bmatrix} 1 & 0 \\ \frac{\frac{1}{n}-1}{R_1} & \frac{1}{n} \end{bmatrix} \\
&= \begin{bmatrix} 1 + \frac{t\left(\frac{1}{n}-1\right)}{R_1} & \frac{t}{n} \\ \frac{n-1}{R_2} - \frac{n-1}{R_1} + \frac{t(n-1)}{R_1 R_2}\left(\frac{1}{n} - 1\right) & \frac{t\left(-\frac{1}{n}+1\right)}{R_2} + 1 \end{bmatrix} \\
&= \begin{bmatrix} 1 + \frac{t\left(\frac{1}{n}-1\right)}{R_1} & t\frac{1}{n} \\ -(n-1)\left(\frac{1}{R_1} - \frac{1}{R_2} + \frac{t}{nR_1R_2}(n-1)\right) & 1 - \frac{t\left(\frac{1}{n}-1\right)}{R_2} \end{bmatrix}
\end{aligned} \tag{1.23}$$

Writing the C term as $-1/f$ we have what's called the Lens makers formula (for a thick lens in this case), and putting back in n and n' we have

$$\frac{1}{f} = \left(\frac{n'}{n} - 1\right)\left(\frac{1}{R_1} - \frac{1}{R_2} + \frac{nt}{n'R_1R_2}\left(\frac{n'}{n} - 1\right)\right). \tag{1.24}$$

Observe that for a thin lens where $t \to 0$ we have the approximation

$$M = \begin{bmatrix} 1 & 0 \\ -1/f & 1 \end{bmatrix}. \tag{1.25}$$

For our focus in eq. (1.24) we get

$$\frac{1}{f} = \frac{n' - n}{n}\left(\frac{1}{R_1} - \frac{1}{R_2}\right). \tag{1.26}$$

This is called the (thin lens) Lens makers formula. This, and $n \approx 1.5$ is enough to construct many lens designs. Our sign conventions for f are illustrated by fig. 1.11. where $f > 0$ is convex and $f < 0$ is concave.

1.2.5 *Properties of the transfer matrix.*

1. The determinant of the transfer matrix is described by index of refraction of just the initial and final media, and $\det M = n_0/n_f$. Examples

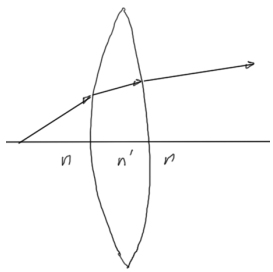

Figure 1.11: Lens.

a) Thin lens

$$M = \begin{bmatrix} 1 & 0 \\ -1/f & 1 \end{bmatrix} \tag{1.27}$$

b) Free propagation

$$M = \begin{bmatrix} 1 & L \\ 0 & 1 \end{bmatrix} \tag{1.28}$$

In particular, if imaging something in air, where we have $n_0 = n_f$ we have $|M| = 1$.

2. How about $|M| = 0$. There are a couple of cases. One is $D = 0$ where

$$\alpha_f = Cy_i + \cancel{D\alpha_i}. \tag{1.29}$$

output $\rightarrow$ input is focus, as illustrated in fig. 1.12.

3. We also have zero determinant when $A = 0$, in which case we have

$$y_f = \cancel{Ay_i} + B\alpha_i. \tag{1.30}$$

The output location is only a function of the input angle as illustrated in fig. 1.13.

4. How about if $B = 0$. Now we have

$$\alpha_f = \cancel{By_i} + C\alpha_i, \tag{1.31}$$

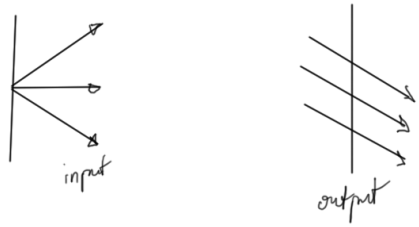

Figure 1.12: $D = 0$.

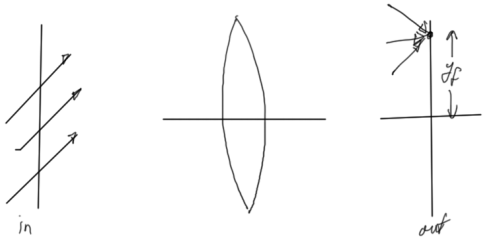

Figure 1.13: $A = 0$.

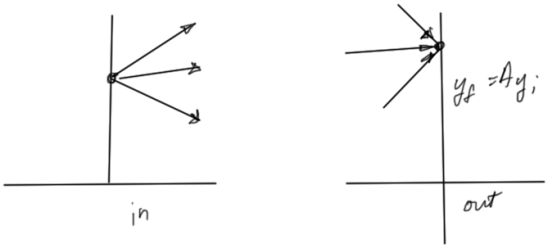

Figure 1.14: $B = 0$.

so that we see that the output is an image of the input, but scaled (a magnifier or reducer). This is illustrated in fig. 1.14.

5. And finally if $C = 0$ we have

$$\alpha_f = \cancel{Cy_i} + D\alpha_i, \tag{1.32}$$

and we find out system is telescopic, magnifying the angle, as illustrated in fig. 1.15.

Figure 1.15: $C = 0$.

1.3 PROBLEMS.

Exercise 1.1 Derive Snell's law.

Fermat's theorem, that light takes the path of least time, can be used to derive Snell's law without resorting to Maxwell's equations.

Note that a proof of Fermat's theorem using the Ray equation can be found in §3.3.2 [2].

Answer for Exercise 1.1

We refer to fig. 1.16, and seek to express the optical path length. Since $n(s) = c/v(s)$, the time spent along any portion of the path is proportional to $n(s)ds$. For the two leg linear route that is

$$OPL = nr + n'r'. \tag{1.33}$$

Since

$$r = \sqrt{h^2 + (L - x)^2}. \tag{1.34a}$$

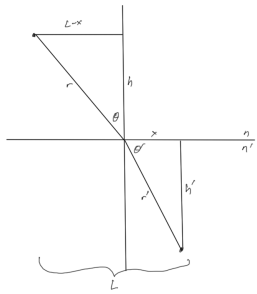

Figure 1.16: Snell's law light paths.

$$r' = \sqrt{h'^2 + x^2}. \tag{1.34b}$$

We want to find x such that

$$\begin{aligned} 0 &= \frac{d(OPL)}{dx} \\ &= \frac{d}{dx}\left(n\sqrt{h^2 + (L-x)^2} + n'\sqrt{h'^2 + x^2}\right) \\ &= n\frac{1}{2r}2(L-x)(-1) + n'\frac{1}{2r'}2x \\ &= -n\sin\theta + n'\sin\theta'. \end{aligned} \tag{1.35}$$

This gives us

$$\boxed{n\sin\theta = n'\sin\theta',} \tag{1.36}$$

as desired.

Exercise 1.2 Image distance for an ideal lens.

For the lens illustrated in fig. 1.17, use geometrical arguments to derive the image location formula

$$\frac{1}{s} + \frac{1}{s'} = \frac{1}{f}. \tag{1.37}$$

Answer for Exercise 1.2

Note that we have

$$\tan\theta = \frac{a}{x} = \frac{b}{f} = \frac{a+b}{s}. \tag{1.38a}$$

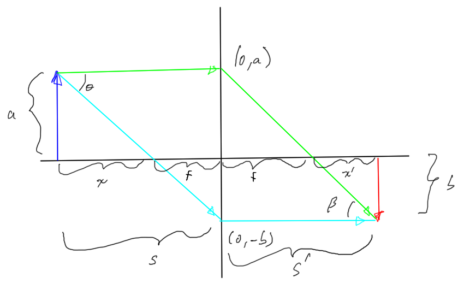

Figure 1.17: Thin paraxial lens with image in input and output conjugate planes.

$$\tan\beta = \frac{b}{x'} = \frac{a}{f} = \frac{a+b}{s'}. \tag{1.38b}$$

We have respectively

$$\frac{b}{a} = \frac{f}{x}. \tag{1.39a}$$

$$\frac{b}{a} = \frac{x'}{f}. \tag{1.39b}$$

Now we can eliminate the x and x' variables using $x = s - f$ and $x' = s' - f$

$$\frac{b}{a} = \frac{f}{s-f} = \frac{s'-f}{f}. \tag{1.40}$$

Rearranging we have

$$f^2 = (s'-f)(s-f) = ss' - fs - s'f + f^2, \tag{1.41}$$

or

$$ss' = fs + fs'. \tag{1.42}$$

Dividing through by fss' we have

$$\frac{1}{f} = \frac{1}{s'} + \frac{1}{s}, \tag{1.43}$$

as expected.

Exercise 1.3 Newton's lens focus formula.

Demonstrate the equivalence of Newton's lens focus formula to the inverse distance result shown above. Also referring to the figure where the distances x and x' were labeled, show that this is equivalent to

$$xx' = f^2. \tag{1.44}$$

Answer for Exercise 1.3

In the class notes, the magnification of the thin lens system described by eq. (1.43) was given by

$$m = -\frac{s'}{s}. \tag{1.45}$$

Other than the negation, this is a logical definition, the ratio of the output size with respect to the input size. I'm guessing that this is defined as negated because the image is inverted. From eq. (1.39b) we have

$$\frac{b}{a} = \frac{x'}{f}. \tag{1.46}$$

Dividing the last two equalities in eq. (1.38) we have

$$\frac{b}{a} = \frac{s'}{s}. \tag{1.47}$$

We can conclude that the magnification, expressed in x' and f is

$$m = -\frac{s'}{s} = \frac{x'}{f}, \tag{1.48}$$

and that fig. 1.17 with the distances as labeled describes the same system. The remainder of the task is therefore algebraic. We have

$$0 = -\frac{1}{f} + \frac{1}{s} + \frac{1}{s'} = -\frac{1}{f} + \frac{1}{x+f} + \frac{1}{x'+f}. \tag{1.49}$$

Multiplying through by $f(x+f)(x'+f)$ we have

$$\begin{aligned} 0 &= -(x+f)(x'+f) + f(x'+f) + f(x+f) \\ &= -xx' - \cancel{fx'} - \cancel{fx} - \cancel{f^2} + \cancel{fx'} + \cancel{f^2} + \cancel{fx} + f^2, \end{aligned} \tag{1.50}$$

or

$$xx' = f^2. \tag{1.51}$$

Let's try to solve exercise 1.4.

Exercise 1.4 Solve the geometry of a convex lens exactly.

Solve this lens geometry exactly and show how we obtain the ABCD matrix result in the limit.

Answer for Exercise 1.4

Our geometry is illustrated in fig. 1.18. We have using the law

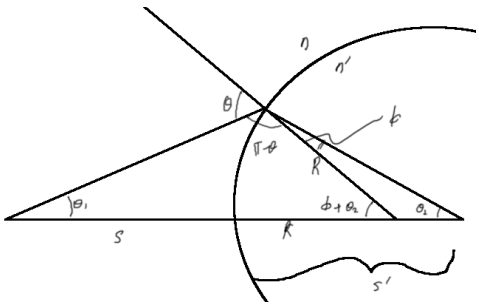

Figure 1.18: Convex lens refraction.

of sines

$$\frac{\sin\theta_1}{R} = \frac{\sin(\pi-\theta)}{s+R} = \frac{\sin\theta}{s+R}, \tag{1.52}$$

and also have

$$\frac{\sin\theta_2}{R} = \frac{\sin\phi}{s'-R}. \tag{1.53}$$

Dividing these we have a Snell's law ratio

$$\frac{n'}{n} = \frac{\sin\theta}{\sin\phi} = \frac{(s+R)\frac{\sin\theta_1}{R}}{(s'-R)\frac{\sin\theta_2}{R}}, \tag{1.54}$$

or

$$\boxed{\frac{n'}{n} = \frac{(s+R)}{(s'-R)}\frac{\sin\theta_1}{\sin\theta_2}.} \tag{1.55}$$

This is the exact result. Let's verify that this matches our paraxial ABCD matrix result.

Introducing signed angles

$$\alpha = \theta_1. \tag{1.56a}$$

$$\alpha' = -\theta_2. \tag{1.56b}$$

The paraxial approximation gives

$$s' = \frac{y}{\sin\theta_2} \sim -\frac{y}{\alpha'} \tag{1.57a}$$

$$s = \frac{y}{\sin\theta_1} \sim \frac{y}{\alpha} \tag{1.57b}$$

$$\frac{n'}{n} = -\frac{(s+R)}{(s'-R)}\frac{\alpha}{\alpha'} = -\frac{\left(\frac{y}{\alpha}+R\right)}{\left(-\frac{y}{\alpha'}-R\right)}\frac{\alpha}{\alpha'} = \frac{(y+R\alpha)}{(y+R\alpha')}. \tag{1.58}$$

We want to solve for α', and find

$$\alpha' = -\frac{y}{R} + \frac{n}{n'}\left(\alpha + \frac{y}{R}\right) = \frac{y}{R}\left(\frac{n}{n'} - 1\right) + \frac{n}{n'}\alpha. \tag{1.59}$$

With $y = y'$ we have in matrix form

$$\boxed{\begin{bmatrix} y' \\ \alpha' \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ \frac{1}{R}\left(\frac{n}{n'} - 1\right) & \frac{n}{n'} \end{bmatrix} \begin{bmatrix} y \\ \alpha \end{bmatrix},} \tag{1.60}$$

which is the ABCD result as desired.

Exercise 1.5 Solve the geometry of a concave spherical mirror.

After solving, apply the paraxial approximation to find the ABCD matrix result.

Answer for Exercise 1.5

Our system is illustrated in fig. 1.19. From the figure, employing the law of sines, we have

$$t\sin\theta = r\sin\alpha = (r - s')\sin\theta_2, \tag{1.61}$$

or

$$\frac{r - s'}{\sqrt{s^2 + y^2}}\sin\theta_2 = \sin(\gamma + \theta_1) = \sin\gamma\cos\theta_1 + \cos\gamma\sin\theta_1, \tag{1.62}$$

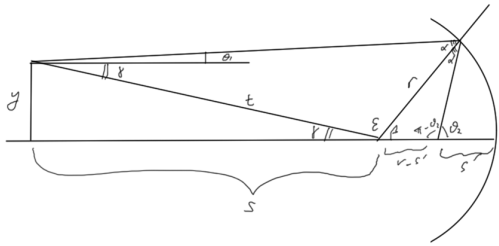

Figure 1.19: Concave spherical reflector.

but since

$$\gamma = \operatorname{atan}\frac{y}{s}, \tag{1.63}$$

and

$$\cos\operatorname{atan} x = \frac{1}{\sqrt{1+x^2}}. \tag{1.64a}$$

$$\sin\operatorname{atan} x = \frac{x}{\sqrt{1+x^2}}, \tag{1.64b}$$

we have

$$\begin{aligned}\frac{r-s'}{\sqrt{s^2+y^2}}\sin\theta_2 &= \frac{y/s}{\sqrt{1+(y/s)^2}}\cos\theta_1 + \frac{1}{\sqrt{1+(y/s)^2}}\sin\theta_1 \\ &= \frac{y}{\sqrt{s^2+y^2}}\cos\theta_1 + \frac{s}{\sqrt{s^2+y^2}}\sin\theta_1,\end{aligned} \tag{1.65}$$

or

$$\boxed{(r-s')\sin\theta_2 = y\cos\theta_1 + s\sin\theta_1.} \tag{1.66}$$

This is the exact result desired. Application of the paraxial approximation gives us

$$(r-s')\theta_2 \sim y + s\theta_1. \tag{1.67}$$

With

$$\theta_2 \sim \frac{y'}{s'}. \tag{1.68a}$$

$$\theta_1 \sim \frac{\Delta y}{r+s}. \tag{1.68b}$$

we have

$$s\theta_1 \sim -r\theta_1 + \Delta y. \tag{1.69}$$

$$s'\theta_2 \sim y'. \tag{1.70}$$

$$r\theta_2 - y' \sim y - r\theta_1 + \Delta y, \tag{1.71}$$

or

$$\theta_2 \sim \frac{2}{r}y' - \theta_1. \tag{1.72}$$

We need to fix the sign conventions for the ABCD matrices, so write

$$\alpha' = -\theta_2. \tag{1.73a}$$

$$R = -r. \tag{1.73b}$$

$$\alpha = \theta_1. \tag{1.73c}$$

A final substitution into eq. (1.72) gives us

$$\alpha' \sim \frac{2}{R}y' + \alpha. \tag{1.74}$$

or in matrix form

$$\begin{bmatrix} y' \\ \alpha' \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ \frac{2}{R} & 1 \end{bmatrix} \begin{bmatrix} y \\ \alpha \end{bmatrix}. \tag{1.75}$$

This is the ABCD matrix we were given in class.

Exercise 1.6 ABCD Matrices. (*2012 Ps1, P1*)

Using the ABCD matrices from the lecture, prove these well-known rules of geometric optics. In each case, **make an illustration**, tracing some important rays that illustrate the rule.

(a) *An image is formed when* $1/f = 1/s_o + 1/s_i$. Solve this problem using the result we found in class: when B=0 for a system matrix, the input and output are conjugate planes.

(b) *An image with magnification* $-x'/f$ *is formed when* $xx' = f^2$. Repeat part (a), but in "Newton's form": replace s_o with $f + x$, and replace s_i with $f + x'$.

(c) *The position distribution at the focus of a lens is the angular position of the incident beam.* (In other words, a lens does a kind of Fourier transform, as you may know already.) Find where the input plane has to be located for $y_o = f\alpha_i$.

(d) *Two identical lenses spaced by* $2f$ *image an object at* f *with unity magnification.*

(e) Two identical lenses spaced by $2f$ are *telecentric*, meaning that an object at $f + x$ from the first lens has a magnification independent of x, in contrast to a simple lens.

(f) A lens and a flat mirror spaced by distance f create a *cat's eye*. What are its properties? Consider, in particular, an emitter located f in front of the Cat's eye and located at $y_i = 0$.

Answer for Exercise 1.6

(a) Our system and the associated transfer matrices labels are illustrated in fig. 1.20. We form the system transfer matrix

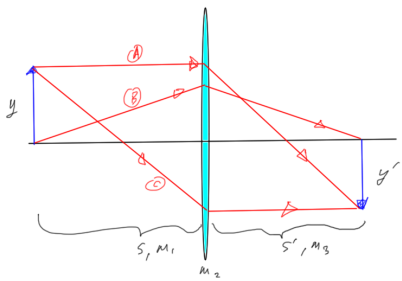

Figure 1.20: Input and output conjugate planes for paraxial thin lens.

by applying first a free propagation matrix, then a thin lens paraxial matrix, and one more free propagation matrix

$$
\begin{aligned}
M &= M_3 M_2 M_1 \\
&= \begin{bmatrix} 1 & s' \\ 01 & \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -1/f & 1 \end{bmatrix} \begin{bmatrix} 1 & s \\ 01 & \end{bmatrix} \\
&= \begin{bmatrix} 1 - s'/f & s' \\ -1/f & 1 \end{bmatrix} \begin{bmatrix} 1 & s \\ 01 & \end{bmatrix} \\
&= \begin{bmatrix} 1 - s'/f & s + s' - ss'/f \\ -1/f & -s/f + 1 \end{bmatrix}.
\end{aligned} \tag{1.76}
$$

Consider ray (B) from the figure, where we have

$$
\begin{bmatrix} 0 \\ \alpha \end{bmatrix} \to \alpha \begin{bmatrix} s + s' - ss'/f \\ -s/f + 1 \end{bmatrix} = \begin{bmatrix} 0 \\ \alpha' \end{bmatrix}. \tag{1.77}
$$

With

$$
y = y' = \alpha(s + s' - ss'/f) = 0, \tag{1.78}
$$

for all α. We must have

$$
s + s' = \frac{ss'}{f}. \tag{1.79}
$$

Dividing through by ss' we have

$$
\boxed{\frac{1}{s'} + \frac{1}{s} = \frac{1}{f},} \tag{1.80}
$$

as expected.

(b) Let's consider the system as the compound action of three transfer matrices as illustrated in fig. 1.21, this time labeling the figure in terms of the variables for this problem. Com-

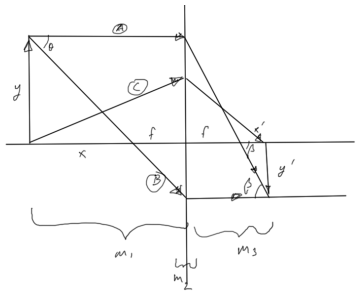

Figure 1.21: Newton's form, an image with magnification.

pounding the transfer matrices we have

$$
\begin{aligned}
M &= M_3 M_2 M_1 \\
&= \begin{bmatrix} 1 & x'+f \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -1/f & 1 \end{bmatrix} \begin{bmatrix} 1 & x+f \\ 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} -x'/f & x'+f \\ -1/f & 1 \end{bmatrix} \begin{bmatrix} 1 & x+f \\ 0 & 1 \end{bmatrix} \\
&= -\frac{1}{f} \begin{bmatrix} x' & xx'-f^2 \\ 1 & x \end{bmatrix}.
\end{aligned} \tag{1.81}
$$

Consider the ray A where the effect is

$$
\begin{bmatrix} y \\ 0 \end{bmatrix} \to -\frac{1}{f} \begin{bmatrix} yx' \\ y \end{bmatrix}. \tag{1.82}
$$

We see that $y' = -yx'/f$ or

$$
\boxed{m = -\frac{x'}{f} = \frac{y'}{y}.} \tag{1.83}
$$

The quantity defined as the magnification is in fact the ratio of the output to the input size as intuitively expected. Now consider a ray C originating at $y = 0$ at the image source,

and landing at $y = 0$ on the conjugate output plane. For this ray we have

$$\begin{bmatrix} 0 \\ \alpha \end{bmatrix} \to -\frac{1}{f} \begin{bmatrix} xx' - f^2 \\ x \end{bmatrix} \theta = \begin{bmatrix} 0 \\ \alpha' \end{bmatrix}. \tag{1.84}$$

Since this holds for all input angles originating at $y = 0$ from the input plane, we must have

$$\boxed{xx' = f^2,} \tag{1.85}$$

as desired.

(c) Here we refer to fig. 1.22, this time considering no ray that passes the focus past the lens. Our system transfer matrix, given the reduced free propagation distance past the lens is

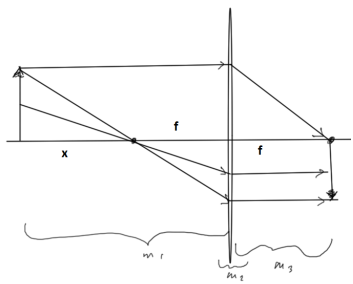

Figure 1.22: Position distribution at the focus of a lens.

$$\begin{aligned} M &= M_3 M_2 M_1 \\ &= \begin{bmatrix} 1 & f \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -1/f & 1 \end{bmatrix} \begin{bmatrix} 1 & x+f \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 0 & f \\ -1/f & 1 \end{bmatrix} \begin{bmatrix} 1 & x+f \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 0 & f \\ -1/f & -x/f \end{bmatrix}. \end{aligned} \tag{1.86}$$

A ray is transformed according to

$$\begin{bmatrix} y \\ \theta \end{bmatrix} \to \begin{bmatrix} 0 & f \\ -1/f & -x/f \end{bmatrix} \begin{bmatrix} y \\ \theta \end{bmatrix} = \begin{bmatrix} f\theta \\ -\frac{1}{f}(y - x\theta) \end{bmatrix}. \tag{1.87}$$

In particular

$$\boxed{y' = f\theta,} \tag{1.88}$$

demonstrating the claim that at the focus, the position is an angular distribution of the incident beam. This is clearly independent of x so the input plane position is irrelevant.

(d) Consider fig. 1.23. The transfer matrix $M = M_5 M_4 M_3 M_2 M_1$

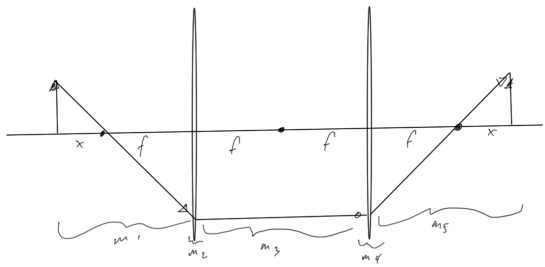

Figure 1.23: Two identical lenses separated by twice focus.

for the system is

$$\begin{aligned}
M &= M_5 M_4 M_3 M_2 M_1 \\
&= \begin{bmatrix} 1 & x+f \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -1/f & 1 \end{bmatrix} \begin{bmatrix} 1 & 2f \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -1/f & 1 \end{bmatrix} \begin{bmatrix} 1 & x+f \\ 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} -x/f & x+f \\ -1/f & 1 \end{bmatrix} \begin{bmatrix} 1 & 2f \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & x+f \\ -1/f & -x/f \end{bmatrix} \\
&= \begin{bmatrix} -x/f & -x+f \\ -1/f & -1 \end{bmatrix} \begin{bmatrix} 1 & x+f \\ -1/f & -x/f \end{bmatrix} \\
&= \begin{bmatrix} -1 & -2x \\ 0 & -1 \end{bmatrix}.
\end{aligned} \tag{1.89}$$

Consider any ray from the source going towards the lens along the horizontal. We have

$$\begin{bmatrix} y \\ 0 \end{bmatrix} \to \begin{bmatrix} -y \\ 0 \end{bmatrix}, \tag{1.90}$$

The ratio of the output to the input height to be

$$\boxed{\frac{y'}{y} = -1,} \tag{1.91}$$

which is the unit magnitude magnification as desired.

(e) This is actually demonstrated above.

(f) Here we consider fig. 1.24. Our system transfer matrix is

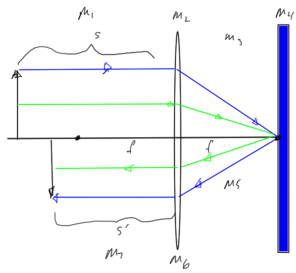

Figure 1.24: Cat's eye. Lens with mirror behind at focus.

$$\begin{aligned}
M &= M_7 M_6 M_5 M_4 M_3 M_2 M_1 \\
&= \begin{bmatrix} 1 & s' \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -1/f & 1 \end{bmatrix} \begin{bmatrix} 1 & f \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & f \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -1/f & 1 \end{bmatrix} \begin{bmatrix} 1 & s \\ 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} 1 - s'/f & s' \\ -1/f & 1 \end{bmatrix} \begin{bmatrix} 1 & 2f \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & s \\ -1/f & -s/f + 1 \end{bmatrix} \\
&= \begin{bmatrix} 1 - s'/f & 2f - s' \\ -1/f & -1 \end{bmatrix} \begin{bmatrix} 1 & s \\ -1/f & -s/f + 1 \end{bmatrix} \\
&= \begin{bmatrix} -1 & 2f - s' - s \\ 0 & -1 \end{bmatrix}
\end{aligned} \tag{1.92}$$

We see that the angle of the output light is unchanged except for sign, so we have no scattering in the paraxial limit. Observe that if the emitter is positioned at $s = f$ we have

$$M = \begin{bmatrix} -1 & f - s' \\ 0 & -1 \end{bmatrix} \tag{1.93}$$

so

$$y' = -y + (f - s')\alpha. \tag{1.94}$$

The image is magnified (negatively) for any position $|s'| > f$ without any angular distortion. In fact, if the observation is also made at the focus, then the image magnification is unity. Notice that at the focus we have both a sign change in the position and the angle coordinate, meaning that the output image is exactly the same as in the input image. In retrospect, this is exactly the same system mathematically as the $2f$ spaced lenses of parts (d) and (e), and we could have done the matrix products just once for all those parts of the problem!

2 GEOMETRIC OPTICS: RAYS AND OPTICS WITH GRADED INDEX.

2.1 READING.

Reading: §3.1.1 [2]

2.2 EIKONAL EQUATION.

We want to find the rays in Maxwell's equations.

$$\begin{aligned}
\nabla \cdot \mathbf{D} &= 0 \\
\nabla \cdot \mathbf{B} &= 0 \\
\nabla \times \mathbf{E} &= -\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t} \\
\nabla \times \mathbf{B} &= \frac{1}{c}\frac{\partial \mathbf{D}}{\partial t}.
\end{aligned} \tag{2.1}$$

Assume

1. Material has no magnetic dependence ($\mu = 1$) so that we have [1]

$$n = \sqrt{\epsilon}. \tag{2.4}$$

We'll neglect loss, with zero for the imaginary part of n

1 Note the choice of units here, with no c in the definition of n. In SI we'd write

$$n = \frac{c}{v} = \frac{c}{1/\sqrt{\epsilon\mu}} \approx c\sqrt{\epsilon}. \tag{2.2}$$

however, with these units (and $\mu = 1$) the wave equation operator takes the form

$$\nabla^2 - \frac{\epsilon}{c^2}\partial_{tt}. \tag{2.3}$$

From this we deduce the wave velocity is $c/\sqrt{\epsilon}$, and then find $n = c/v$ matches eq. (2.4) above.

2. Short wavelength limit $\lambda \ll d$, any other length scale in problem.

If n = constant, we know that plane waves are solutions. Try

$$\begin{bmatrix} \mathbf{E} \\ \mathbf{B} \end{bmatrix} = \begin{bmatrix} \mathbf{E}_0(\mathbf{r}) \\ \mathbf{B}_0(\mathbf{r}) \end{bmatrix} e^{i\phi(\mathbf{r}) - i\omega t}. \tag{2.5}$$

We know that for plane waves we'll have

$$\begin{aligned} \mathbf{E}_0(\mathbf{r}) &\to \mathbf{E} \\ \mathbf{B}_0(\mathbf{r}) &\to \mathbf{B} \\ \phi(\mathbf{r}) &\to \mathbf{k} \cdot \mathbf{r}. \end{aligned} \tag{2.6}$$

The time derivatives are

$$\frac{1}{c}\frac{\partial}{\partial t}\begin{bmatrix} \mathbf{E} \\ \mathbf{B} \end{bmatrix} = -i\frac{\omega}{c}\begin{bmatrix} \mathbf{E} \\ \mathbf{B} \end{bmatrix} = -ik_0 \begin{bmatrix} \mathbf{E} \\ \mathbf{B} \end{bmatrix}. \tag{2.7}$$

For the spatial derivatives we have

$$\boldsymbol{\nabla} \cdot \mathbf{E} = e^{-i\omega t}\left(\overbrace{\boxed{e^{i\phi(\mathbf{r})}\boldsymbol{\nabla} \cdot \mathbf{E}_0(\mathbf{r})}}^{\text{neglect this}} + \mathbf{E}_0(\mathbf{r}) \cdot \left(\boldsymbol{\nabla} e^{i\phi(\mathbf{r})}\right)\right), \tag{2.8}$$

and

$$\boldsymbol{\nabla} \times \mathbf{E} = e^{-i\omega t}\left(\overbrace{\boxed{e^{i\phi(\mathbf{r})}\boldsymbol{\nabla} \times \mathbf{E}_0(\mathbf{r})}}^{\text{neglect this}} - \mathbf{E}_0(\mathbf{r}) \times \left(\boldsymbol{\nabla} e^{i\phi(\mathbf{r})}\right)\right). \tag{2.9}$$

We can computing the phase gradient directly

$$\begin{aligned} \boldsymbol{\nabla} e^{i\phi} &= \mathbf{e}_m \partial_m e^{i\phi} \\ &= i\mathbf{e}_m \partial_m \phi e^{i\phi} \\ &= i(\boldsymbol{\nabla}\phi)e^{i\phi}, \end{aligned} \tag{2.10}$$

and use this to justify the neglect of the gradients products of $\mathbf{E}_0$ above. Since

$$\frac{1}{E_0^n}\frac{dE_0^n}{dx_p} \ll \frac{1}{\lambda}. \tag{2.11}$$

for all $x_p \in \{x, y, z\}$. Since

$$\nabla\phi \approx k_0 \sim \frac{2\pi}{\lambda}. \tag{2.12}$$

we see that the non-neglected terms above are of order $|\mathbf{E}_0|/\lambda$, which justifies the action.

This leaves us with

$$\nabla \cdot \mathbf{E} \approx ie^{i\phi - i\omega t}\mathbf{E}_0(\mathbf{r}) \cdot \nabla\phi. \tag{2.13}$$

$$\nabla \times \mathbf{E} \approx -ie^{i\phi - i\omega t}\mathbf{E}_0(\mathbf{r}) \times \nabla\phi. \tag{2.14}$$

Maxwell's equations now take the form

$$\mathbf{E}_0 \cdot \nabla\phi = 0. \tag{2.15a}$$

$$\mathbf{B}_0 \cdot \nabla\phi = 0. \tag{2.15b}$$

$$\nabla\phi \times \mathbf{E}_0 = k_0\mathbf{B}_0. \tag{2.15c}$$

$$\nabla\phi \times \mathbf{B}_0 = -\epsilon k_0\mathbf{E}_0. \tag{2.15d}$$

Crossing $\nabla\phi$ with eq. (2.15c) we have

$$\begin{aligned} \nabla\phi \times (\nabla\phi \times \mathbf{E}_0) &= k_0(\nabla\phi \times \mathbf{B}_0) \\ \nabla\phi\,(\cancel{\nabla\phi \cdot \mathbf{E}_0}) - \mathbf{E}_0(\nabla\phi)^2 &= -\epsilon k_0^2\mathbf{E}_0 \end{aligned} \tag{2.16}$$

This is called the Eikonal equation and can be written as

$$|\nabla\phi|^2 = k_0^2\epsilon(\mathbf{r}). \tag{2.17}$$

or

$$\boxed{|\nabla\phi| = k_0 n(\mathbf{r})} \tag{2.18}$$

If n = constant

$$|\nabla\phi| = k_0 n. \tag{2.19}$$

This can be illustrated as in fig. 2.1. If $n \neq$ constant only locally would we have plane waves as in fig. 2.2.

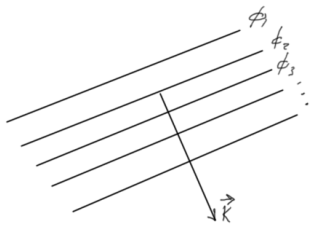

Figure 2.1: Plane waves for constant index of refraction.

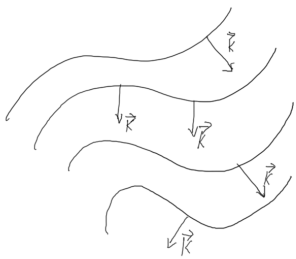

Figure 2.2: Plane waves only locally with variation of index of refraction.

2.3 POYNTING VECTOR.

How about the Poynting vector? This is the direction of the "ray", the direction of the transport of energy and momentum. That is

$$\mathbf{S} = \frac{c}{4\pi} \operatorname{Re} \mathbf{E} \times \operatorname{Re} \mathbf{B}, \tag{2.20}$$

and after some math, taking the average we have

$$\langle \mathbf{S} \rangle_{\text{time}} = \frac{c}{8\pi k_0} |\mathbf{E}_0|^2 \nabla \phi. \tag{2.21}$$

We see that the rays point along $\nabla \phi$.

2.4 RAY EQUATION.

Referring to fig. 2.3 we let

$$s = \text{distance along ray}. \tag{2.22}$$

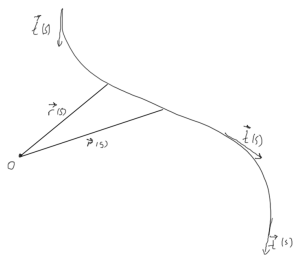

Figure 2.3: Unit tangents on a curve.

$$\mathbf{t} = \text{tangent} = \frac{d\mathbf{r}(s)}{ds}. \tag{2.23}$$

The unit vector, parallel to $\nabla\phi$ is

$$\begin{aligned} \frac{d\mathbf{r}(s)}{ds} = \mathbf{t} &= \frac{\nabla\phi}{|\nabla\phi|} \\ &= \frac{\nabla\phi}{n(\mathbf{r})k_0}, \end{aligned} \tag{2.24}$$

So we have

$$n(\mathbf{r})\frac{d\mathbf{r}}{ds} = \frac{1}{k_0}\nabla\phi. \tag{2.25}$$

We'd like to get rid of the pesky dependence on the phase. Let's take another derivative to attempt to get rid of $\nabla\phi$. Will this work?

$$\begin{aligned} \frac{d}{ds}\left(n(\mathbf{r})\frac{d\mathbf{r}}{ds}\right) &= \frac{1}{k_0}\frac{d}{ds}\nabla\phi \\ &= \frac{1}{k_0}\left(\frac{d\mathbf{r}}{ds}\cdot\nabla\right)\nabla\phi \\ &= \frac{1}{k_0}\left(\frac{1}{k_0 n(\mathbf{r})}\nabla\phi\cdot\nabla\right)\nabla\phi \end{aligned} \tag{2.26}$$

Here we've used the convective derivative

$$\frac{d}{ds} = \frac{\partial}{\partial s} + \frac{\partial\mathbf{r}}{\partial s}\cdot\nabla = \frac{d\mathbf{r}}{ds}\cdot\nabla. \tag{2.27}$$

In exercise 2.2 we show that

$$(\nabla\phi \cdot \nabla)\nabla\phi = k_0^2 n \nabla n, \tag{2.28}$$

which gives us the Ray equation

$$\boxed{\frac{d}{ds}\left(n(\mathbf{r})\frac{d\mathbf{r}}{ds}\right) = \nabla n(\mathbf{r})} \tag{2.29}$$

Note that this almost looks like a $F = ma$ type of equation with time parameterization replaced by arc length along the ray (should we ignore the index of refraction on the LHS), and also ignore the lack of a minus sign.

The lack of minus sign we can interpret as something like "bending to higher n".

2.5 GRIN (GRADED REFRACTIVE INDEX) OPTICS.

With a constant index we have

$$\frac{d}{ds}\left(n\frac{d\mathbf{r}}{ds}\right) = \nabla n = 0. \tag{2.30}$$

So

$$\frac{d^2}{ds^2}\mathbf{r}(s) = 0. \tag{2.31}$$

Integrating twice we see this is the straight ray that we expect

$$\mathbf{r} = s\mathbf{a} + \mathbf{r}_0. \tag{2.32}$$

We can compute the unit tangent

$$\frac{d\mathbf{r}}{ds} = \mathbf{a} = \frac{1}{nk_0}\nabla\phi, \tag{2.33}$$

finding that our constant vector $\mathbf{a}$, in this case, is a unit vector in the direction of the gradient. This shows that the gradient of the phase lies along the ray path of wave front.

The ray $\mathbf{r}(t)$ for a fixed phase front (not a general expression) can be implicitly defined by

$$\phi(\mathbf{r}, t) = \omega t, \tag{2.34}$$

Or more generally

$$\mathbf{r} \cdot \nabla \phi(\mathbf{r}) = \omega t. \tag{2.35}$$

We find

$$\mathbf{a} \cdot \mathbf{r} = \frac{\omega t}{n\omega/c} = \frac{c}{n} t = vt, \tag{2.36}$$

so the phase front of the wave moves with speed c/n along the ray direction.

2.6 TRAP A RAY.

Let's have some fun with non-constant n. Can we trap a ray of light as in fig. 2.4? If we have a circular trajectory

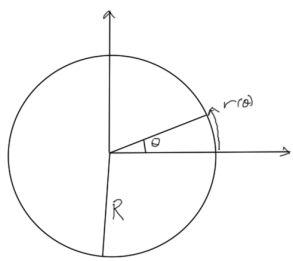

Figure 2.4: Ray trap.

$$\mathbf{r} = R \begin{bmatrix} \cos\theta(s) \\ \sin\theta(s) \\ 0 \end{bmatrix}. \tag{2.37}$$

We can imagine any sort of variation of n with $\mathbf{r}$, such as fig. 2.5, but we want to figure out exactly what $n(\mathbf{r})$ has to be. We can do so by plugging into eq. (2.29). We'll assume that $n(\mathbf{r})$ is radially symmetric, so that given the constant radius of the ray (a circle) we have $dn/ds = 0$. This gives

$$\frac{d^2}{ds^2}\mathbf{r}(s) = \frac{\nabla n(\mathbf{r})}{n(\mathbf{r})}, \tag{2.38}$$

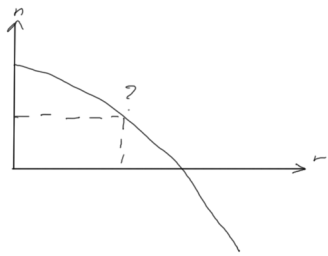

Figure 2.5: Imagined possible relationship between index of refraction and position..

Taking derivatives, we have

$$\frac{d\mathbf{r}}{ds} = R \begin{bmatrix} -\sin\theta(s) \\ \cos\theta(s) \\ 0 \end{bmatrix} \frac{d\theta}{ds}. \tag{2.39}$$

Since $s = R\theta$, we have $d\theta/ds = 1/R$, and

$$\frac{d\mathbf{r}}{ds} = \begin{bmatrix} -\sin\theta(s) \\ \cos\theta(s) \\ 0 \end{bmatrix}. \tag{2.40}$$

Taking the next derivative we have

$$\frac{d^2\mathbf{r}}{ds^2} = \begin{bmatrix} -\cos\theta(s) \\ -\sin\theta(s) \\ 0 \end{bmatrix} \frac{1}{R} = -\frac{1}{R^2}\mathbf{r}. \tag{2.41}$$

We find

$$\frac{d^2\mathbf{r}}{ds^2} = \frac{\nabla n}{n}. \tag{2.42}$$

or

$$\nabla n = -n\frac{\mathbf{r}}{R^2}. \tag{2.43}$$

Trap your own ray today!

2.7 GRADIUM LENS.

Reading : §6.4 [8].

We'll use Fermat's theorem

Definition 2.1: Fermat's theorem

The pathlength is the same for all rays.

Looking to fig. 2.6 we will now consider a cylindrical lens constructed out of a non-uniform index material. Suppose we have an index of refraction that is distributed parabolicly

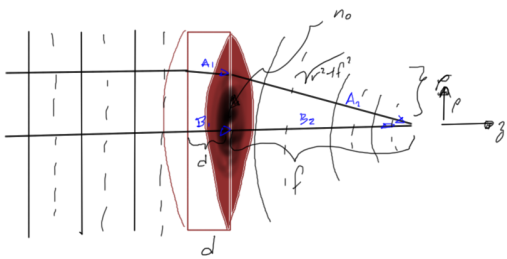

Figure 2.6: Gradium lens.

$$n(\boldsymbol{\rho}) = n_0 - \frac{1}{2}\alpha\boldsymbol{\rho}^2. \tag{2.44}$$

Consider the ray paths that pass through the lens at $\rho = 0$. These will have pathlength

$$OPL|_{\rho=0} = OPL_{B_1} + OPL_{B_2} = dn_0 + f. \tag{2.45}$$

Note that we assume $n = 1$ outside of the lens so that portion of the pathlength is f and not $n_{\text{outside}}f$. A ray passing through the lens at $\rho \neq 0$ will have a pathlength of approximately (neglecting any curvature or ray angle within the lens)

$$OPL|_{\rho\neq 0} = OPL_{A_1} + OPL_{A_2} = dn(\rho) + \sqrt{f^2 + \rho^2}. \tag{2.46}$$

For these to come together to the same focal point, the two pathlengths must be equal (Fermat's theorem), so we have

$$n(\rho) = -\frac{1}{d}\sqrt{f^2 + \rho^2} + n_0 + \frac{f}{d}. \tag{2.47}$$

Assuming a paraxial approximation where $\rho \ll f$ we have

$$\begin{aligned} n(\rho) &= n_0 + \frac{f}{d}\left(1 - \sqrt{1 + \left(\frac{\rho}{f}\right)^2}\right) \\ &\sim n_0 + \frac{f}{d}\left(1 - \left(1 + \frac{1}{2}\left(\frac{\rho}{f}\right)^2\right)\right) \\ &= n_0 - \frac{1}{2}\frac{\rho^2}{fd}. \end{aligned} \tag{2.48}$$

We write

$$\alpha = \frac{1}{2fd}, \tag{2.49}$$

and seek to solve the Ray equation

$$\frac{d}{ds}\left(n(\rho)\frac{d}{ds}\left(\boldsymbol{\rho} + z\hat{\mathbf{z}}\right)\right) = \boldsymbol{\nabla} n(\rho). \tag{2.50}$$

With $\boldsymbol{\nabla} n(\rho) \cdot \hat{\mathbf{z}} = 0$, we can consider only the radial portion of this equation

$$\frac{d}{ds}\left(n(\rho)\frac{d\boldsymbol{\rho}}{ds}\right) = \boldsymbol{\nabla} n(\rho). = -\alpha\boldsymbol{\rho}. \tag{2.51}$$

We seek a relationship as potentially illustrated in fig. 2.7 where given a paraxial approximation we have

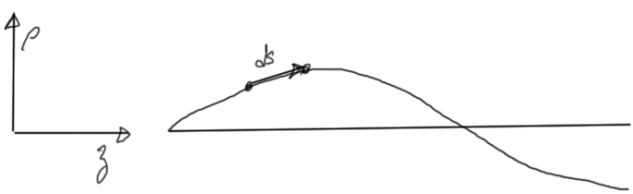

Figure 2.7: Arc length.

$$ds = dz\sqrt{1 + \left|\frac{d\rho}{dz}\right|^2} \sim dz, \tag{2.52}$$

and can reduce the Ray equation to

$$\frac{d}{dz}\left(n_0 - \frac{\alpha}{2}\rho^2\right)\frac{d\boldsymbol{\rho}}{dz} = -\alpha\boldsymbol{\rho}. \tag{2.53}$$

We'd like to drop the ρ^2 term above, and can do that provided

$$\rho \ll \sqrt{\frac{2n_0}{\alpha}} = 2\sqrt{n_0 f d}. \tag{2.54}$$

This leaves us with a plain old SHO

$$\boxed{n_0\frac{d^2}{dz^2}\boldsymbol{\rho} = -\alpha\boldsymbol{\rho}} \tag{2.55}$$

Let's write this out

$$\boldsymbol{\rho}(z) = \begin{bmatrix} x(z) \\ y(z) \end{bmatrix} = \begin{bmatrix} A\cos\left(\sqrt{\frac{\alpha}{n_0}}z\right) + B\sin\left(\sqrt{\frac{\alpha}{n_0}}z\right) \\ C\cos\left(\sqrt{\frac{\alpha}{n_0}}z\right) + D\sin\left(\sqrt{\frac{\alpha}{n_0}}z\right) \end{bmatrix} \tag{2.56}$$

We have 4 constants determined by the initial conditions

$$\boldsymbol{\rho}(0) = \begin{bmatrix} A \\ C \end{bmatrix} \tag{2.57}$$

$$\left.\frac{d\boldsymbol{\rho}}{dz}\right|_0 = \sqrt{\frac{\alpha}{n_0}}\begin{bmatrix} B \\ D \end{bmatrix}. \tag{2.58}$$

If we set $\boldsymbol{\rho}(0) = 0$, then we are left with just

$$\boldsymbol{\rho}(z) = \begin{bmatrix} B \\ D \end{bmatrix}\sin\left(\sqrt{\frac{\alpha}{n_0}}z\right). \tag{2.59}$$

This vectoral solution $\mathbf{r} = \boldsymbol{\rho} + z\hat{\mathbf{z}} + \mathbf{z}_0$ (with $\mathbf{z}_0 = 0$) is illustrated in fig. 2.8. Observe that the nodes are placed at $\sqrt{\alpha/n_0} = n\pi$ with a complete cycle in distance as illustrated in fig. 2.9.

$$L = 2\pi\sqrt{\frac{n_0}{\alpha}}. \tag{2.60}$$

If the length is an integer L as in fig. 2.10. If the length is a half integer L as in fig. 2.11. If length $= \left(n + \frac{1}{4}\right)L$ as in fig. 2.12. It was mentioned that this solved a problem with regular fiber optic cables illustrated in fig. 2.13 so that for the GRIN configuration we have something more like fig. 2.14. This phenomena is well described in §5.6.1 [8].

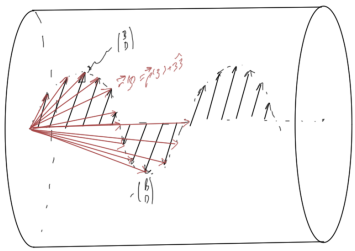

Figure 2.8: Ray variation with position through GRIN material.

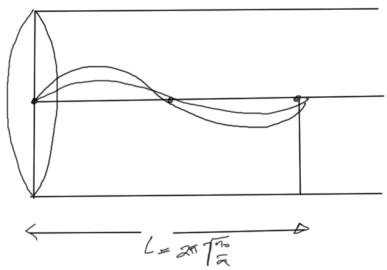

Figure 2.9: Nodal distribution.

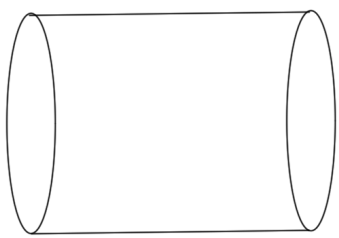

Figure 2.10: first order solution.

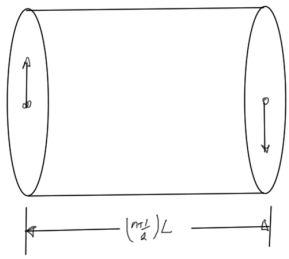

Figure 2.11: second order solution.

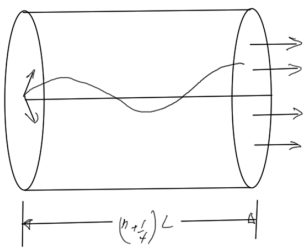

Figure 2.12: third order solution.

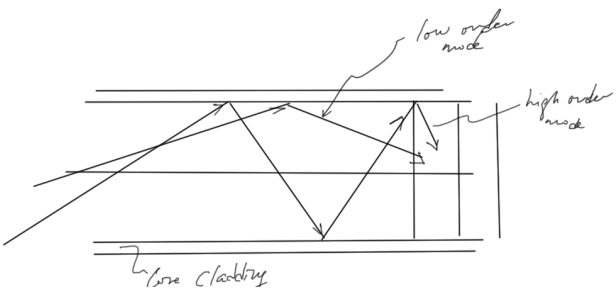

Figure 2.13: regular fiber effects.

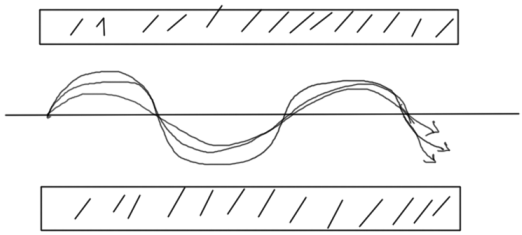

Figure 2.14: step index fiber.

2.7.1 *Phase delay in GRIN lens?*

Are path lengths equal? Instead of dropping all but the dz term in our ds approximation eq. (2.52), how about we retain the first order Taylor expansion

$$\begin{aligned}
\tau &= \int_0^L \frac{ds}{c/n(\mathbf{r})} \\
&= \frac{1}{c}\int_0^L dz \sqrt{1+\left|\frac{d\rho}{dz}\right|^2}\left(n_0 - \frac{\alpha}{2}\rho^2\right) \\
&\approx \frac{n_0}{c}\int_0^L dz \left(1+\frac{1}{2}\left|\frac{d\boldsymbol{\rho}}{dz}\right|^2\right)\left(1-\frac{\alpha}{2n_0}\rho^2\right) \\
&\approx \frac{n_0}{c}\int_0^L dz \left(1+\frac{1}{2}\left|\frac{d\boldsymbol{\rho}}{dz}\right|^2 - \frac{\alpha}{2n_0}\boldsymbol{\rho}^2\right) + O(\text{higher order corrections}).
\end{aligned} \tag{2.61}$$

But

$$\frac{d}{dz}\left(\boldsymbol{\rho}\cdot\frac{d\boldsymbol{\rho}}{dz}\right) \approx \left|\frac{d\boldsymbol{\rho}}{dz}\right|^2 + \boldsymbol{\rho}\cdot\frac{d^2\boldsymbol{\rho}}{d^2z}, \tag{2.62}$$

so our approximate path length can be written

$$\tau = \frac{n_0}{c}\int_0^L dz \left(1+\frac{1}{2}\frac{d}{dz}\left(\boldsymbol{\rho}\cdot\frac{d\boldsymbol{\rho}}{dz}\right) - \frac{1}{2}\rho\cdot\frac{d^2\boldsymbol{\rho}}{d^2z} - \frac{1}{2}\frac{\alpha}{n_0}\rho^2\right). \tag{2.63}$$

But

$$-\frac{1}{2}\boldsymbol{\rho}\cdot\frac{d^2\boldsymbol{\rho}}{d^2z}-\frac{1}{2}\frac{\alpha}{n_0}\boldsymbol{\rho}^2=-\frac{1}{2}\overbrace{\boxed{\left(\frac{d^2}{dz^2}\boldsymbol{\rho}-\frac{\alpha}{n_0}\boldsymbol{\rho}\right)}}^{=0\text{ by Eikonal}}\cdot\boldsymbol{\rho}. \tag{2.64}$$

so

$$\tau=\frac{n_0L}{c}+\frac{n_0}{2c}\overbrace{\boxed{\boldsymbol{\rho}\cdot\frac{d\boldsymbol{\rho}}{dz}\bigg|_0^L}}^{=0\text{ if refocused}}. \tag{2.65}$$

Here if refocused means $\boldsymbol{\rho}=0$ at both sides, as we had for $L=2\pi\sqrt{n_0/\alpha}$. The conclusion is that when light traverses focal point to focal point within the GRIN material of this sort, it propagates without any sort of phase delay.

2.8 RAY EQUATION AND ACTION MINIMIZATION.

Reading : §3 of [2] for details on this topic.

Ray equation gives paths of stationary action

In general our action is

$$S=\int_{t_1}^{t_2}dt\mathcal{L}. \tag{2.66}$$

where $\mathcal{L}$ is the Lagrangian. We recall Hamilton's principle which states that if

$$\delta S=0, \tag{2.67}$$

(a path variation as illustrated in fig. 2.15), then the statement eq. (2.67) gives us Hamiltonian dynamics (Hamilton's equations).

Why does this work? One explanation is that we have in quantum mechanics the most general action

$$\text{amplitude(path)}=\exp(iS[\text{path}]/\hbar), \tag{2.68}$$

This is called Feynman's Path Integral.

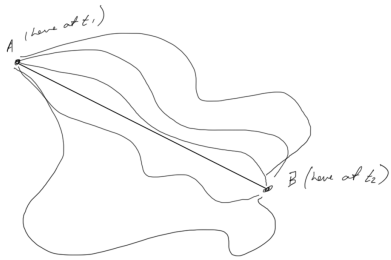

Figure 2.15: Action minimization.

We have a Lagrangian for electromagnetism

$$\mathcal{L}_{\text{EM}} = \frac{1}{c}\rho\phi + \frac{1}{c}\mathbf{j}\cdot\mathbf{A} + \frac{1}{8\pi}\mathbf{E}^2 - \frac{1}{8\pi}\mathbf{B}^2. \tag{2.69}$$

Using this we can derive Maxwell's equations.

As a problem we are going to calculate the amplitude for the Cornu spiral fig. 2.16. Propose some experiments

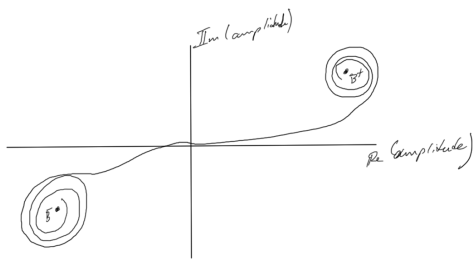

Figure 2.16: Cornu spiral path of interest.

1) Block the primary path fig. 2.17. 2) Block 2 paths fig. 2.18. 3) Block half paths fig. 2.19. 4) Allow paths with the same phase.

Figure 2.20. We'll see that the Action principle does in fact provide us all the real physical effects (Fresnel diffraction, multi-slit diffraction, ...)

2.9 PROBLEMS.

Exercise 2.1 Calculate the Poynting vector time average.

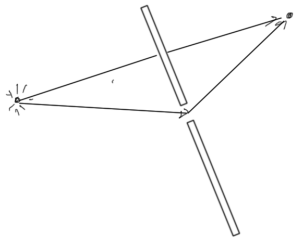

Figure 2.17: Single slit.

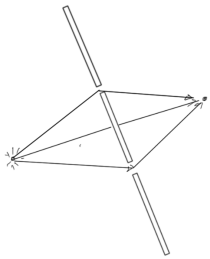

Figure 2.18: Double slit.

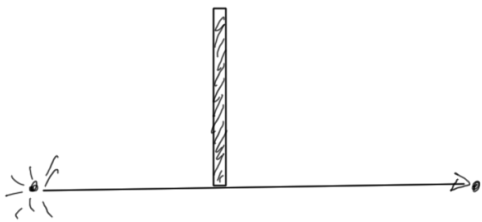

Figure 2.19: Wall blocking half path.

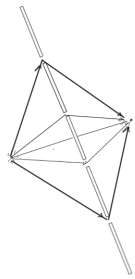

Figure 2.20: Many slits.

Demonstrate eq. (2.21).

Answer for Exercise 2.1

In exercise B.1 we have shown that

$$\langle \mathbf{S} \rangle = \langle \mathbf{E} \times \mathbf{B} \rangle = \frac{1}{2} \mathrm{Re}(\mathbf{E}_0 \times \mathbf{B}_0^*), \tag{2.70}$$

where the fields were specified by the phasor relations

$$\mathbf{E} = \mathrm{Re}\left(\mathbf{E}_0 e^{i\mathbf{k}\cdot\mathbf{x}-\omega t}\right) \tag{2.71}$$

$$\mathbf{B} = \mathrm{Re}\left(\mathbf{B}_0 e^{i\mathbf{k}\cdot\mathbf{x}-\omega t}\right). \tag{2.72}$$

Given that, switching to cgs units, the exercise is reduced to showing that

$$\mathrm{Re}(\mathbf{E}_0 \times \mathbf{B}_0^*) = \frac{1}{k_0} |\mathbf{E}_0|^2 \boldsymbol{\nabla}\phi. \tag{2.73}$$

From back to Maxwell's equation, in particular eq. (2.15c), we have

$$\begin{aligned}
\mathbf{E}_0 \times \mathbf{B}_0^* &= \mathbf{E}_0 \times \left(\frac{1}{k_0}\boldsymbol{\nabla}\phi \times \mathbf{E}_0\right)^* \\
&= \frac{1}{k_0}\mathbf{E}_0 \times (\boldsymbol{\nabla}\phi \times \mathbf{E}_0^*) \\
&= -\frac{1}{k_0}\left(\cancel{(\mathbf{E}_0 \cdot \boldsymbol{\nabla}\phi)}\mathbf{E}_0^* - |\mathbf{E}_0|^2 \boldsymbol{\nabla}\phi\right) \\
&= \frac{1}{k_0}|\mathbf{E}_0|^2 \boldsymbol{\nabla}\phi,
\end{aligned} \tag{2.74}$$

as desired.

Exercise 2.2 Second order product in Ray equation.

Derive eq. (2.28).

Answer for Exercise 2.2

Let's expand out the gradients in Cartesian coordinates

$$\begin{aligned}(\boldsymbol{\nabla}\phi\cdot\boldsymbol{\nabla})\,\boldsymbol{\nabla}\phi &= (\partial_k\phi\partial_k)\,\mathbf{e}_m\partial_m\phi \\ &= \mathbf{e}_m\partial_k\phi\partial_m\partial_k\phi \\ &= \frac{1}{2}\mathbf{e}_m\partial_m\left(\partial_k\phi\partial_k\phi\right) \\ &= \frac{1}{2}\boldsymbol{\nabla}\left(\boldsymbol{\nabla}\phi\right)^2.\end{aligned} \tag{2.75}$$

Now we use Eikonal eq. (2.17)

$$\begin{aligned}(\boldsymbol{\nabla}\phi\cdot\boldsymbol{\nabla})\,\boldsymbol{\nabla}\phi &= \frac{1}{2}\boldsymbol{\nabla}(k_0 n)^2 \\ &= k_0^2 n\boldsymbol{\nabla}n.\end{aligned} \tag{2.76}$$

Exercise 2.3 Solve the Ray trapping equation.

Can we easily solve eq. (2.43)?

Answer for Exercise 2.3

We have $n = n(r)$ so that

$$\boldsymbol{\nabla}n = \left(\hat{\mathbf{r}}\frac{\partial}{\partial r} + \frac{\hat{\boldsymbol{\theta}}}{r}\frac{\partial}{\partial\theta} + \hat{\mathbf{z}}\frac{\partial}{\partial z}\right)n(r) = \hat{\mathbf{r}}\frac{\partial n}{\partial r} = -n\frac{\hat{\mathbf{r}}}{R}. \tag{2.77}$$

Cancelling $\hat{\mathbf{r}}$ factors, we find that this is separable

$$\int\frac{dn}{n} = -\int\frac{dr}{R}, \tag{2.78}$$

or

$$\ln n = -r + \ln n(0). \tag{2.79}$$

Exponentiating we have

$$n = n(0)e^{-r}. \tag{2.80}$$

Exercise 2.4 Eikonal equation, Geometric Algebra.

Express eq. (2.15) in it's Geometric Algebra form, and figure out how eq. (2.17) follows from that directly.

Answer for Exercise 2.4

We write Maxwell's equations for the assumed phasor solution more symmetrically (and re-introducing μ temporarily for generality)

$$\nabla \cdot \epsilon \mathbf{E} = 0. \tag{2.81a}$$

$$\nabla \cdot \mathbf{B} = 0. \tag{2.81b}$$

$$\nabla \times \mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t}. \tag{2.81c}$$

$$\nabla \times \mathbf{B} = \frac{\mu\epsilon}{c}\frac{\partial \mathbf{E}}{\partial t}. \tag{2.81d}$$

Observe that we have $[\nabla] = [(1/c)\partial/\partial t] = 1/L$, so $\mathbf{E}$ and $\mathbf{B}$ have the same dimensions in cgs and $\mu\epsilon$ is dimensionless.

Employing the usual notation for the 3D unit pseudoscalar $I = \mathbf{e}_1\mathbf{e}_2\mathbf{e}_3$ and the identity

$$\mathbf{ab} = \mathbf{a}\cdot\mathbf{b} + I(\mathbf{a}\times\mathbf{b}), \tag{2.82}$$

we can perform a first amalgamation of Maxwell's equations into two multivector equations

$$\nabla\mathbf{E} = -\frac{I}{c}\frac{\partial \mathbf{B}}{\partial t}. \tag{2.83a}$$

$$\nabla\mathbf{B} = \frac{I\mu\epsilon}{c}\frac{\partial \mathbf{E}}{\partial t}. \tag{2.83b}$$

With knowledge that the wave velocity in cgs units is $v = c/\sqrt{\mu\epsilon}$ (which will be confirmed shortly) we can rescale

$$\nabla\mathbf{E} = -\frac{\sqrt{\mu\epsilon}}{c}\frac{\partial}{\partial t}\frac{I\mathbf{B}}{\sqrt{\mu\epsilon}}. \tag{2.84a}$$

$$\nabla \frac{I\mathbf{B}}{\sqrt{\mu\epsilon}} = -\frac{\sqrt{\mu\epsilon}}{c}\frac{\partial \mathbf{E}}{\partial t}. \tag{2.84b}$$

and do a final amalgamation of Maxwell's equations into multivector form

$$\boxed{\left(\nabla + \frac{\sqrt{\mu\epsilon}}{c}\frac{\partial}{\partial t}\right)\left(\mathbf{E} + \frac{I}{\sqrt{\mu\epsilon}}\mathbf{B}\right) = 0.} \tag{2.85}$$

For the multivector electrodynamic field we write

$$F = \mathbf{E} + \frac{I}{\sqrt{\mu\epsilon}}\mathbf{B} = \mathbf{E} + \frac{I}{n}\mathbf{B}. \tag{2.86}$$

Observe that the wave equation can be found by left multiplying with $\nabla - (\sqrt{\mu\epsilon}/c)\partial_t$

$$\left(\nabla^2 - \frac{\mu\epsilon}{c^2}\partial_{tt}\right) F = 0. \tag{2.87}$$

This confirms that the velocity of the wave is $\sqrt{\mu\epsilon}/c$. Note that since this scalar operator equation must hold separately for both the vector and pseudoscalar components of this equation, it also applies separately to $\mathbf{E}$ and $\mathbf{B}$ independently as expected.

Okay, we've got our starting point. Let's now assume a phasor solution for F as in class

$$F = F_0(\mathbf{r})e^{i\phi(\mathbf{r})-i\omega t}. \tag{2.88}$$

Application of Maxwell's equation eq. (2.85) to this we find

$$\begin{aligned}
&\left(\nabla + \frac{\sqrt{\mu\epsilon}}{c}\frac{\partial}{\partial t}\right) F_0 e^{i\phi - i\omega t} \\
&= e^{i\phi-i\omega t}\nabla F_0 + e^{-i\omega t}\mathbf{e}_k F_0 \partial_k e^{i\phi} + e^{i\phi}F_0(-i\omega)\frac{\sqrt{\mu\epsilon}}{c}e^{-i\omega t} \\
&= e^{i\phi-i\omega t}\left(\nabla + i(\nabla\phi) - i\omega\frac{\sqrt{\mu\epsilon}}{c}\right) F_0.
\end{aligned} \tag{2.89}$$

Neglecting the ∇F_0 term, writing $\omega/c = k_0$ and requiring this to hold for any phase we have

$$(\nabla\phi)F_0 = k_0 n(\mathbf{r})F_0. \tag{2.90}$$

Left multiplication by $\boldsymbol{\nabla}\phi$ gives us

$$(\boldsymbol{\nabla}\phi)^2 F_0 = k_0 n(\boldsymbol{\nabla}\phi)F_0 = k_0^2 n^2 F_0, \tag{2.91}$$

yielding the Eikonal equation as desired

$$(\boldsymbol{\nabla}\phi)^2 = k_0^2 n^2. \tag{2.92}$$

From this we observe that $\hat{\mathbf{k}}$ propagation direction unit vector that we are used to is generalized to a spatially dependent form

$$\hat{\mathbf{k}}(\mathbf{r}) = \frac{\boldsymbol{\nabla}\phi}{k_0 n}. \tag{2.93}$$

Using that, the equation to solve takes the form

$$\hat{\mathbf{k}} F_0 = F_0. \tag{2.94}$$

Let's check that this matches our expectations by multiplying out this equation explicitly

$$\begin{aligned} \mathbf{E} + \frac{I}{n}\mathbf{B} &= \hat{\mathbf{k}}\left(\mathbf{E} + \frac{I}{n}\mathbf{B}\right) \\ &= \hat{\mathbf{k}}\cdot\mathbf{E} + \frac{I}{n}\hat{\mathbf{k}}\cdot\mathbf{B}\hat{\mathbf{k}}\wedge\mathbf{E} + \frac{I}{n}\hat{\mathbf{k}}\wedge\mathbf{B}. \end{aligned} \tag{2.95}$$

Grouping by scalar, trivector, vector, and bivector terms we find

$$\hat{\mathbf{k}}\cdot\mathbf{E}_0 = 0. \tag{2.96a}$$

$$\hat{\mathbf{k}}\cdot\mathbf{B}_0 = 0. \tag{2.96b}$$

$$\mathbf{E}_0 = -\hat{\mathbf{k}}\times\frac{\mathbf{B}_0}{n}. \tag{2.96c}$$

$$\frac{\mathbf{B}_0}{n} = \hat{\mathbf{k}}\times\mathbf{E}_0. \tag{2.96d}$$

Good, this matches with eq. (2.15) as derived in class.

As another check, observe that the wave equation can be found by left multiplying with $\boldsymbol{\nabla} - (\sqrt{\mu\epsilon}/c)\partial_t$

$$\left(\boldsymbol{\nabla}^2 - \frac{\mu\epsilon}{c^2}\partial_{tt}\right) F = 0. \tag{2.97}$$

This confirms that the velocity of the wave is $\sqrt{\mu\epsilon}/c$. Note that since this scalar operator equation must hold separately for both the vector and pseudoscalar components of this equation, it also applies separately to $\mathbf{E}$ and $\mathbf{B}$ independently as expected.

Okay, with some sanity checking done, we've got our starting point. Let's now assume a phasor solution for F as in class

$$F = F_0(\mathbf{r})e^{i\phi(\mathbf{r})-i\omega t}. \tag{2.98}$$

Application of Maxwell's equation eq. (2.85) to this we find

$$\begin{aligned}
&\left(\boldsymbol{\nabla} + \frac{\sqrt{\mu\epsilon}}{c}\frac{\partial}{\partial t}\right) F_0 e^{i\phi-i\omega t} \\
&= e^{i\phi-i\omega t}\boldsymbol{\nabla} F_0 + e^{-i\omega t}\mathbf{e}_k F_0 \partial_k e^{i\phi} + e^{i\phi}F_0(-i\omega)\frac{\sqrt{\mu\epsilon}}{c}e^{-i\omega t} \\
&= e^{i\phi-i\omega t}\left(\boldsymbol{\nabla} + i(\boldsymbol{\nabla}\phi) - i\omega\frac{\sqrt{\mu\epsilon}}{c}\right) F_0.
\end{aligned} \tag{2.99}$$

Neglecting the $\boldsymbol{\nabla} F_0$ term, writing $\omega/c = k_0$ and requiring this to hold for any phase we have

$$(\boldsymbol{\nabla}\phi)F_0 = k_0 n(\mathbf{r})F_0. \tag{2.100}$$

Left multiplication by $\boldsymbol{\nabla}\phi$ gives us

$$(\boldsymbol{\nabla}\phi)^2 F_0 = k_0 n(\boldsymbol{\nabla}\phi)F_0 = k_0^2 n^2 F_0, \tag{2.101}$$

yielding the Eikonal equation as desired

$$(\boldsymbol{\nabla}\phi)^2 = k_0^2 n^2. \tag{2.102}$$

Note that it almost looks like eq. (2.102) could be observed directly from eq. (2.93). However, at that point we hadn't actually proved that $\hat{\mathbf{k}}$ was a unit vector, but had sneakily just used notation that implied it.

Having shown now that $\hat{\mathbf{k}}$ is a unit vector, and because eq. (2.94) is now no longer any different than the result for linear media

with a phasor assumed to be of the form $F_0 e^{i\mathbf{k}\cdot\mathbf{x}-i\omega t}$, we must also have as the general solution to Maxwell's equation given our approximation

$$F_0 = (1 + \hat{\mathbf{k}})\mathbf{C}. \tag{2.103}$$

where $\mathbf{C} \cdot \hat{\mathbf{k}} = 0$. However, in this case, we have to allow $\hat{\mathbf{k}}$ and $\mathbf{C}$ to covary spatially.

Exercise 2.5 Ray in a linear index gradient. (*2012 Ps1, P2*)

What is the shape of a ray moving into a linear index gradient fig. 2.21? You'd expect something like a parabola from the intuition that the Ray Equation is 'Newton-like'. Find out what you actually get! To establish some conventions: take $n(y) = n_0 - \beta y$; choose parameterization of the ray so that $s = 0$ at the top of the trajectory: $\mathbf{r}(0) = 0$, and $d\mathbf{r}/ds = \hat{x}$ at $s = 0$. In this case the ray will remain in the xy plane, so your task is to find $x(s)$ and $y(s)$.

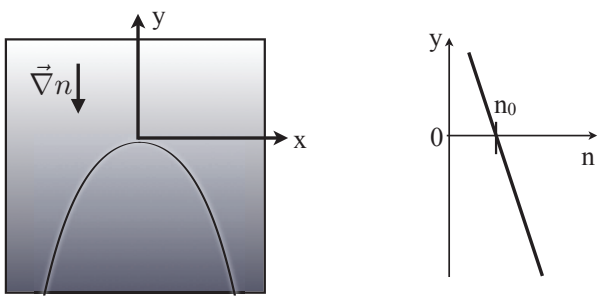

Figure 2.21: .

(a) Start with the Ray Equation $\frac{d}{ds}\{n\frac{d}{ds}\mathbf{r}\} = \boldsymbol{\nabla} n$. Integrate both sides with respect to s, and use initial conditions to determine constants of integration. You should be left with two first-order differential equations.

(b) Solve the dy/ds equation first, by integrating again with respect to s. Give an exact expression for $z(s)$. Also give approximate expressions for $y(s)$ in two limits: small s, and large s.

(c) Now solve the dx/ds equation. Again, give an exact expression for $x(s)$, and approximate expressions for $x(s)$ in two limits: small s, and large s.

(d) Combine your results to give $x(y)$. (This may seem strange, but an exact result for $y(x)$ is hard to write down. You'll have to restrict yourself to $x > 0$ for this curve to be functional.) You can again find an exact result, an small-s approximation, and a large-s approximation.

(e) Is the trajectory of an optical ray a parabola in any limit? If so, what is gravitational acceleration?

(f) Use your favourite software (Mathematica, ...) to make a plot of $x(s)$, $y(s)$, and $x(y)$. In each plot, compare the exact expression (as a solid line) to the two limiting expressions (as dashed lines). Nondimensionalize in terms of $L = n_0/\beta$: in other words, use the variables x/L, y/L, and s/L.

Answer for Exercise 2.5

(a) Our ray equation, after computation of the gradient of the index of refraction for the material becomes

$$\frac{d}{ds}\left(n(\mathbf{r})\frac{d\mathbf{r}}{ds}\right) = \boldsymbol{\nabla} n(\mathbf{r}) = \boldsymbol{\nabla}\left(n_0 - \beta y\right) = -\beta\hat{\mathbf{y}}. \tag{2.104}$$

In components this is

$$\begin{aligned}
\frac{d}{ds}\left(\left(n_0 - \beta y\right)\frac{dx}{ds}\right) &= 0 \\
\frac{d}{ds}\left(\left(n_0 - \beta y\right)\frac{dy}{ds}\right) &= -\beta \\
\frac{d}{ds}\left(\left(n_0 - \beta y\right)\frac{dz}{ds}\right) &= 0
\end{aligned} \tag{2.105}$$

Integrating once, with the introduction of n_0 factors in our integration constant (which will clearly make life easier), we have

$$\begin{aligned}
\left(n_0 - \beta y\right)\frac{dx}{ds} &= An_0 \\
\left(n_0 - \beta y\right)\frac{dy}{ds} &= -\beta s + Bn_0 \\
\left(n_0 - \beta y\right)\frac{dz}{ds} &= Cn_0
\end{aligned} \tag{2.106}$$

In particular, at $s = 0$, where $x(0) = y(0) = z(0) = 0$, $x'(0) = 1$ and $y'(0) = z'(0) = 0$, we have

$$\begin{aligned} n_0(1) &= An_0 \\ n_0(0) &= Bn_0 \\ n_0(0) &= Cn_0 \end{aligned} \tag{2.107}$$

Our equations of motion become

$$\boxed{\begin{aligned} (n_0 - \beta y)\frac{dx}{ds} &= n_0 \\ (n_0 - \beta y)\frac{dy}{ds} &= -\beta s \\ (n_0 - \beta y)\frac{dz}{ds} &= 0. \end{aligned}} \tag{2.108}$$

We have two non-trivial differential equations to solve.

(b) First observe that unless $n_0 = \beta y(s)$ for all s, then $z(s)$ must be constant. However, our boundary condition $\mathbf{r}(0) = 0$ means that this constant is zero

$$z(s) = \text{constant} = z(0) = 0. \tag{2.109}$$

Solving for $y(s)$ next we have after rearranging

$$\int (n_0 - \beta y)\, dy = -\beta \int s ds. \tag{2.110}$$

This yields

$$n_0 y - \frac{\beta}{2}y^2 = -\frac{\beta}{2}s^2 + C. \tag{2.111}$$

Noting that $y(0) = 0$ we have $C = 0$

$$y^2 - s^2 - 2\frac{n_0}{\beta}y = 0. \tag{2.112}$$

Completing the square

$$\left(y - \frac{n_0}{\beta}\right)^2 = s^2 + \left(\frac{n_0}{\beta}\right)^2. \tag{2.113}$$

or

$$y = \frac{n_0}{\beta} \pm \sqrt{s^2 + \left(\frac{n_0}{\beta}\right)^2}. \tag{2.114}$$

Given the $y(0) = 0$ boundary constraint, we can only pick the negative root. Borrowing the $L = n_0/\beta$ notation from later in the problem, we have

$$\boxed{y(s) = L\left(1 - \sqrt{\left(\frac{s}{L}\right)^2 + 1}\right).} \tag{2.115}$$

Let's look at the small limit where $s \ll L$

$$y(s) \sim L\left(1 - \left(1 + \frac{1}{2}\left(\frac{s}{L}\right)^2\right)\right), \tag{2.116}$$

$$\boxed{y(s) \sim -\frac{s^2}{2L} \quad \text{when } s \ll L.} \tag{2.117}$$

In the large limit for $s \gg L$ the s^2 term dominates, leaving

$$\boxed{y(s) \sim -s \quad \text{when } s \gg L.} \tag{2.118}$$

A plot of y/L, $-s/L$, and $-s^2/2L^2$ can be found in fig. 2.22 and fig. 2.23.

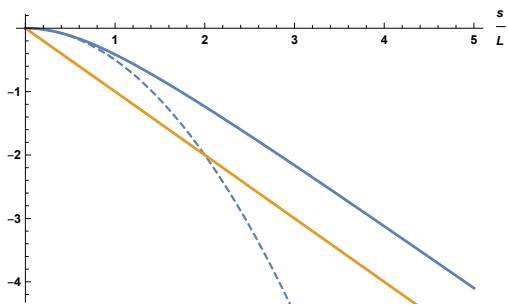

Figure 2.22: Plots of $y(s)$ and corresponding big and small limiting forms, scaled for small limit.

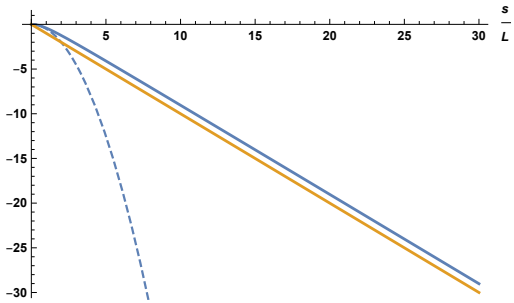

Figure 2.23: Plots of $y(s)$ and corresponding big and small limiting forms, scaled for large limit.

(c) We are now set to solve our x component ray equation

$$(L-y)\frac{dx}{ds} = L, \tag{2.119}$$

or

$$\sqrt{s^2+L^2}\frac{dx}{ds} = L. \tag{2.120}$$

Integrating we have

$$\begin{aligned} x &= L\int_0^s \frac{ds'}{\sqrt{s'^2+L^2}} \\ &= L\int_0^s \frac{ds'}{\sqrt{s'^2+L^2}} \\ &= L\int_0^{s/L} \frac{dt}{\sqrt{t^2+1}} \\ &= L\ln\left(t+\sqrt{t^2+1}\right)\Big|_0^{s/L}. \end{aligned} \tag{2.121}$$

This is

$$\boxed{x(s) = L\ln\left(\frac{s}{L}+\sqrt{\left(\frac{s}{L}\right)^2+1}\right).} \tag{2.122}$$

In the large limit for $s \gg L$ the s^2 term in the square root dominates, leaving

$$\boxed{x(s) \sim L\ln\left(\frac{2s}{L}\right) \quad \text{when } s \gg L.} \tag{2.123}$$

In the small limit $s \ll L$

$$x(s) \sim L \ln\left(\frac{s}{L}+1\right) = L\left(\frac{s}{L} - \frac{1}{2}\left(\frac{s}{L}\right)^2 + \frac{1}{3}\left(\frac{s}{L}\right)^3 - \cdots\right), \tag{2.124}$$

or

$$\boxed{x(s) \sim s \quad \text{when } s \ll L.} \tag{2.125}$$

With $t = s/L$, we have a plot of $u(t) = x(Lt)/L$, and the small and large limit approximations above in fig. 2.24.

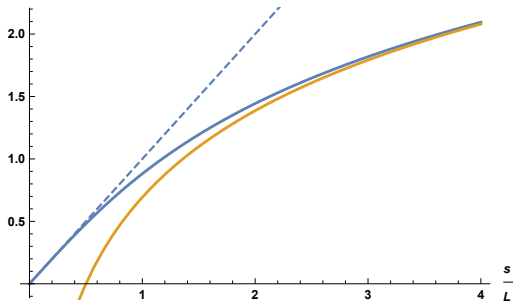

Figure 2.24: Plots of $x(s)$ and corresponding big and small limiting forms.

(d) With $t = s/L$, $u = x/L$, and $v = y/L$ we have

$$u = \ln\left(t + \sqrt{t^2+1}\right). \tag{2.126a}$$

$$v = 1 - \sqrt{t^2+1}. \tag{2.126b}$$

Rearranging for t and $\sqrt{1+t^2}$, we have

$$\sqrt{t^2+1} = 1 - v. \tag{2.127a}$$

$$t = \sqrt{(1-v)^2 - 1}, \tag{2.127b}$$

so

$$u(v) = \ln\left(\sqrt{v^2 - 2v} + 1 - v\right). \tag{2.128}$$

or

$$\boxed{x(y) = L\ln\left(\sqrt{\left(\frac{y}{L}\right)^2 - 2\frac{y}{L} + 1} - \frac{y}{L}\right).} \tag{2.129}$$

Now, for the approximations. Noting that the range of y is $(-\infty, 0]$ let's write $w = -v = |v|$ in eq. (2.128) so that we have

$$u(w) = \ln\left(\sqrt{w^2 + 2w} + 1 + w\right). \tag{2.130}$$

For $w >> 1$ we have

$$\sqrt{w^2 + w} \sim \sqrt{w^2} = w = -y/L. \tag{2.131}$$

so that

$$\boxed{x(y) \sim L\ln\left(-\frac{2y}{L}\right) \quad \text{when } -y/L \gg 1.} \tag{2.132}$$

In the small limit $w << 1$ we also have $w^2 \ll w$, so that

$$u(w) \sim \ln(\sqrt{w} + 1) \sim \sqrt{w} - \frac{1}{2}(\sqrt{w})^2 + \frac{1}{3}(\sqrt{w})^3 - \cdots \sim \sqrt{w}, \tag{2.133}$$

or

$$\boxed{x(y) \sim L\sqrt{-\frac{y}{L}} \quad \text{when } -y/L \ll 1.} \tag{2.134}$$

A plot of $x(y/L)/L$, and the small and large limit approximations can be found in fig. 2.25 and fig. 2.26.

(e) In the small limit we found

$$x(s) \sim s. \tag{2.135a}$$

$$y(s) \sim -\frac{s^2}{2L}, \tag{2.135b}$$

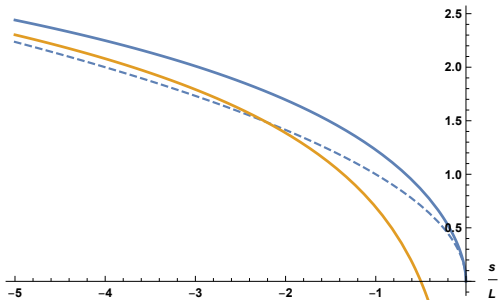

Figure 2.25: Plots of $x(y/L)/L$ and corresponding big and small limiting forms, scaled for small limit.

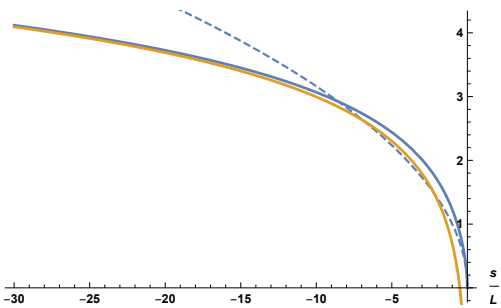

Figure 2.26: Plots of $x(y/L)/L$ and corresponding big and small limiting forms, scaled for large limit.

so we have

$$y \sim -\frac{x^2}{2L}, \tag{2.136}$$

a parabolic trajectory. Comparing to $y'' = g$, where $y = gt^2/2 + y_0' t + y_0$, the quantity that's analogous to the gravitational acceleration in eq. (2.136) is

$$\boxed{-\frac{1}{L} = -\frac{\beta}{n_0} \to g.} \tag{2.137}$$

(f) These plots were included above. Good asymptotic matching in the large limit was found to be fairly range dependent, also shown above. This can be observed in modernOpticsProblemSet1.cdf , where dynamic (Manipulate) graphs are available for each of the graphs above, where the range is slider parameterized.

Exercise 2.6 Ray equation at a surface. (*2012 Ps1, P3*)

Show that Snell's law can be derived from the *transverse* component of the ray equation applied at an index step. Set up the problem with an index step from n_1 in the half-plane $x < 0$; and n_2 in the half-plane $x > 0$ fig. 2.27. Define your rays according to two straight-line trajectories: a ray in the xy plane defined by $x = s\cos\theta_1$ and $y = s\sin\theta_1$ for $x < 0$; and $x = s\cos\theta_2$ and $y = s\sin\theta_2$ for $x > 0$.

(a) Solve the transverse (or y-) component of the Ray Equation. Show that it gives Snell's law.

(b) Show that the *normal* (or x-) component of the Ray Equation is contradictory, unless the limit of a small index step is taken. Why is this? What is missing?

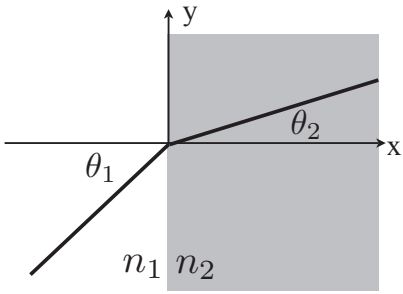

Figure 2.27: .

Answer for Exercise 2.6

(a) The index of refraction $n(x)$ has no y-component, so we have

$$\hat{\mathbf{y}} \cdot \boldsymbol{\nabla} n = 0. \tag{2.138}$$

The y-component of the Ray equation

$$\frac{d}{ds}\left(n(x)\frac{dy}{ds}\right) = 0, \tag{2.139}$$

can therefore be integrated directly

$$n(x)\frac{dy}{ds} = \text{constant}. \tag{2.140}$$

With the chosen ray parameterization we have for $x < 0$ the y-component of the ray "velocity"

$$\hat{\mathbf{y}} \cdot \frac{d\mathbf{r}_1}{ds} = \hat{\mathbf{y}} \cdot \frac{d}{ds}s(\cos\theta_1, \sin\theta_1) = \hat{\mathbf{y}} \cdot (\cos\theta_1, \sin\theta_1) = \sin\theta_1. \tag{2.141}$$

Similarly for the y-component in the $x > 0$ region we have

$$\hat{\mathbf{y}} \cdot \frac{d\mathbf{r}_2}{ds} = \sin\theta_2. \tag{2.142}$$

We want to use this in the integrated Ray equation eq. (2.140) which takes the form

$$n_1\frac{dy_1}{ds} = \text{constant} = n_2\frac{dy_2}{ds}, \tag{2.143}$$

but since we have found that $dy_1/ds = \sin\theta_1$ and $dy_2/ds = \sin\theta_2$, we have Snell's law

$$\boxed{n_2 \sin\theta_2 = n_1 \sin\theta_1.} \tag{2.144}$$

(b) We can produce a contradictory result if we avoid the origin when treating the x-component of the Ray equation. Repeating the argument above for $|x| > 0$ where $\boldsymbol{\nabla} n = 0$ would give us

$$n\frac{dx}{ds} = \text{constant}. \tag{2.145}$$

With $dx_1/ds = \cos\theta_1$ and $dx_2/ds = \cos\theta_2$ we would have

$$n_2 \cos\theta_2 = n_1 \cos\theta_1, \tag{2.146}$$

which contradicts Snell's law.

This conclusion isn't valid because we have avoided the origin, where the index of refraction is not continuous. What is missing is proper treatment of this step discontinuity. To frame this properly, let's express the index of refraction a bit more precisely. That is

$$n(x) = n_1 + \Delta n \theta(x). \tag{2.147}$$

where $\Delta n = n_2 - n_1$. We now have a non-zero gradient

$$\boldsymbol{\nabla} n = \hat{\mathbf{x}} \Delta n \delta(x). \tag{2.148}$$

The Ray equation, split by coordinates, now takes the form

$$\boxed{\begin{aligned} \frac{d}{ds}\left((\theta(x)n_2 + \theta(-x)n_1)\frac{dx}{ds}\right) &= \Delta n \delta(x) \\ (\theta(x)n_2 + \theta(-x)n_1)\frac{dy}{ds} &= \text{constant}. \end{aligned}} \tag{2.149}$$

Note that any solution of the above must also take into account the dependence between s, x and y

$$ds^2 = dx^2 + dy^2, \tag{2.150}$$

or

$$1 = (dx/ds)^2 + (dy/ds)^2. \tag{2.151}$$

While we can still directly integrate the y-component equation once (as done above), our original assumed parameterization of $\mathbf{r}(s) = s(\cos\theta, \sin\theta)$ looses it's convenient form since

we now have $\theta = \theta(s)$ in the neighborhood of the origin. Once we choose to not neglect the step discontinuity, we have a coupled, much more difficult, system to deal with.

Can this system, or one for which a limiting form of the unit step and delta functions is used (i.e. the sinc representation of the delta function), be solved exactly?

$$n\frac{d\mathbf{r}}{ds} = \nabla\phi. \tag{2.152}$$

Because $\boldsymbol{\nabla} \times \boldsymbol{\nabla}\phi = 0$ we have

$$\boldsymbol{\nabla} \times \left(n\frac{d\mathbf{r}}{ds}\right) = 0, \tag{2.153}$$

and can now apply a differential loop boundary condition to find that only the tangential component contributes, which leads to Snell's law.

3 DIFFRACTION.

3.1 CONTEXT.

We start the class with a green laser setup, where the light is displayed on the screen, then also after going through a single and double slit, as illustrated in fig. 3.1. We see also that a blue laser

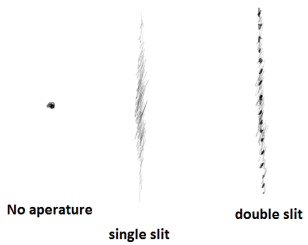

Figure 3.1: Laser on screen.

diffracts less. The bigger the wavelength, the harder it is to ignore. We can consider this a breakdown of geometric optics.

3.2 DIFFRACTION.

We'll want to consider systems of this sort (light source, object in between, goes some distance, then observed) mathematically. We consider the geometry of fig. 3.2 where $\mathbf{R} = \mathbf{r} - \mathbf{r}'$, and $\mathbf{R}_s = \mathbf{r}_s - \mathbf{r}'$, and $R = |\mathbf{R}|$, $R_s = |\mathbf{R}_s|$. We have two approximations to the full problem

1. A scalar theory can suffice.
2. The region of interest (and source) are paraxial.

Why a scalar theory? If we have a plane wave polarization

$$\mathbf{E}(\mathbf{r}, t) = (E_1\hat{\mathbf{x}} + E_2\hat{\mathbf{y}})\, e^{i\mathbf{k}\cdot\mathbf{r} - i\omega t}. \tag{3.1}$$

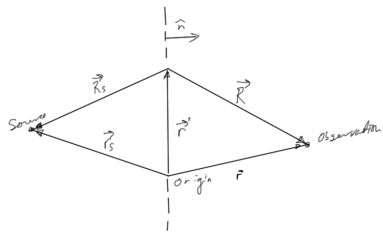

Figure 3.2: Diffracting object (i.e. aperture).

With the principle of superposition

1. Solve for the x polarization.
2. Solve for the y polarization.
3. Vector addition of result.

We will assume no mixing, so that we can treat just one component.

Reading: §8.3.1 [2], §9.8 [9]. The first goes and proves that the scalar theory is sufficient under this conditions.

We'd like to solve the wave equation with these approximations.

$$\nabla^2 E = \frac{1}{c^2}\frac{\partial^2 E}{\partial t^2}. \tag{3.2}$$

We will use a monochromatic wave so that we can write the electric field magnitude as a vector function times a time phase term

$$E = \Psi(\mathbf{r})e^{-i\omega t}. \tag{3.3}$$

We find

$$\boxed{\left(\nabla^2 + \mathbf{k}^2\right)\Psi(\mathbf{r}) = 0} \tag{3.4}$$

This is called the Helmholtz equation.

It turns out that the solution to this equation is generally written out as the surface integral

$$\Psi(\mathbf{r}) = \iint da' \left(\Psi(\mathbf{r}')\boldsymbol{\nabla}' G - G\boldsymbol{\nabla}'\Psi(\mathbf{r}')\right) \cdot \hat{\mathbf{n}}. \tag{3.5}$$

Here $\hat{\mathbf{n}}$ is the unit normal perpendicular to the surface, and the Green function of the Helmholtz equation is

$$G(\mathbf{r}, \mathbf{r}') = -\frac{e^{ikR}}{4\pi R} = -\frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{4\pi|\mathbf{r}-\mathbf{r}'|}, \tag{3.6}$$

It is somewhat messy, but relatively straightforward to demonstrate [10] that this Green's function works to solve the forced Helmoltz equation

$$\left(\boldsymbol{\nabla}^2 + \mathbf{k}^2\right)\Psi_{\mathbf{k}}(\mathbf{r}) = s(\mathbf{r}), \tag{3.7}$$

with solution

$$\Psi_{\mathbf{k}}(\mathbf{r}) = \int G(|\mathbf{r}-\mathbf{r}'|)s(\mathbf{r}')d^3\mathbf{r}'. \tag{3.8}$$

However, it is far from obvious how to apply this to the homogeneous Helmoltz equation. The tricks involved (application of Green's theorem to a spherical volume with the center deleted) can be found in §A.2, §10.4 of [8], and also in §5.2 of [5]. The end result of that trickery is called the Kirchhoff Integral Theorem.

Is this Green's function reasonable seeming? As illustrated in fig. 3.3) this isn't an entirely unsurprising seeming Green's function for this problem. We have the e^{ikR} type of phase factor that we expected (and guessed in the geometric optics treatment, and also have the $1/R$ factor that we need to retain power at a distance R. Also note that the primed gradient is taken with respect to the coordinates of $\mathbf{r}'$

$$\boldsymbol{\nabla}' = \mathbf{e}_m \frac{\partial}{\partial x'_m}. \tag{3.9}$$

If we take the gradient of the Green's function we find

$$\boldsymbol{\nabla}\left(\frac{e^{ikr}}{r}\right) = \hat{\mathbf{r}}\left(ik - \frac{1}{r}\right)\frac{e^{ikr}}{r}. \tag{3.10}$$

Applying this to our problem we find

$$\boxed{\Psi(\mathbf{r}) = -\frac{1}{4\pi}\iint \frac{e^{ikR}}{R}\hat{\mathbf{n}} \cdot \left(\boldsymbol{\nabla}'\Psi(\mathbf{r}') + \left(ik - \frac{1}{R}\right)\frac{\mathbf{R}}{R}\Psi(\mathbf{r}')\right) da'.}$$

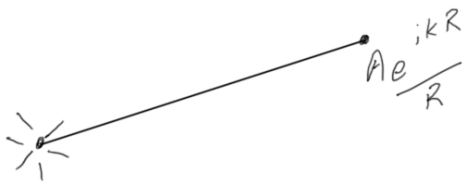

Figure 3.3: Wave function at the aperture.

(3.11)

Here $\mathbf{R} = \mathbf{r} - \mathbf{r}'$ and $da' = dx'dy'$ or $\rho' d\rho' d\theta'$. We are going to neglect the surface at ∞ as illustrated in fig. 3.4. This neglect is

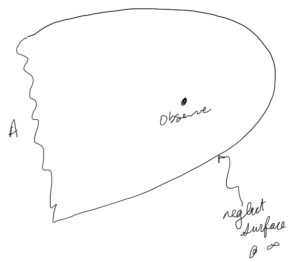

Figure 3.4: Neglecting the surface at infinity.

justified for example in Jackson, cited above.

3.3 A CALCULATED EXAMPLE: PINHOLE.

With placement of our origin at the pinhole, so that $\mathbf{r}' = 0$, $\mathbf{R} = \mathbf{r}$, $\mathbf{R}_s = \mathbf{r}_s$, we want to consider the geometry of fig. 3.5. Our spherical wave function at the aperture is

$$\Psi(\mathbf{r}') = A\frac{e^{ikR_s}}{R_s}. \tag{3.12}$$

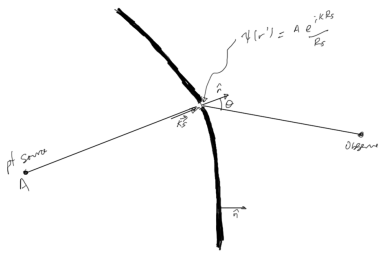

Figure 3.5: Source, aperture and observation point.

so that

$$\begin{aligned}\nabla'\Psi &= -A\frac{\mathbf{r}_s - \mathbf{r}'}{|\mathbf{r}_s - \mathbf{r}'|}\left(ik - \frac{1}{R_s}\right)\frac{e^{ikR_s}}{R_s} \\ &= -A\frac{\mathbf{r}_s}{|\mathbf{r}_s|}\left(ik - \frac{1}{R_s}\right)\frac{e^{ikR_s}}{R_s} \\ &= A\hat{\mathbf{n}}\left(ik - \frac{1}{R_s}\right)\frac{e^{ikR_s}}{R_s}.\end{aligned} \tag{3.13}$$

Our resulting wave function is then

$$\Psi(\mathbf{r}) = -\frac{A}{4\pi}\iint da'\frac{e^{ik(R+r_s)}}{Rr_s}\left(\hat{\mathbf{n}}\cdot\hat{\mathbf{n}}\left(ik - \frac{1}{r_s}\right) + \hat{\mathbf{n}}\cdot\hat{\mathbf{r}}\left(ik - \frac{1}{r}\right)\right). \tag{3.14}$$

Now, in all these $ik - 1/r_s$ we have k of order $1/\lambda$ and $1/r_s$ is of order $1/r$ or $1/r_s$.

Recall from geometric optics that we used

$$\nabla\left(\mathbf{E}_0 e^{i\phi(\mathbf{r})}\right) \approx i(\mathbf{E}_0\cdot\nabla\phi)e^{i\phi(\mathbf{r})}, \tag{3.15}$$

With an assumption

$$\lambda \ll r, r_s, \tag{3.16}$$

and

$$\lambda \ll d \ll r, r_s, \tag{3.17}$$

where d is the "typical object size", so that we have

$$\Psi(\mathbf{r}) = -\frac{A}{4\pi}\iint da'\frac{e^{ik(R+r_s)}}{Rr_s}\left(\hat{\mathbf{n}}\cdot\hat{\mathbf{n}}\left(ik - \cancel{\frac{1}{r_s}}\right) + \hat{\mathbf{n}}\cdot\hat{\mathbf{r}}\left(ik - \cancel{\frac{1}{r}}\right)\right).$$

(3.18)

or with θ as illustrated,

$$\boxed{\Psi(\mathbf{r}) = \frac{A}{\lambda i} \iint da' \frac{e^{ik(r+r_s)}}{rr_s} k(\theta),} \tag{3.19}$$

where

$$k(\theta) = \frac{1}{2}\left(1 + \cos\theta\right), \tag{3.20}$$

is the "obliquity factor".

This is called the Huygens-Fresnel principle.

3.4 FRESNEL AND FRAUNHOFER DIFFRACTION.

Last time we got as far as finding

$$\Psi(\mathbf{r}) = -\frac{1}{4\pi} \int da' \frac{e^{ikR}}{R} \left(\boldsymbol{\nabla}'\Psi' + \left(ik - \frac{1}{R} \right) \hat{\mathbf{R}}\Psi \right) \cdot \hat{\mathbf{n}}. \tag{3.21}$$

We want to consider non-pinhole apertures.

Length scales

- Wavelength $\lambda = 2\pi/k$
- Object size d, where the object is larger than λ
- Distance of observation $r \gg d \gg \lambda$
- Sources $r_s \gg d \gg \lambda$

Observe that $r \gg \lambda$ we have

$$\left(ik - \frac{1}{r} \right) = \frac{2\pi i}{\lambda} \left(1 - \frac{\lambda}{r} \right) \approx ik, \tag{3.22}$$

Another consequence is that in an integral like

$$\int \frac{\cdots}{|\mathbf{r}_s - \mathbf{r}'|} da' \approx \frac{1}{R_s} \int \cdots da', \tag{3.23}$$

because $r \gg d$.

We still have to be careful with something like

$$\int \cdots e^{ik\mathbf{r}'^2/r}. \tag{3.24}$$

since this exponential could matter very much. Recalling that $\mathbf{R} = \mathbf{r} - \mathbf{r}'$ and expanding in Taylor series to second order

$$\begin{aligned}
R &= \sqrt{\mathbf{R}^2} \\
&= \sqrt{(\mathbf{r}-\mathbf{r}')^2} \\
&= \sqrt{\mathbf{r}^2 + \mathbf{r}'^2 - 2\mathbf{r}\cdot\mathbf{r}'} \\
&= r\sqrt{1 + \frac{\mathbf{r}'^2}{\mathbf{r}^2} - 2\frac{\hat{\mathbf{r}}}{r}\cdot\mathbf{r}'} \\
&\approx r\left(1 + \frac{1}{2}\left(\frac{\mathbf{r}'^2}{\mathbf{r}^2} - 2\frac{\hat{\mathbf{r}}}{r}\cdot\mathbf{r}'\right) - \frac{1}{8}\left(\frac{\mathbf{r}'^2}{\mathbf{r}^2} - 2\frac{\hat{\mathbf{r}}}{r}\cdot\mathbf{r}'\right)^2 + \cdots\right) \\
&= r\left(1 + \frac{1}{2}\frac{\mathbf{r}'^2}{\mathbf{r}^2} - \frac{\hat{\mathbf{r}}}{r}\cdot\mathbf{r}' - \frac{1}{8}\frac{\mathbf{r}'^4}{\mathbf{r}^4} - \frac{1}{2}\left(\frac{\hat{\mathbf{r}}}{r}\cdot\mathbf{r}'\right)^2 + \frac{1}{2}\frac{\mathbf{r}'^2}{\mathbf{r}^2}\frac{\hat{\mathbf{r}}}{r}\cdot\mathbf{r}' + \cdots\right) \\
&= r\left(1 + \frac{1}{2r^2}r'^2 - \frac{\hat{\mathbf{r}}}{r}\cdot\mathbf{r}' - \frac{1}{8r^4}r'^4 - \frac{1}{2r^2}\left(\hat{\mathbf{r}}\cdot\mathbf{r}'\right)^2 + \frac{1}{2r^3}r'^2\hat{\mathbf{r}}\cdot\mathbf{r}' + \cdots\right).
\end{aligned} \tag{3.25}$$

Grouping by order of significance we have

$$kR = \underbrace{\boxed{kr}}_{O(r\lambda)} - \underbrace{\boxed{k\hat{\mathbf{r}}\cdot\mathbf{r}'}}_{O(d/\lambda)} + \underbrace{\boxed{\frac{k}{2r}\left(\mathbf{r}'^2 - \left(\hat{\mathbf{r}}\cdot\mathbf{r}'\right)^2\right)}}_{O(d^2/\lambda r)} + \underbrace{\boxed{\cdots}}_{O(d^3/\lambda r^2)}. \tag{3.26}$$

We need any exponential to be small with respect to 1 to neglect.

- In the above if $r \gg d^2/\lambda$ we can neglect this third term, which is the Fraunhofer case.
- Fresnel diffraction retains the $\mathbf{r}'^2$ term! (with $r^2 \gg d^3/\lambda$, we'll stop at this $\mathbf{r}'^2$ term).

We'll treat the Fraunhofer case now killing the $1/R$ term here:

$$\Psi(\mathbf{r}) = -\frac{1}{4\pi}\int da' \frac{e^{ikR}}{R}\left(\boldsymbol{\nabla}'\Psi' + \left(ik - \cancel{\frac{1}{R}}\right)\hat{\mathbf{R}}\Psi\right)\cdot\hat{\mathbf{n}}. \tag{3.27}$$

We'll use a point source

$$\Psi(\mathbf{r}') = \frac{e^{ikR_s}}{R_s}. \tag{3.28}$$

where

$$\boldsymbol{\nabla}'\Psi' = -\hat{\mathbf{R}}_s \left(ik - \cancel{\frac{1}{R_s}} \right) \frac{e^{ikR_s}}{R_s} \approx \hat{\mathbf{n}} ik \frac{e^{ikR_s}}{R_s}. \tag{3.29}$$

Our diffraction integral becomes

$$\Psi(\mathbf{r}) = -\frac{ik}{4\pi} \int da' \frac{e^{ikR}}{R} \frac{e^{ikR_s}}{R_s} \left(1 + \hat{\mathbf{n}} \cdot \mathbf{R}\right). \tag{3.30}$$

With a small enough object $d \ll r, r_s$, and writing $k/2\pi = 1/\lambda$, we'll be able to pull the obliquity factor out of the integral

$$\begin{aligned} \Psi(\mathbf{r}) &= \frac{A}{\lambda i} \iint da' \frac{e^{ik(R+R_s)}}{R_s R} k(\theta) \\ &\approx \frac{Ak(\theta)}{\lambda i} \iint da' \frac{e^{ik(R+R_s)}}{R_s R} \\ &\approx \frac{A}{\lambda i} \iint da' \frac{e^{ik(R+R_s)}}{R_s R}. \end{aligned} \tag{3.31}$$

We've also made the paraxial approximation, recalling that

$$k(\theta) = \frac{1 + \cos\theta}{2}, \tag{3.32}$$

so that for $\theta \approx 0$, in the region illustrated in fig. 3.6 and fig. 3.7.

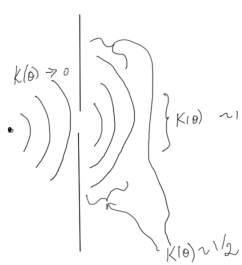

Figure 3.6: Obliquity factor.

we have $k(\theta) \approx 1$.

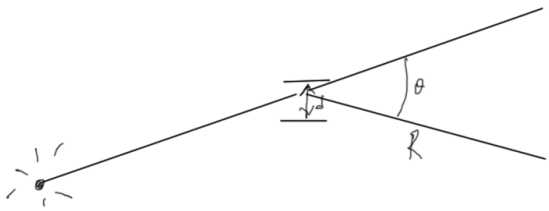

Figure 3.7: Obliquity factor.

Our problem is now reduced to

$$\Psi(\mathbf{r}) = \frac{A}{\lambda i}\frac{e^{ik(r_s+r)}}{r_s r}\iint_{\text{aperture}} da' e^{ikf(\mathbf{r}')}. \tag{3.33}$$

where

$$\begin{aligned} f(\mathbf{r}') = &-(\hat{\mathbf{r}}+\hat{\mathbf{r}}_s)\cdot\mathbf{r}' \\ &+\frac{1}{2r}\left(\mathbf{r}'^2-(\hat{\mathbf{r}}\cdot\mathbf{r}')^2\right)+\frac{1}{2r_s}\left((\mathbf{r}')^2-(\hat{\mathbf{r}}_s\cdot\mathbf{r}')^2\right). \end{aligned} \tag{3.34}$$

the first term is the Fraunhofer term and the last two are the Fresnel contributions.

Referring to fig. 3.8 we find

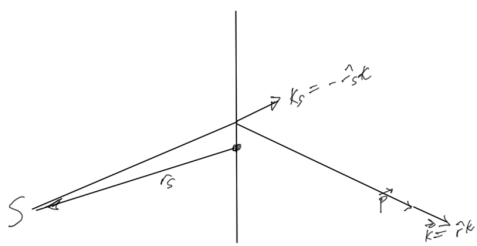

Figure 3.8: Defining k vectors.

$$kf = -k(\hat{\mathbf{r}}+\hat{\mathbf{r}}_s)\cdot\mathbf{r}' = -(\mathbf{k}-\mathbf{k}_s)\cdot\mathbf{r}' = (\mathbf{k}_s-\mathbf{k})\cdot\mathbf{r}'. \tag{3.35}$$

putting things back into the diffraction integral, we have something of the form

$$\Psi(\mathbf{r}) = \text{constant}\iint_{\text{aperture}} d^2\mathbf{r}' e^{i(\mathbf{k}-\mathbf{k}_s)\cdot\mathbf{r}'} g(\mathbf{r}'). \tag{3.36}$$

where $g(\mathbf{r}')$ is an "aperture" function defined in the open portion as illustrated in fig. 3.9. If we define $g(\mathbf{r}')$ to be zero outside of the

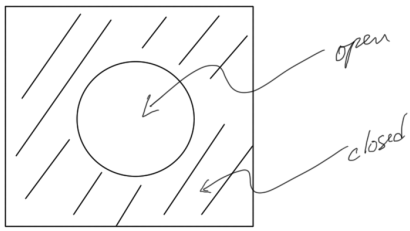

Figure 3.9: Circular aperture.

aperture

$$g(\mathbf{r}') = \begin{cases} 1 & \text{open} \\ 0 & \text{blocked} \end{cases} \tag{3.37}$$

then we can just write

$$\Psi(\mathbf{r}) = \text{constant} \iint d^2\mathbf{r}' e^{i(\mathbf{k}-\mathbf{k}_s)\cdot\mathbf{r}'} g(\mathbf{r}'). \tag{3.38}$$

so that

$$\Psi = (\text{constant}) G(\mathbf{k}_s - \mathbf{k}). \tag{3.39}$$

where

$$G(\mathbf{k}) = \iint e^{-i\mathbf{k}\cdot\mathbf{r}'} g(\mathbf{r}') d^2\mathbf{r}', \tag{3.40}$$

which is just a Fourier transform! Our amplitude is

$$I(\mathbf{r}) = |\Psi(\mathbf{r})|^2 = (\text{constant})^2 |G(\mathbf{k}_s - \mathbf{k})|^2. \tag{3.41}$$

Note that if the amplitude

$$|\Psi(\mathbf{r}')| = \Psi_0. \tag{3.42}$$

then this constant is

$$\frac{1}{2}\left(\frac{k\Psi_0}{2\pi r}\right)^2. \tag{3.43}$$

Calculating this for a circular pattern is done in the class notes handout, where the result involved J_1 Bessel functions.

We can deal with double slit by doing a convolution of a rectangle aperture with a pair of delta functions and then just multiply the Fourier transforms.

We will be applying this diffraction result to investigate coherence. We'll find that if the source is not coherent, the chance of observing the fringe oscillations far from the source becomes very small.

3.5 FRESNEL DIFFRACTION FROM AN EDGE.

Consider the experiments illustrated in fig. 3.10, and fig. 3.11.

Figure 3.10: Intensity observed with no blockages.

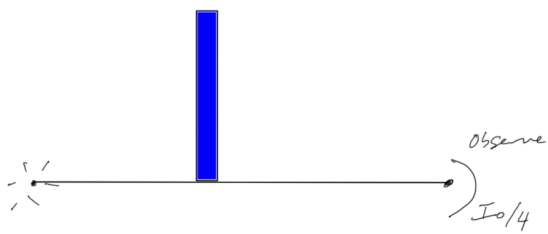

Figure 3.11: Intensity observed with blockage just above line of sight.

Why, with such a carefully placed barrier, do we end up with $I_0/4$? If we consider that the light takes all paths, and we have blocked half the paths, so that the amplitude of the wave function

$|\Psi_0/2|$ results in the factor of $1/4$. Let's do the math to see why this is the case more precisely.

We found

$$\Psi(\mathbf{r}) = \frac{A}{i\lambda}\frac{e^{ik(r_s+r)}}{r_s r}\int_{\text{aperture}} e^{ikf(\mathbf{r}')}da'. \tag{3.44}$$

In the Fraunhofer limit (the far field) we found eq. (3.35) that

$$kf \to (\mathbf{k}_s - \mathbf{k})\cdot\mathbf{r}', \tag{3.45}$$

where $r \gg d^2/\lambda$, and d is a typical aperture size. Recalling that the exact expression for f was

$$f(\mathbf{r}') = -\left(\hat{\mathbf{r}} + \hat{\mathbf{r}}_s\right)\cdot\mathbf{r}' + \frac{1}{2r}\left(\mathbf{r}'^2 - (\hat{\mathbf{r}}\cdot\mathbf{r}')^2\right) + \frac{1}{2r_s}\left((\mathbf{r}')^2 - (\hat{\mathbf{r}}_s\cdot\mathbf{r}')^2\right). \tag{3.46}$$

We'll now consider the <u>Fresnel</u> limit where $\mathbf{k}_s = \mathbf{k}$, and

$$kf = \frac{k}{2}\left(\frac{1}{r} + \frac{1}{r_s}\right)r'^2. \tag{3.47}$$

These Fresnel terms are generally important when $r \sim d^2/\lambda$ even if $r \gg d$ (because $\lambda \ll d$). We'd like to massage this kf expression

$$kf = \frac{k}{2}\left(r_s^{-1} + r^{-1}\right)\left(x'^2 + y'^2\right) = \frac{\pi}{2}\left(u^2 + v^2\right), \tag{3.48}$$

where we have made a change of variables

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \sqrt{\frac{\pi/k}{r_s^{-1} + r^{-1}}}\begin{bmatrix} u \\ v \end{bmatrix} = \sqrt{\frac{\lambda/2}{r_s^{-1} + r^{-1}}}\begin{bmatrix} u \\ v \end{bmatrix} \tag{3.49}$$

Our area element is then

$$dx'dy' = \frac{\lambda/2}{r_s^{-1} + r^{-1}}dudv. \tag{3.50}$$

Our integral is now

$$\begin{aligned}\Psi(\mathbf{r}) &= \frac{A}{i\lambda}\frac{e^{ik(r_s+r)}}{r_s r}\int_{\text{aperture}} e^{ikf(\mathbf{r}')}da' \\ &= \frac{A}{i\lambda}\frac{e^{ik(r_s+r)}}{r_s r}\frac{\lambda/2}{r_s^{-1} + r^{-1}}\int_{\text{aperture}} e^{i\frac{\pi}{2}(u^2+v^2)}dudv \\ &= \frac{A}{i\lambda}e^{ik(r_s+r)}\frac{\lambda/2}{r_s + r}\int_{\text{aperture}} e^{i\frac{\pi}{2}(u^2+v^2)}dudv.\end{aligned} \tag{3.51}$$

Referring to fig. 3.12, let's do this integral. Putting in our limits we have

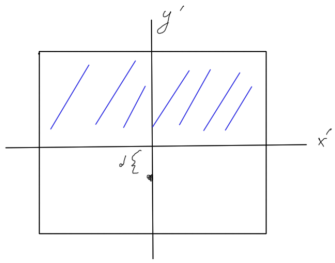

Figure 3.12: Region of integration.

$$\Psi(\mathbf{r}) = \frac{A}{2i}\frac{e^{ik(r_s+r)}}{r_s+r}\int_{-\infty}^{\infty} du \int_{-\infty}^{w} dv e^{i\frac{\pi}{2}\left(u^2+v^2\right)} du dv. \tag{3.52}$$

where

$$w = \sqrt{\frac{2}{\lambda}\left(\frac{1}{r_s}+\frac{1}{r}\right)}. \tag{3.53}$$

Evaluating $\int -z^2 dz$ over a pizza contour it can be demonstrated [12] that

$$\int_{-\infty}^{\infty} dv e^{i\frac{\pi}{2}v^2} = 1+i = \sqrt{2}e^{i\pi/4}. \tag{3.54}$$

$$\int_{-\infty}^{w} dv e^{i\frac{\pi}{2}v^2} = \int_{-\infty}^{0}\int_{0}^{w} dv e^{i\frac{\pi}{2}v^2} = \frac{1+i}{2} + C(w) + S(w), \tag{3.55}$$

where

$$S(w) = \int_0^w \sin\left(\frac{\pi}{2}u^2\right) du. \tag{3.56a}$$

$$C(w) = \int_0^w \cos\left(\frac{\pi}{2}u^2\right) du. \tag{3.56b}$$

Parametrically plotting these we get the Cornu Spiral as plotted in fig. 3.13. There are some interesting features of this curve.

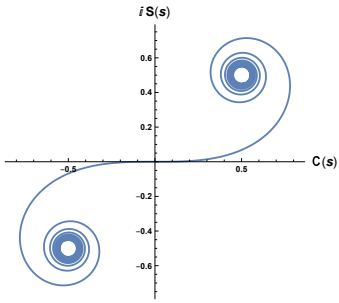

Figure 3.13: Cornu Spiral.

1. The length along the curve is

$$\begin{aligned} dl^2 &= dS^2 + dC^2 \\ &= \left(\frac{dS}{dw}\right)^2 + \left(\frac{dC}{dw}\right)^2 \\ &= \left(\sin^2\left(\frac{\pi}{2}w^2\right) + \cos^2\left(\frac{\pi}{2}w^2\right)\right) dw^2 \\ &= dw^2. \end{aligned} \tag{3.57}$$

so that

$$\boxed{dl = dw} \tag{3.58}$$

2. How about the angle along the curve. Stating the result, where the angle is given by

$$\boxed{\tan\theta = \frac{dy}{dx}} \tag{3.59}$$

one can find that

$$\theta = \frac{\pi}{2}w^2. \tag{3.60}$$

Going back to our diffraction integral we find

$$\Psi(\mathbf{r}) = \frac{A}{2i}\frac{e^{ik(r_s+r)}}{r_s+r}(1+i)\left(\frac{1+i}{2} + C(w) + S(w)\right). \tag{3.61}$$

Check: No obstruction? We've got $w \to \infty$, so that $C(w) = 1/2$ and $S(w) = 1/2$. This gives us

$$\Psi(\mathbf{r}) = A\frac{e^{ik(r_s+r)}}{r_s + r} \equiv \Psi_\infty(\mathbf{r}). \tag{3.62}$$

Now let's consider our obstruction right along the line of sight ($w = 0$). Now we have, since $C(0) = S(0) = 0$

$$\Psi(\mathbf{r}) = \Psi_\infty(\mathbf{r})\frac{1}{2i}(1+i)\left(\frac{1+i}{2} + 0\right) = \frac{1}{2}\Psi_\infty(\mathbf{r}). \tag{3.63}$$

We do see that we end up with half the amplitude, so that as claimed our intensity (which squares the amplitude) results in a factor of $1/4$

In general, for a barrier offset by d, and a value of w that corresponds to that, our Intensity is

$$I = |\Psi|^2 = |\Psi_\infty|^2\frac{1}{2}\left(\left(\frac{1}{2} + C(w)\right)^2 + \left(\frac{1}{2} + S(w)\right)^2\right). \tag{3.64}$$

Check, again with $w = 0$, we have

$$I_\infty\frac{1}{2}\left(\frac{1}{4} + \frac{1}{4}\right) = \frac{I_\infty}{4}. \tag{3.65}$$

Other examples.

Diffraction from an edge $w = d$: fig. 3.14. Poisson spot. Poisson

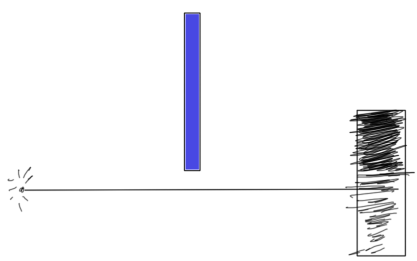

Figure 3.14: Diffraction spectrum with partial blockage above line of sight (brutally rough illustration).

crafted a counter argument for the wave theory of light stating that

if it was true, then you should be able to see a spot behind a circular blockage, as if some of the light was going around the blockage. This is illustrated in fig. 3.15, and can in fact be observed with the right setup. Once we understand that light does in fact take all

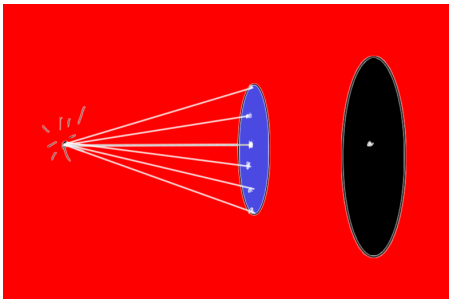

Figure 3.15: Poisson spot.

the paths, we can utilize this to build a Fresnel lens by blocking selectively as illustrated very roughly in fig. 3.16. A diffraction

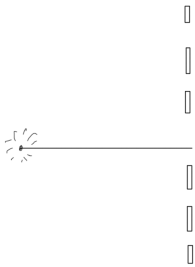

Figure 3.16: Diffraction grating (imagined to have been constructed to focus x-rays).

setup like allows us block all the portions of the phase that negatively interfere. This can be used for example to focus x-rays. That application will be explored in more detail in the problem set.

3.6 PROBLEMS.

Exercise 3.1 Diffraction patterns. (*2012 Ps2, P1*)

Give the intensity pattern at the back plane of a lens of focal length f for the following aperture distributions. Assume a point source

that is infinitely far away, on-axis, and quasimonochromatic with wavelength λ.

This geometry is illustrated in fig. 3.17.

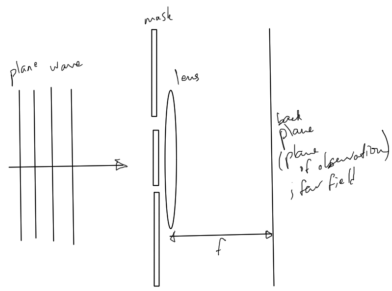

Figure 3.17: Geometry for the masked diffraction problems.

a. A rectangular aperture of size L by W
b. Three slits, each a wide, spaced by b. {Do this part in 1D, ignoring the other axis.}
c. A mask whose transmission function is $\exp\left(-x^2/\sigma_x^2 - y^2/\sigma_y^2\right)$.

Answer for Exercise 3.1

Let's first consider the geometrical optics of the lens and transmission to the focal plane. With an ABCD matrix of M_1 for the lens and M_2 for the transmission we have for the composite operation

$$M = M_2 M_1 = \begin{bmatrix} 1 & f \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -1/f & 0 \end{bmatrix} = \begin{bmatrix} 0 & f \\ -1/f & 1 \end{bmatrix}. \tag{3.66}$$

So an initial position and angle pair is transformed as

$$\begin{bmatrix} y \\ \alpha \end{bmatrix} \to \begin{bmatrix} 0 & f \\ -1/f & 1 \end{bmatrix} \begin{bmatrix} y \\ \alpha \end{bmatrix} = \begin{bmatrix} f\alpha \\ -y/f + \alpha \end{bmatrix} \tag{3.67}$$

The new position $y' = f\alpha$ is strictly a function of the output angle at the mask, or the input to the lens.

Now consider the geometry of the aperture and the position of the ray on the focal plane as illustrated in fig. 3.18. The position $\mathbf{r}'$

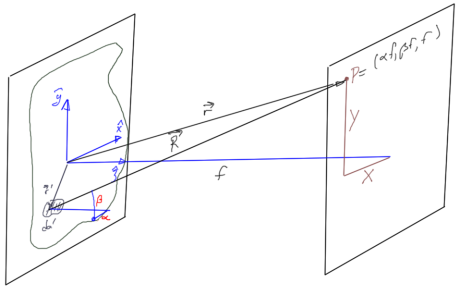

Figure 3.18: Diffraction geometry with lens.

of the area element in the diffraction integral is

$$\mathbf{r}' = \begin{bmatrix} x' \\ y' \\ 0 \end{bmatrix}, \tag{3.68}$$

and the position P of the ray on the focal plane is

$$\mathbf{r} = f \begin{bmatrix} \alpha \\ \beta \\ 1 \end{bmatrix}. \tag{3.69}$$

Our diffraction integral is of the form

$$\Psi(\mathbf{r}) = \frac{A}{i\lambda} \frac{e^{ikR_s}}{R_s} \iint_A \frac{e^{ikR}}{R} da'. \tag{3.70}$$

Writing for the incident wavefunction

$$\Psi_s = A \frac{e^{ikR_s}}{R_s}, \tag{3.71}$$

which we will consider to be approximately plane wave anyhow, we calculate our approximation for the distance from the area element to the point on the focal plane

$$\begin{aligned} kR &= k\sqrt{(\mathbf{r}-\mathbf{r}')^2} \\ &= kr\sqrt{1+\frac{r'^2}{r^2}-\frac{2}{r^2}\mathbf{r}\cdot\mathbf{r}'} \\ &\approx kr\left(1+\frac{1}{2}\frac{r'^2}{r^2}-\frac{1}{r^2}\mathbf{r}\cdot\mathbf{r}'\right) \\ &= kr+\frac{k}{2}\frac{r'^2}{r}-\hat{\mathbf{r}}\cdot\mathbf{r}'. \end{aligned} \tag{3.72}$$

With an assumption that $r \approx f \gg r'^2/\lambda$, and noting that $R \approx f$ in the denominator of the integrand where any variation will not matter as much we have

$$\Psi(\mathbf{r}) = \frac{\Psi_s}{i\lambda}\frac{e^{ikf}}{f}\iint_A e^{-ik\mathbf{r}'\cdot\hat{\mathbf{r}}}da'. \tag{3.73}$$

Noting from eq. (3.69) that $\mathbf{r}^2 \approx f^2$ if α and β are small, then we have

$$\hat{\mathbf{r}} \sim \begin{bmatrix}\alpha\\ \beta\\ 1\end{bmatrix}, \tag{3.74}$$

so that our diffraction integral becomes

$$\Psi(\mathbf{r}) = \frac{\Psi_s}{i\lambda}\frac{e^{ikf}}{f}\iint_A e^{-ik(\alpha x'+\beta y')}da'. \tag{3.75}$$

We are now ready to consider the specific geometries of this problem.

Part a. Single rectangular slit. Let's setup our coordinates as in fig. 3.19. Let's write $\mu = -ik\alpha$ and $\nu = -ik\beta$ so that we want to

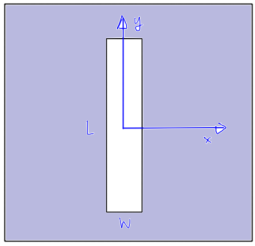

Figure 3.19: Single rectangular slit.

compute

$$\begin{aligned}\iint e^{-ik(x'\alpha+y'\beta)}dx'dy' &= \int_{-W/2}^{W/2} dx' e^{i\mu x'} \int_{-L/2}^{L/2} dy' e^{i\nu y'} \\ &= \left.\frac{e^{i\mu x'}}{i\mu}\right|_{-W/2}^{W/2} \left.\frac{e^{i\nu y'}}{i\nu}\right|_{-L/2}^{L/2} \\ &= \frac{4}{\mu\nu}\sin(\mu W/2)\sin(\nu L/2) \\ &= LW\,\mathrm{sinc}(\mu W/2)\,\mathrm{sinc}(\nu L/2)\,.\end{aligned} \tag{3.76}$$

So our wave function is

$$\Psi(\mathbf{r}) = \frac{LW\Psi_s}{i\lambda}\frac{e^{ikf}}{f}\,\mathrm{sinc}(\mu W/2)\,\mathrm{sinc}(\nu L/2)\,. \tag{3.77}$$

With the time averaged intensity at the aperture of

$$I_s = \langle \Psi_s \rangle = \frac{A^2}{2R_s^2}, \tag{3.78}$$

We have for the time averaged intensity at the focal plane position $\mathbf{r} = f(\alpha, \beta, 1)$

$$\boxed{I(\mathbf{r}) = \Psi(\mathbf{r}) = I_s\left(\frac{LW}{\lambda f}\right)^2 \mathrm{sinc}^2\left(\frac{\pi\alpha W}{\lambda}\right)\mathrm{sinc}^2\left(\frac{\pi\beta L}{\lambda}\right).} \tag{3.79}$$

Part b. Three slits We now consider the geometry of a three slit setup fig. 3.20. We form the integral

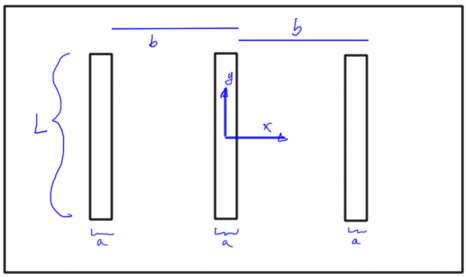

Figure 3.20: Three slit diffraction aperture.

$$
\begin{aligned}
&\iint_A e^{-ik(\alpha x'+\beta y')}dx'dy' \\
&\quad = \int_{-L/2}^{L/2} dy' e^{-ik\beta y'} \left(\left(\int_{-b-a/2}^{-b+a/2} + \int_{-a/2}^{a/2} + \int_{b-a/2}^{b+a/2} \right) dx' e^{-ik\alpha x'} \right) \\
&\quad = L \operatorname{sinc}\left(\frac{\beta k L}{2}\right) \sum_{n=-1}^{n=1} \left. \frac{e^{-ik\alpha x'}}{-ik\alpha} \right|_{nb-a/2}^{nb+a/2} \\
&\quad = L \operatorname{sinc}\left(\frac{\beta k L}{2}\right) \frac{2}{k\alpha} \sum_{n=-1}^{n=1} \left(\frac{e^{-ik\alpha(nb-a/2)} - e^{-ik\alpha(nb+a/2)}}{2i} \right) \\
&\quad = L \operatorname{sinc}\left(\frac{\beta k L}{2}\right) \frac{2}{k\alpha} \sin(k\alpha a/2) \sum_{n=-1}^{n=1} e^{-ik\alpha nb} \\
&\quad = La \operatorname{sinc}\left(\frac{\beta k L}{2}\right) \operatorname{sinc}\left(\frac{\alpha k a}{2}\right) \sum_{n=-1}^{n=1} e^{-ik\alpha nb}.
\end{aligned}
\tag{3.80}
$$

With $a = e^{ik\alpha b}$, and noting that

$$
\frac{1}{a} + 1 + a = \frac{a^3 - 1}{a - 1}, \tag{3.81}
$$

we have

$$
\sum_{n=-1}^{n=1} e^{-ik\alpha nb} = \frac{e^{3ik\alpha b} - 1}{e^{ik\alpha b} - 1} = \frac{e^{3ik\alpha b/2}}{e^{ik\alpha b/2}} \frac{\sin(3k\alpha b/2)}{\sin(k\alpha b/2)} = e^{ik\alpha b} \frac{\sin(3k\alpha b/2)}{\sin(k\alpha b/2)}. \tag{3.82}
$$

We can now write our wavefunction

$$
\Psi(\mathbf{r}) = \frac{La\Psi_s}{i\lambda} \frac{e^{ikf}}{f} \operatorname{sinc}\left(\frac{\beta k L}{2}\right) \operatorname{sinc}\left(\frac{\alpha k a}{2}\right) e^{ik\alpha b} \frac{\sin\left(\frac{3k\alpha b}{2}\right)}{\sin\left(\frac{k\alpha b}{2}\right)}. \tag{3.83}
$$

We see our single slit terms become an envelope for the resulting waveform, with faster frequency terms due to the ratio of sinusoidal terms.

Our time averaged intensity, again in terms of the time averaged intensity of the plane waves at the aperture I_s, is

$$I = I_s \left(\frac{La}{\lambda f}\right)^2 \operatorname{sinc}^2\left(\frac{\beta k L}{2}\right) \operatorname{sinc}^2\left(\frac{\alpha k a}{2}\right) \frac{\sin^2\left(\frac{3k\alpha b}{2}\right)}{\sin^2\left(\frac{k\alpha b}{2}\right)}. \tag{3.84}$$

We were asked to consider this as a 1D problem, but it was no harder as a 2D problem. For a 1D only result, looking say at the horizon where $\beta = 0$, we have

$$\boxed{I = I_s \left(\frac{La}{\lambda f}\right)^2 \operatorname{sinc}^2\left(\frac{\alpha k a}{2}\right) \frac{\sin^2\left(\frac{3k\alpha b}{2}\right)}{\sin^2\left(\frac{k\alpha b}{2}\right)}.} \tag{3.85}$$

Part c. Transmission function We've been implicitly evaluating diffraction integrals of the form

$$\iint_A e^{-ik(x'\alpha + y'\beta)} dx'dy' = \iint_{-\infty}^{\infty} g(x', y') e^{-ik(x'\alpha + y'\beta)} dx'dy'. \tag{3.86}$$

where

$$g(x', y') = \begin{cases} 1 & \text{if } x' \text{ and } y' \text{ lie within the aperture} \\ 0 & \text{otherwise} \end{cases} \tag{3.87}$$

We are now asked to consider a more general aperture function

$$g(x', y') = e^{-x'^2/\sigma_x^2 - y'^2/\sigma_y^2}. \tag{3.88}$$

which fully allows transmission at the origin where $g(0, 0) = 1$, and then gradually lets less and less light through the aperture as illustrated in fig. 3.21. Our task is to evaluate the integral

$$\iint_{-\infty}^{\infty} g(x', y') e^{-ik(x'\alpha + y'\beta)} dx'dy' = \int dx' e^{-x'^2/\sigma_x^2 - ikx'\alpha} \int dy' e^{-y'^2/\sigma_y^2 - iky'\beta}. \tag{3.89}$$

Figure 3.21: Gaussian transmission function aperture.

Since these have the same form, it is sufficient to just look at one of them.

$$\int dx' e^{-x'^2/\sigma_x^2 - ikx'\alpha} = \sigma_x \int dx' e^{-x'^2/\sigma_x^2 - ik\alpha\sigma_x x'/\sigma_x} = \sigma_x \int dx' e^{-u^2 - ik\alpha\sigma_x u}. \tag{3.90}$$

With $2b = k\alpha\sigma_x$ we have

$$\begin{aligned} \int dx' e^{-x'^2/\sigma_x^2 - ikx'\alpha} &= \sigma_x \int dx' e^{-u^2 - 2ibu} \\ &= \sigma_x \int du e^{-(u+ib)^2 + (ib)^2} \\ &= \sigma_x e^{-b^2} \int dv e^{-v^2} \\ &= \sigma_x e^{-b^2} \sqrt{\pi}. \end{aligned} \tag{3.91}$$

Our aperture integral is

$$\iint_{-\infty}^{\infty} g(x', y') e^{-ik(x'\alpha + y'\beta)} dx' dy' = \pi\sigma_x\sigma_y e^{-(k\alpha\sigma_x/2)^2 - (k\beta\sigma_y/2)^2}, \tag{3.92}$$

and our wave function evaluated at point $\mathbf{r} = f(\alpha, \beta, 1)$, given plane wave function Ψ_s at the aperture is

$$\Psi(\mathbf{r}) = \frac{\pi\sigma_x\sigma_y\Psi_s}{i\lambda} \frac{e^{ikf}}{f} e^{-(k\alpha\sigma_x/2)^2 - (k\beta\sigma_y/2)^2}, \tag{3.93}$$

with time averaged intensity

$$\boxed{I(\mathbf{r}) = I_s \left(\frac{\pi\sigma_x\sigma_y}{\lambda f} \right)^2 e^{-(k\alpha\sigma_x)^2/2 - (k\beta\sigma_y)^2/2}.} \tag{3.94}$$

Exercise 3.2 Fresnel Lens. (*2012 Ps2, P2*)

In this problem you will design a Fresnel lens that focuses a plane wave onto a line (ie, a cylindrical lens). Design it for wavelength λ and a focal length f. The on-axis Fresnel diffraction integral we found in class was $\Psi = \frac{\Psi_\infty}{2i} \int\int du\, dv \exp\left[i\frac{\pi}{2}(u^2+v^2)\right]$, where $u = x'\sqrt{2(r_S^{-1}+r^{-1})/\lambda}$, and a similar expression applied for v as a function of y'. The goal of the design is to maximize the intensity at the focus of the lens, located at $\mathbf{r} = \langle 0, 0, f\rangle$. We'll call this amplitude Ψ_f and this intensity I_f.

The variables to be used for the Fresnel mask transitions is illustrated in fig. 3.22.

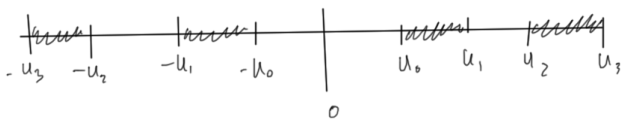

Figure 3.22: Notation for Fresnel lens mask.

a. Integrate out the y' direction, so that we are only dealing with an integral of u. Give an expression for the contribution to Ψ_f from an open segment from $u = a$ to $u = b$, in terms of the Fresnel Integrals $\mathcal{S}(u)$ and $\mathcal{C}(u)$.

b. As a first step in the design, consider a Fresnel lens with only one opening, from $-u_0$ to u_0. Plot the resultant intensity I_f/I_∞ versus u_0. *You will need numerical evaluation of the Fresnel integrals for parts b and e of this problem.*

c. The next step is to find subsequent open regions (I'll call these zones) of the mask that most increase I_f. These zones need to chosen to have a phase that matches the amplitude passing through the central zone. **Illustrate this principle with a drawing of the Cornu spiral**, showing what segments should be blocked.

d. From (c), show that this criterion results in zone edges at $\pm u_m$, where $u_m = \sqrt{3/2 + 2m}$, and $m = \{0, 1, 2, \ldots\}$. *Hint: recall that the angle of the spiral is* $\beta = \frac{\pi}{2}u^2$.

e. Calculate the increase in intensity with three open zones and with five open zones.

f. Give a pathlength interpretation of the expression for u_m.

g. If this is an x-ray Fresnel lens, where $\lambda = 10\,\text{nm}$ and $f = 10\,\text{cm}$, how big is the central opening? Take the plane wave limit $r_s \to \infty$. If we can only fabricate zones as small as $1\mu\text{m}$, how many zones could we make?

Answer for Exercise 3.2

Part a. Performing the y (v) integration we find for our wavefunction

$$\begin{aligned}\Psi &= \frac{\Psi_\infty}{2i}\int due^{i\pi u^2/2}du\int_{-\infty}^{\infty} due^{i\pi u^2/2}du \\ &= \frac{\Psi_\infty}{2i}\int due^{i\pi u^2/2}du\left(2\int_0^{\infty} due^{i\pi u^2/2}du\right) \\ &= \frac{\Psi_\infty}{2i}\int due^{i\pi u^2/2}du2\left(\frac{1}{2}+\frac{i}{2}\right) \\ &= \frac{\Psi_\infty}{2}(1-i)\int due^{i\pi u^2/2}du,\end{aligned} \tag{3.95}$$

so for an interval $[a, b]$ we have for the wavefunction observed at $\mathbf{r} = (0, 0, f)$

$$\Psi_f = \frac{\Psi_\infty}{2}(1-i)\left(C(s)+iS(s)\right)\Big|_a^b. \tag{3.96}$$

Part b. Setting the interval to $[-u_0, u_0]$ we have, noting that $S(-s) = -S(s)$ and $C(-s) = -C(s)$

$$\Psi_f = \Psi_\infty(1-i)\left(C(u_0)+iS(u_0)\right). \tag{3.97}$$

The ratio of intensities is

$$\frac{I_f}{I_\infty} = 2\left(C^2(u_0)+S^2(u_0)\right), \tag{3.98}$$

which is plotted in fig. 3.23. Numerically, we find that the peak of the first lobe falls at $u_0 \sim 1.21$. A plot of the Cornu spiral up to this point of maximum intensity is found in fig. 3.24. Visually it appears that the angle at the termination of this region is $3\pi/4$ which is consistent with $u_0 = \sqrt{3/2}$, since the angle at u_0 is $\pi(3/2)/2$.

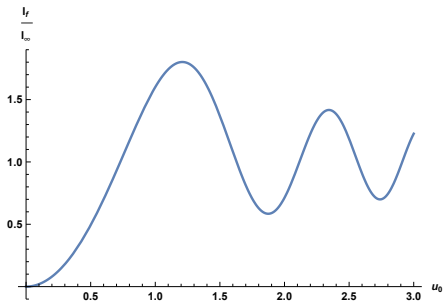

Figure 3.23: Intensity ratio vs u_0.

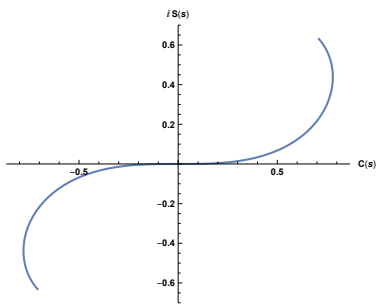

Figure 3.24: Cornu spiral segment up to the point of the max intensity ratio.

Parts c, d. For an aperture open in the interval $[-u_2, -u_1]$ and $[u_1, u_2]$ we have

$$\begin{aligned}\Psi_f &= \frac{\Psi_\infty}{2}(1-i)\Big(C(u_2)+iS(u_2)-C(u_1)-iS(u_1)\\ &\qquad +C(-u_1)+iS(-u_1)-C(-u_2)-iS(-u_2)\Big)\\ &= \Psi_\infty(1-i)\left(C(u_2)+iS(u_2)-C(u_1)-iS(u_1)\right).\end{aligned} \tag{3.99}$$

Should we with to add this non-destructively to our wavefunction for the $[-u_0, u_0]$ range, we need to match the phases of these functions, or

$$\frac{S[u_0]}{C[u_0]} = \frac{S[u_2]-S[u_1]}{C[u_2]-C[u_1]}. \tag{3.100}$$

For Part d we are asked to show that the zone edges are found at $u_m = \sqrt{3/2+2m}$, given that the tangential angles at those points are $\pi u_m^2/2$. That is

$$\theta(u_m) = \frac{\pi}{2}u_m^2 = \frac{\pi}{2}\left(\frac{3}{2}+2m\right) = \frac{3\pi}{4}+m\pi. \tag{3.101}$$

This gives us

$$\begin{aligned}\theta(u_0) &= \frac{3\pi}{4}\\ \theta(u_1) &= \frac{3\pi}{4}+\pi\\ \theta(u_2) &= \frac{3\pi}{4}+2\pi\ldots\end{aligned} \tag{3.102}$$

so that the difference between the tangential angle at u_m and u_{m-1} is π. In the upper right quadrant of the spiral we see that blocking the intervals $[u_{2k}, u_{2k+1}]$ will provide, approximately, the desired phase matching of eq. (3.100). This masking is illustrated in fig. 3.25.

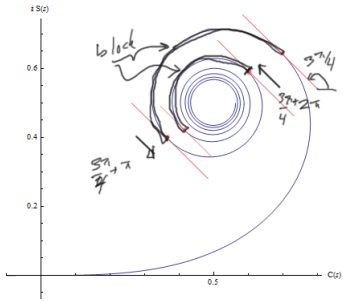

Figure 3.25: Cornu Spiral with regions blocked for equal phase differences.

Part e. The respective wave functions for each of the non-blocked intervals chosen are

- $[-u_0, u_0]$:

$$\Psi_{f0} = \Psi_\infty(1-i)\left(C\left(\sqrt{3/2}\right) + iS\left(\sqrt{3/2}\right)\right). \tag{3.103}$$

- $[-u_2, -u_1] \cup [u_1, u_2]$:

$$\begin{aligned}\Psi_{f2} = \Psi_\infty(1-i)\Bigg(&C\left(\sqrt{11/2}\right) - C\left(\sqrt{7/2}\right)\\ &+ iS\left(\sqrt{11/2}\right) - iS\left(\sqrt{7/2}\right)\Bigg).\end{aligned} \tag{3.104}$$

- $[-u_4, -u_3] \cup [u_3, u_4]$:

$$\begin{aligned}\Psi_{f4} = \Psi_\infty(1-i)\Bigg(&C\left(\sqrt{19/2}\right) - C\left(\sqrt{15/2}\right)\\ &+ iS\left(\sqrt{19/2}\right) - iS\left(\sqrt{15/2}\right)\Bigg).\end{aligned} \tag{3.105}$$

Numerically, we find the values

$$0.742832, 0.713766, 0.747819, \tag{3.106}$$

which. As expected, these $\arg(\Delta C + i\Delta S)$ values are not perfect matches, as we expected by looking at the graph of the Cornu spiral. They are however good approximations, and the destructive interference with summation should be minimal.

We want to compare the intensities of the sums of these (one, three, and five open zones respectively)

$$\begin{aligned}\Psi_0 = \Psi_{f0} &= \Psi_\infty 1.34171 e^{-0.0425663i} \\ \Psi_{f0} + \Psi_{f2} &= \Psi_\infty 1.76859 e^{-0.0495833i} \\ \Psi_{f0} + \Psi_{f2} + \Psi_{f4} &= \Psi_\infty 2.0779 e^{-0.0477963i}\end{aligned} \tag{3.107}$$

The phase change with each addition is because the $\Delta C + i\Delta S$ values evaluated over the $[u_{2k-1}, u_{2k}]$ intervals with $u_m = \sqrt{3/2 + 2m}$ were based on tangential angles, and picking those tangential angles with this π separation, will rotate slightly with each iteration into the spiral. A more exact numerical choice for the u_m end points is required to avoid this, but we see that this was good enough to increase the magnitude (and thus the intensity) with each additional pair of opened apertures.

Squaring the absolute values in the wave functions above we find with $1, 3, and 5$ openings, the following intensity ratios

$$\begin{aligned}\frac{I_1}{I_\infty} &= 1.80018 \\ \frac{I_3}{I_\infty} &= 3.12791 \\ \frac{I_5}{I_\infty} &= 4.31767\end{aligned} \tag{3.108}$$

So, the first pair of additional openings results in a 1.7 times increase in intensity compared to the single opening, and the second pair of additional openings results in a 2.4 times intensity increase compared to the single opening.

Part f. To relate this back to pathlength we note that we have for $r_s \to \infty$

$$u_m = x_m \sqrt{\frac{2}{\lambda}\left(\frac{1}{r_s} + \frac{1}{r_m}\right)} \sim x_m \sqrt{\frac{2}{\lambda r_m}} \sim x_m \sqrt{\frac{2}{\lambda f}}, \tag{3.109}$$

or

$$x_m \sim u_m \sqrt{\frac{\lambda f}{2}}. \tag{3.110}$$

Looking to fig. 3.26 we see that we have

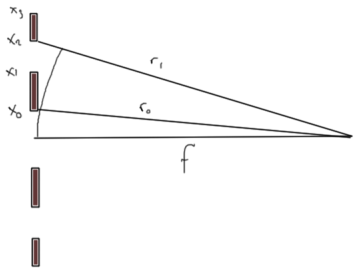

Figure 3.26: Geometry for path length interpretation.

$$
\begin{aligned}
r_m &= \sqrt{f^2 + x_m^2} \\
&= f\sqrt{1 + \frac{x_m^2}{f^2}} \\
&\sim f\left(1 + \frac{1}{2}\frac{x_m^2}{f^2}\right) \\
&= f + \frac{1}{2}\frac{x_m^2}{f}.
\end{aligned}
\tag{3.111}
$$

Illustrating first with the representative $[x_1, x_2]$ interval, observe that the ray distance from the midpoint is

$$
\begin{aligned}
\frac{1}{2}(r_2 + r_1) &\sim \frac{1}{2}\left(f + \frac{1}{2}\frac{x_2^2}{f} + f + \frac{1}{2}\frac{x_1^2}{f}\right) \\
&= f + \frac{1}{2f}\left(x_1^2 + x^2\right) \\
&= f + \frac{\lambda f}{8f}(u_1^2 + u_2^2) \\
&= f + \frac{\lambda}{8}(u_1^2 + u_2^2).
\end{aligned}
\tag{3.112}
$$

A bit more generally we see that the average additional pathlength at the midpoint of the aperture is

$$\begin{aligned}\frac{r_{2m}+r_{2m-1}}{2}-f &= \frac{\lambda}{8}\left(u_{2m}^2+u_{2m-1}^2\right)\\ &= \frac{\lambda}{8}\left(\frac{3}{2}+4m+\frac{3}{2}+2(2m-1)\right)\\ &= \frac{\lambda}{8}\left(3+8m-1\right)\\ &= \lambda\left(m+\frac{1}{8}\right).\end{aligned} \tag{3.113}$$

We are adding about an additional λ of pathlength from each aperture, resulting in constructive instead of destructive interference.

Part g. The central opening for these values is

$$\begin{aligned}2x_0 &\sim 2u_0\sqrt{\lambda f/2}\\ &= 2\sqrt{\frac{3}{2}\frac{\lambda f}{2}}\\ &= \sqrt{3\lambda f}\\ &= \sqrt{3(10\times 10^{-9})(10^{-1})}\mathrm{m}\\ &\sim 55\mu\mathrm{m}.\end{aligned} \tag{3.114}$$

To determine how many zones we can make, we note that the size of one of the openings in each pair of additional zones is

$$\begin{aligned}x_{2k}-x_{2k-1} &\sim (u_{2k}-u_{2k-1})\sqrt{\frac{\lambda f}{2}}\\ &= \sqrt{\frac{\lambda f}{2}}\left(\sqrt{\frac{3}{2}+4k}-\sqrt{\frac{3}{2}+4k-2}\right) > 10^{-6},\end{aligned} \tag{3.115}$$

which after substitution of our numbers is the numerical problem of finding the biggest integer k for which

$$\sqrt{8k+3}-\sqrt{8k-1} > \frac{1}{5\sqrt{10}}. \tag{3.116}$$

We find the largest value is $k = 124$ (can make 124 additional pairs of openings after the central opening), or 249 zones in total.

Exercise 3.3 Convolution theorem.

Find the form of the Fourier transform

$$\tilde{f}(\omega) = \int f(y)e^{-i\omega y}dy. \tag{3.117}$$

of a convolution integral

$$f(y) = \int dy' g(y')h(y-y') = g(y)*h(y). \tag{3.118}$$

Answer for Exercise 3.3

This is basically just an application of change of variables

$$\begin{aligned}
\mathcal{F}f(y) &= \int dy e^{-i\omega y} dy' g(y')h(y-y') \\
&\qquad\qquad\qquad\qquad\qquad\qquad y-y'=u \\
&= \int dy' g(y') \int dy e^{-i\omega y} \boxed{h(y-y')} \\
&= \int dy' g(y') \int du e^{-i\omega(u+y')} h(u) \\
&= \tilde{h}(\omega) \int dy' g(y') e^{-i\omega y'} \\
&= \tilde{h}(\omega)\tilde{g}(\omega).
\end{aligned} \tag{3.119}$$

Exercise 3.4 Fraunhofer diff., 4 circles. (*2010 final, q3*)

Calculate the diffraction pattern for the geometry of fig. 3.27.

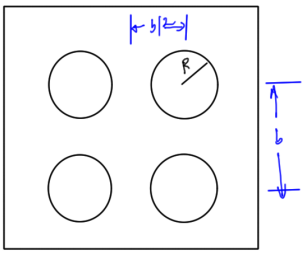

Figure 3.27: Four circular apertures.

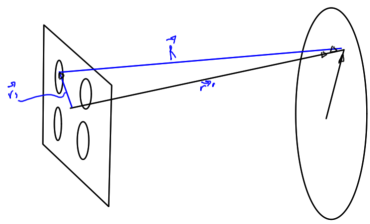

Figure 3.28: Four apertures with observation point and distances.

Answer for Exercise 3.4

We are working with distances illustrated in fig. 3.28. As usual we write

$$\mathbf{R} = \mathbf{r}' - \mathbf{r}_s. \tag{3.120a}$$

$$R = r'\left(1 + \frac{r_s^2}{r'^2} - 2\frac{\mathbf{r}_s \cdot \mathbf{r}'}{r'^2}\right)^{1/2} \approx r' + \frac{r_s^2}{2r'} - \mathbf{r}_s \cdot \hat{\mathbf{r}}'. \tag{3.120b}$$

so that

$$\Psi(\mathbf{r}') = \frac{\Psi_0}{e^{ikr'}} i\lambda r' \int_A e^{-ik\mathbf{r}_s \cdot \hat{\mathbf{r}}'}. \tag{3.121}$$

Let's write

$$\hat{\mathbf{r}} \cdot \hat{\mathbf{x}} = \alpha. \tag{3.122a}$$

$$\hat{\mathbf{r}} \cdot \hat{\mathbf{y}} = \beta, \tag{3.122b}$$

and introduce an aperture function

$$g(x, y) = \begin{cases} 1 & \text{if } x^2 + y^2 \leq R^2 \\ 0 & \text{otherwise} \end{cases} \tag{3.123}$$

This allows us to write our diffraction integral as

$$
\begin{aligned}
\Psi(\mathbf{r}') &= \frac{\Psi_0}{e^{ikr'}} i\lambda r' \int dudv \Bigg(e^{-ik((u+b/2)\alpha+(v+b/2)\beta)} + e^{-ik((u-b/2)\alpha+(v-b/2)\beta)} \\
&\qquad + e^{-ik((u+b/2)\alpha+(v-b/2)\beta)} + e^{-ik((u-b/2)\alpha+(v+b/2)\beta)} \Bigg) \\
&= \frac{\Psi_0}{e^{ikr'}} i\lambda r' \Bigg(e^{-ik(\alpha b/2+\beta b/2)} + e^{-ik(-\alpha b/2-\beta b/2)} \\
&\qquad + e^{-ik(\alpha b/2-\beta b/2)} + e^{-ik(-\alpha b/2+\beta b/2)} \Bigg) \times \\
&\qquad \int dudv e^{-ik(u\alpha+v\beta)} \\
&= 2\frac{\Psi_0}{e^{ikr'}} i\lambda r' \left(e^{-ik\alpha b/2} \cos(k\beta b/2) + e^{ik\alpha b/2} \cos(k\beta b/2) \right) \times \\
&\qquad \int dudv e^{-ik(u\alpha+v\beta)} \\
&= 4\frac{\Psi_0}{e^{ikr'}} i\lambda r' \cos(k\alpha b/2) \cos(k\beta b/2) \times \\
&\qquad \int_{\rho=0}^{R} \int_{\theta=0}^{2\pi} \rho d\rho d\theta e^{-ik\rho(\cos\theta\alpha+\sin\theta\beta)}.
\end{aligned}
\tag{3.124}
$$

This last integral isn't something that we can evaluate in just Bessel functions unless one of α or β is zero. For example, if $\beta = 0$, so that the observation axis lies in the one of the perpendicular planes, then we have

$$
\begin{aligned}
\Psi &\sim \frac{e^{ikr'}}{r'} \cos(k\alpha b/2) \int_{\rho=0}^{R} \int_{\theta=0}^{2\pi} \rho d\rho d\theta e^{-ik\rho\cos\theta\alpha} \\
&= \frac{e^{ikr'}}{r'} \cos(k\alpha b/2) 2\pi R \frac{J_1(-k\alpha R)}{-k\alpha} \\
&= \frac{e^{ikr'}}{r'} \cos(k\alpha b/2) 2\pi R \frac{J_1(k\alpha R)}{k\alpha}.
\end{aligned}
\tag{3.125}
$$

This looks fairly sinc like fig. 3.29. We can also solve for the case when $\alpha = \beta$, because we can write

$$
\cos\theta \pm \sin\theta = \sqrt{2}\cos(\theta \mp \pi/4). \tag{3.126}
$$

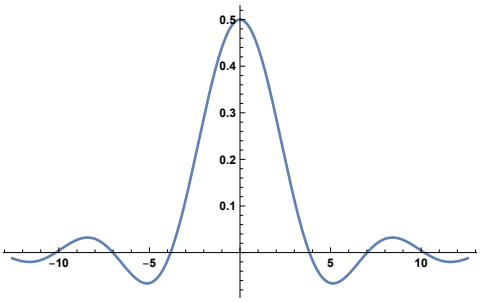

Figure 3.29: Plot of $J_1(x)/x$.

The phase shift doesn't make a difference when we are integrating over $[0, 2\pi]$, so we are left with

$$\begin{aligned}\Psi &\sim \frac{e^{ikr'}}{r'}\cos(k\alpha b/2)\int_{\rho=0}^{R}\int_{\theta=0}^{2\pi} \\ &= \frac{e^{ikr'}}{r'}\cos(k\alpha b/2)2\pi R\frac{J_1(\sqrt{2}k\alpha R)}{\sqrt{2}k\alpha}.\end{aligned} \tag{3.127}$$

For arbitrary α and β there's no such obvious change of variables. Mathematica calls the result a regularized hypergeometric function

$$\Psi \sim \frac{e^{ikr'}}{r'}\cos(k\alpha b/2)\pi R^2\,{}_0\tilde{F}_1\left(;2;-\frac{1}{4}k^2R^2\left(\alpha^2+\beta^2\right)\right), \tag{3.128}$$

The hypergeometric function itself looks fairly sinc like, but not with the R^2 multiplicative factor (plotted as a function of R). This is plotted in fig. 3.30, but curiously, this appears to be a divergent function? On the exam, I expect that the expectation was just to look on axis, but it would probably also be useful to plug in some actually representative numbers.

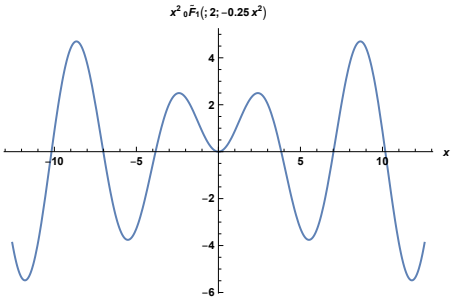

Figure 3.30: Plot of hypergeometric function.

4 COHERENCE.

4.1 INTERFERENCE.

Given a number of sources as illustrated in fig. 4.1, we can have interference (fringes, ... due to motion or polarization. In general we can consider these multiple sources as a sum over all the electric fields

Figure 4.1: Multiple sources potentially interfering.

$$\mathbf{E} = \mathbf{E}_1 + \mathbf{E}_2 + \mathbf{E}_3 + \cdots . \tag{4.1}$$

While there are some experiments that are sensitive to the actual fields, we will typically not care about those specifically, but instead will care about Intensity which has the form

$$I = (\text{const}) \left\langle |\mathbf{E}|^2 \right\rangle . \tag{4.2}$$

here the constant is unit dependent, such as $c\epsilon_0$, but since we will typically be looking at ratios, we can ignore those. We will move to an intensity based description, but first start with a field description.

Consider two sources

$$\mathbf{E}_1 = \boldsymbol{\epsilon}_1 E_1(\mathbf{r}) e^{i\phi_1(\mathbf{r},t)} \tag{4.3}$$

$$\mathbf{E}_2 = \boldsymbol{\epsilon}_2 E_2(\mathbf{r}) e^{i\phi_2(\mathbf{r},t)} \tag{4.4}$$

Here ϵ_i is a polarization constant, which we allow to be complex. The values E_i are the amplitudes which we constrain to have real values here, and ϕ_i is the (real) phase angle.

We observe

$$I = \left\langle |\mathbf{E}|^2 \right\rangle . \tag{4.5}$$

where

$$\langle f(t) \rangle = \lim_{T\to\infty} \frac{1}{T} \int_0^T dt' f(t'). \tag{4.6}$$

This time averaging method makes sense for optics where we may have response times $T \gtrsim 10^{-9}$s, so that $wT \gg 1$ (slow!)

Forming the magnitude of the field square we have

$$\begin{aligned} \left\langle |\mathbf{E}|^2 \right\rangle &= \langle (\mathbf{E}_1 + \mathbf{E}_2) \cdot (\mathbf{E}_1 + \mathbf{E}_2)^* \rangle \\ &= \left\langle |\mathbf{E}_1|^2 \right\rangle + \left\langle |\mathbf{E}_2|^2 \right\rangle + \langle \mathbf{E}_1 \cdot \mathbf{E}_2^* + \mathbf{E}_2 \cdot \mathbf{E}_1^* \rangle \\ &= I_1 + I_2 + 2\,\mathrm{Re}\left(\langle \mathbf{E}_1 \cdot \mathbf{E}_2^* \rangle \right). \end{aligned} \tag{4.7}$$

Consider the cross term. We have

$$\mathbf{E}_1 \cdot \mathbf{E}_2^* = \epsilon_1 \cdot \epsilon_2^* E_1 E_2 e^{i\phi_1 - i\phi_2}. \tag{4.8}$$

We see immediately that if the polarization vectors ϵ_1 and ϵ_2 are orthogonal, we have no interference. Let's consider some examples of some polarization vectors

- Linear polarization

$$\epsilon_1 = \hat{\mathbf{x}} \tag{4.9}$$

$$\epsilon_2 = \hat{\mathbf{y}} \tag{4.10}$$

or

$$\epsilon_1 = \frac{1}{\sqrt{2}}(\hat{\mathbf{x}} + \hat{\mathbf{y}}) \tag{4.11}$$

$$\epsilon_2 = \frac{1}{\sqrt{2}}(\hat{\mathbf{x}} - \hat{\mathbf{y}}) \tag{4.12}$$

- circular polarization

$$\epsilon_1 = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\ i\end{bmatrix} = \sigma^+ \tag{4.13}$$

$$\epsilon_2 = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\ -i\end{bmatrix} = \sigma^- \tag{4.14}$$

(here we require the conjugation to make $\epsilon_1 \cdot \epsilon_2^* = 0$)

READING: Polarization: §5 [8] and §2 [5]. This will be considered background material and not covered here.

Two sources in a scalar theory are

$$\Psi_1 = \sqrt{I_1(\mathbf{r})}e^{i\phi_1(\mathbf{r},t)} \tag{4.15}$$

$$\Psi_2 = \sqrt{I_2(\mathbf{r})}e^{i\phi_2(\mathbf{r},t)}. \tag{4.16}$$

Here $\sqrt{I_2(\mathbf{r})}$ are the amplitudes (real).

Interference (scalar or identical polarizations)

$$\begin{aligned} I &= |\Psi_1 + \Psi_2|^2 \\ &= |\Psi_1|^2 + |\Psi_2|^2 + 2\,\mathrm{Re}\,\langle \Psi_1 \Psi_2^* \rangle \\ &= I_1 + I_2 + 2\sqrt{I_1 I_2}\,\mathrm{Re}\left\langle e^{i\phi_1 - i\phi_2}\right\rangle \\ &= I_1 + I_2 + 2\sqrt{I_1 I_2}\,\langle \cos(\phi_1 - \phi_2)\rangle\,. \end{aligned} \tag{4.17}$$

Here we've made use of the fact that Re(.) and $\langle . \rangle$ are both linear operators so we can reverse their order of operation.

Question: Isn't the average of cosine just zero? Answer: we are considering phases that can vary with time. We don't necessarily have a constant phase difference here that would be wiped out in an average over one period.

Example: Diffraction

$$\Psi = \sum_{\text{paths}} \Psi_i \to \int \sqrt{I_i}e^{i\phi_i} = (\text{prefactor}) \iint_{\text{aperture}} e^{ikf}\,da'. \tag{4.18}$$

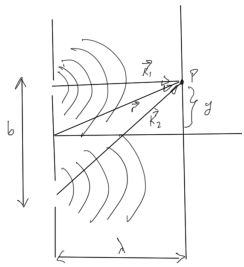

Figure 4.2: A diffraction geometry to consider.

Did we have interference in the diffraction example as illustrated in fig. 4.2. We had

$$\phi_1 = \mathbf{k}_1 \cdot \mathbf{r} - \omega t + \theta_1 \tag{4.19}$$

$$\phi_2 = \mathbf{k}_2 \cdot \mathbf{r} - \omega t + \theta_2 \tag{4.20}$$

This was monochromatic light (ω was the same). In this diffraction case we had

$$\phi_1 - \phi_2 = (\mathbf{k}_1 - \mathbf{k}_2) \cdot \mathbf{r} + \Delta\theta. \tag{4.21}$$

We can figure out from the geometry, and using the far-field limit ($x \gg b$) that we have a time independent phase difference

$$(\mathbf{k}_1 - \mathbf{k}_2) \cdot \mathbf{r} = -2\pi \frac{by}{r\lambda}. \tag{4.22}$$

What is our intensity?

$$I = I_1 + I_2 + 2\sqrt{I_1 I_2} \cos\left(-2\pi \frac{b}{r\lambda} y + \Delta\theta\right). \tag{4.23}$$

Here y is the position of observation, as illustrated in fig. 4.3.

Definition 4.1: Visibility

$$\boxed{\mathcal{V} \equiv \frac{I_{\max} - I_{\min}}{I_{\max} + I_{\min}}.} \tag{4.24}$$

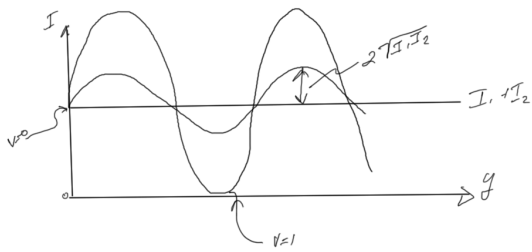

Figure 4.3: Some intensity variation with visibility.

This is a quantity that is easy to measure in the lab.
In this (diffraction) case we have

$$\mathcal{V} = 2\frac{\sqrt{I_1 I_2}}{I_1 + I_2}. \tag{4.25}$$

We illustrate this in fig. 4.4, where after zoom we see the same image. This is called Heterodyne amplification.

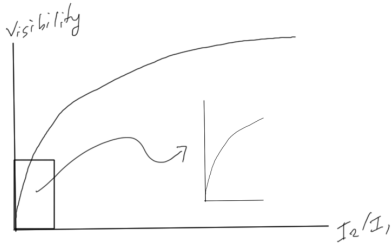

Figure 4.4: Heterodyne detection.

Definition 4.2: Heterodyne Detection

Measure a phase of a weak beam: Interfere with a strong beam "local oscillator" (I_{L0})! Interference : $2\sqrt{I_1 I_2}$. Even if I_1 is small this interference term can be big.

$$2\sqrt{I_p I_{LO}} \gg I_p. \tag{4.26}$$

$$4 I_p I_{LO} \gg I_p^2. \tag{4.27}$$

$$I_{\mathrm{LO}} \gg I_p/4. \tag{4.28}$$

4.2 ZOOLOGY OF INTERFEROMETERS.

Definition 4.3: Coherence

(Operational definition) Something measured by an interferometer

Types of dual path interferometers Some types of single path interferometers

- Young's fig. 4.5.
- Michaelson's fig. 4.6.
- Fresnel Biprism fig. 4.7.
- Lloyd's mirror fig. 4.8.
- Mach-Zender fig. 4.9.

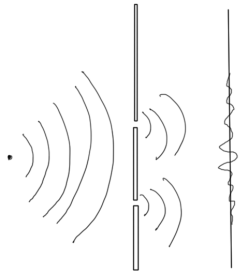

Figure 4.5: Wavefront splitting. Young's interferometer.

Types of multi-path interferometers Some types of multiple path interferometers

- Wavefront splitting fig. 4.10.

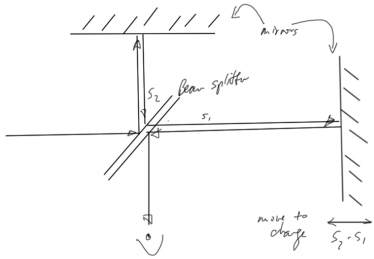

Figure 4.6: Amplitude splitting. Michaelson's interferometer.

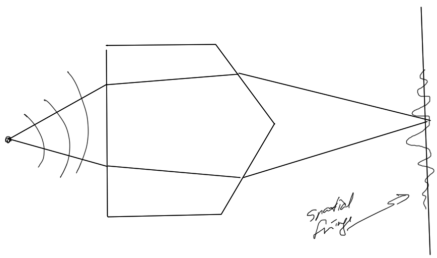

Figure 4.7: Fresnel Biprism (wavefront splitter).

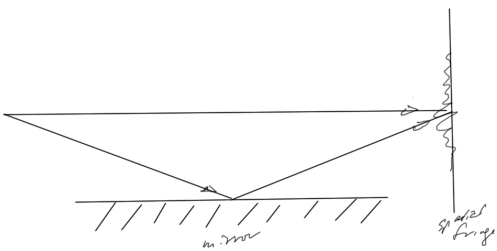

Figure 4.8: Lloyd's mirror. Interference from different path lengths.

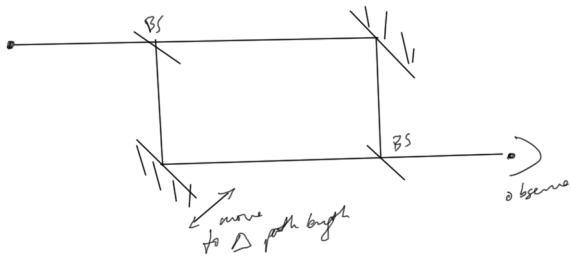

Figure 4.9: Mach-Zender interferometer. Temporal fringe if moving mirror.

- Infinite reflection in multiple mirrors (accidental) fig. 4.11.
- Infinite reflection in multiple mirrors (Fabry-Perot Cavity) fig. 4.12.

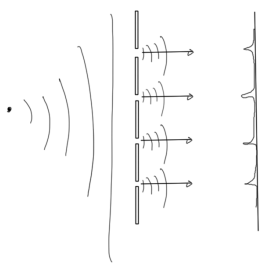

Figure 4.10: Wavefront splitting.

4.3 LLOYD'S INTERFEROMETER.

Using a virtual ray we can think of the Lloyd's interferometer setup as equivalent to a Young's double slit setup as illustrated in fig. 4.13 and fig. 4.14. Consider two sources as in fig. 4.15. Looking at this mathematically we have

$$\begin{aligned} I &= \left\langle |\Psi|^2 \right\rangle \\ &= \left\langle |\Psi(\mathbf{r}_1, t) + \Psi(\mathbf{r}_2, t)|^2 \right\rangle \\ &= I(\mathbf{r}_1) + I(\mathbf{r}_2) + 2\,\mathrm{Re}\,\langle \Psi(\mathbf{r}_1, t)\Psi^*(\mathbf{r}_2, t)\rangle\,. \end{aligned} \tag{4.29}$$

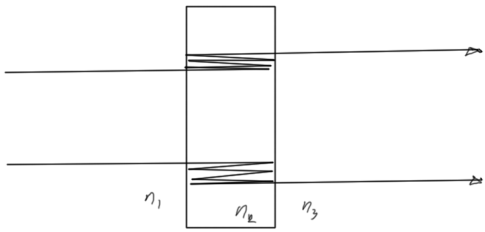

Figure 4.11: Bathroom cabinet setup, with reflection within reflection within ....

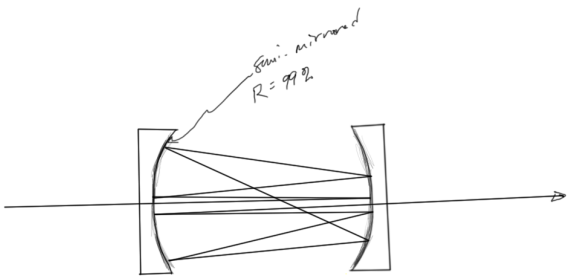

Figure 4.12: Fabry-Perot Cavity (repeated reflection on purpose).

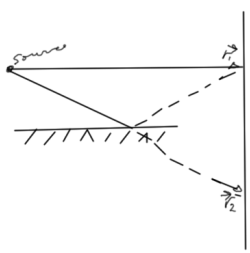

Figure 4.13: Virtual beam with mirror.

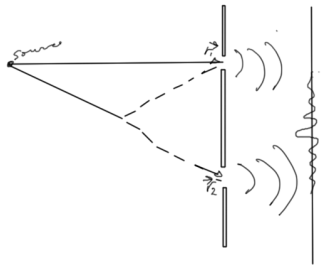

Figure 4.14: Virtual beam as diffraction source.

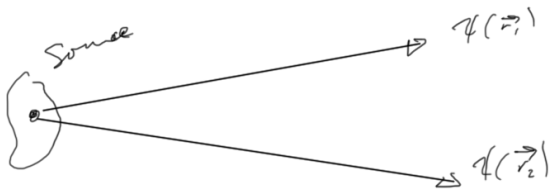

Figure 4.15: Extended source.

All the action is in the cross term. The portion of this that is hard to calculate, we call the Mutual coherence

$$\Gamma_{12} \equiv \langle \Psi(\mathbf{r}_1, t)\Psi^*(\mathbf{r}_2, t)\rangle \,. \tag{4.30}$$

4.4 TYPES OF COHERENCE.

4.4.1 *Longitudinal coherence.*

Consider the measurement of the relative interference at two points as in fig. 4.16. where we have a device that measures the relative interference at these points as in fig. 4.17 and fig. 4.18. where we suppose that there's something that has introduced a small amount of delay or path length. The extra pathlength like a time delay

$$\tau = \frac{s_2 - s_1}{c}. \tag{4.31}$$

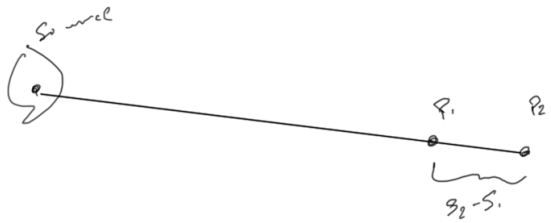

Figure 4.16: Multiple paths along one ray direction.

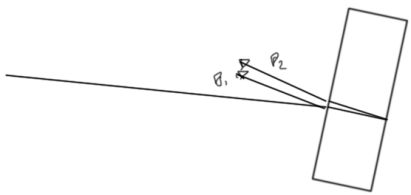

Figure 4.17: Imagine exaggerated refraction and reflection from cavity at end of ray.

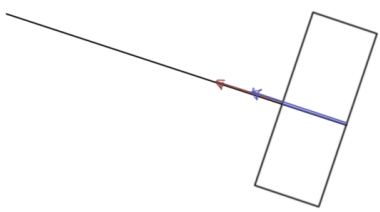

Figure 4.18: But with cavity aligned.

With a coherence time defined as

$$\tau_{\text{coh}} = \frac{1}{\Delta w}. \tag{4.32}$$

where Δw is the spectral width of the source.

We will show that if

$$s_2 - s_1 \ll c\tau_{\text{coh}}. \tag{4.33}$$

we have good visibility.

We want to think about what happens when the source gets broad as in fig. 4.19.

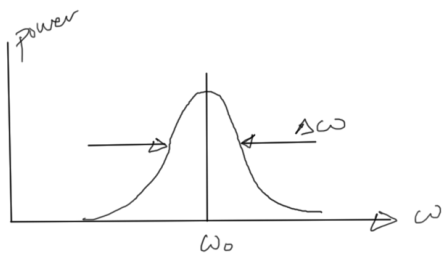

Figure 4.19: Power distribution with interference due to extended source.

4.4.2 *Transverse coherence.*

As illustrated in fig. 4.20. with

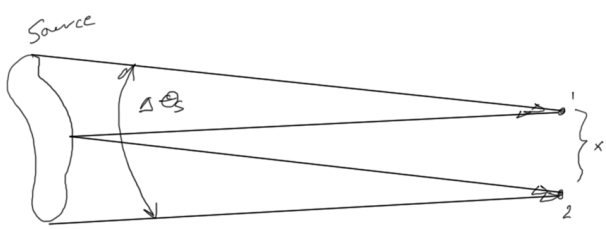

Figure 4.20: Interference from extended source.

$$\text{length} = \frac{\lambda}{\Delta\theta_s}. \tag{4.34}$$

we will show that we get a good fringe if

$$x \ll \frac{\lambda}{\Delta\theta_s}. \tag{4.35}$$

A point source is one for which $\Delta\theta_s \to 0$, so that $\lambda/\Delta\theta_s \to \infty$.

We want to think about what happens when the source gets big.

Doesn't the intensity loss in the P_1, P_2 linear interference setup matter? Consider fig. 4.21. If R is small, then the resulting intensities are

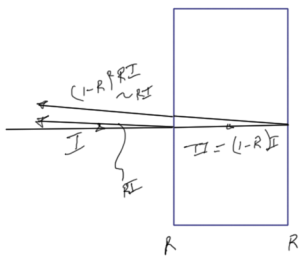

Figure 4.21: Intensity differences after cavity reflection.

similar.

Observing intensity in a two-path interferometer

$$I = I_1 + I_2 + 2\,\mathrm{Re}\,\Gamma_{12}. \tag{4.36}$$

where

$$I_1 = \langle \Psi_1 \Psi_1^* \rangle\,. \tag{4.37a}$$

$$I_2 = \langle \Psi_2 \Psi_2^* \rangle\,. \tag{4.37b}$$

$$\Gamma_{12} = \langle \Psi_1 \Psi_2^* \rangle\,. \tag{4.37c}$$

Here Γ_{12} is the mutual coherence and $\langle ... \rangle$ indicates the time average.

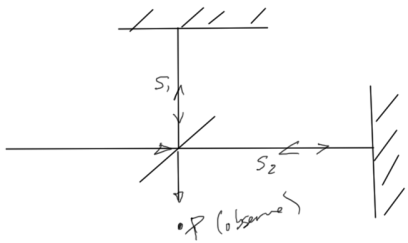

Figure 4.22: Michaelson.

Figure 4.23: Equivalent to Michaelson.

We'll consider a Michaelson interferometer setup as illustrated in fig. 4.22. The net effect of this is as if a phase delay in a linear system had been introduced, as illustrated in fig. 4.23. With

$$\Psi(\mathbf{r}_k, t) = \sqrt{I(\mathbf{r}_k)} \exp\left(i\phi(\mathbf{r}_k, t)\right). \tag{4.38a}$$

$$\Psi_{1,2} = \Psi(\mathbf{r}_{1,2}, t). \tag{4.38b}$$

We work with $|\mathbf{r}| \gg \lambda$, so that we neglect any small intensity change, and make the approximation

$$I(\mathbf{r}_1) \sim I(\mathbf{r}_2). \tag{4.39}$$

and define

$$\boxed{\tau = \frac{s_2 - s_1}{c}} \tag{4.40}$$

We write

$$\gamma_{12} = \frac{\Gamma_{12}}{\sqrt{I_1}\sqrt{I_2}}. \tag{4.41}$$

So that our total intensity is just

$$I = 2I_1(1 + \mathrm{Re}\,\gamma_{12}). \tag{4.42}$$

Our task is to calculate

$$\gamma_{12} = \left\langle e^{i\phi_1 - i\phi_2} \right\rangle. \tag{4.43}$$

With

$$\phi(t) = \omega t + \Delta(t), \tag{4.44}$$

we suppose that we have some sort of system, perhaps due to atomic interactions, we have random discrete phase jumps at regular intervals, as in fig. 4.24 and fig. 4.25. This isn't necessarily a

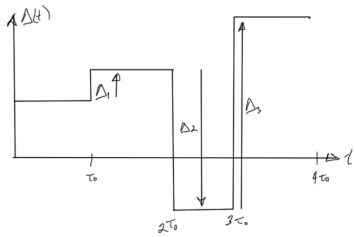

Figure 4.24: Random step phase changes.

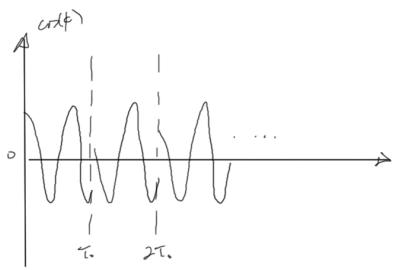

Figure 4.25: Effect of random phase changes.

realistic system, but it one that we can calculate.

$$\gamma_{12} = \left\langle e^{i\phi(t)} e^{-i\phi(t+\tau)} \right\rangle = e^{i\omega\tau} \lim_{T\to\infty} \int_0^T e^{i\Delta(t) - i\Delta(t+\tau)} dt. \tag{4.45}$$

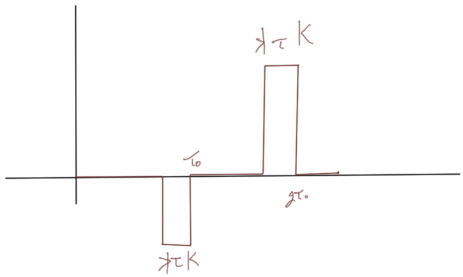

Figure 4.26: Differences of random phases after time delay.

The phase transitions above are illustrated in fig. 4.26. Integrating across one time interval and then summing we have

$$\frac{1}{N\tau_0}\sum_{n=1}^{N}\int_{n\tau_0}^{(n+1)\tau_0} e^{i(\Delta(t)-\Delta(t+\tau)}dt = \frac{\sum n = 1^N}{N\tau_0}(\tau_0-\tau) + \cancel{\frac{\sum_{n=1}^{N}}{N\tau_0}\tau e^{-i\Delta_i}}. \tag{4.46}$$

To account for the cancellation, note that we are summing over a number of complex numbers, like

$$e^{i\Delta_1} + e^{i\Delta_2} + e^{i\Delta_3}. \tag{4.47}$$

where the Δ's are random. This is illustrated in fig. 4.27. (we do

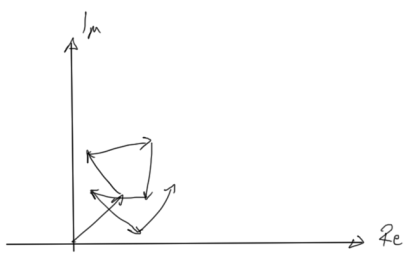

Figure 4.27: Random walk evolution.

get somewhere in a random walk, but it is approximately $\sqrt{N}$ on average, so we have $\sqrt{N}/N$ in the sum which goes to zero). Putting results together we have

$$\gamma_{12} = \begin{cases} \left(1 - \frac{|\tau|}{\tau_0}\right) e^{-i\omega\tau} & \text{if } \tau \leq \tau_0 \\ 0 & \text{if } \tau \geq \tau_0 \end{cases} \tag{4.48}$$

Also see §3.5 in [5] for this derivation.

The intensity output of the interferometer is

$$
\begin{aligned}
I &= 2I_0 + 2I_0 \operatorname{Re} \gamma_{12} \\
&= 2I_0 + 2I_0 |\gamma_{12}| \cos(\omega\tau) \\
&= 2I_0 + 2I_0 \cos(\omega\tau) \left(1 - \frac{\tau}{\tau_0}\right) e^{-i\omega\tau} \begin{cases} 1 - |\tau|\tau_0 & \tau \leq \tau_0 \\ 0 & \tau \geq \tau_0 \end{cases}
\end{aligned}
\tag{4.49}
$$

as in fig. 4.28. One fringe is at

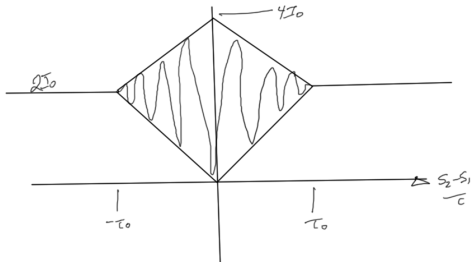

Figure 4.28: Resulting interference intensity.

$$\omega\tau = 2\pi, \tag{4.50}$$

or

$$\omega \frac{s_2 - s_1}{c} = 2\pi. \tag{4.51}$$

With

$$2\pi \frac{c}{\omega} = \frac{\omega\lambda}{\omega} = \lambda. \tag{4.52}$$

we have

$$s_2 - s_1 = \lambda. \tag{4.53}$$

typically require 500 nm for visible light. We need micron scale control.

4.5 MORE GENERAL MUTUAL COHERENCE.

What do we know about $\langle \Psi_1 \Psi_2^* \rangle$?

$$\langle \Psi_1(t)\Psi_2^*(t+\tau)\rangle = \lim_{T\to\infty} \int_0^T dt \Psi_1(t)\Psi_2(t+\tau). \tag{4.54}$$

This (integral) is just a convolution, so we can compute this by performing Fourier transforms and inverse Fourier transforms. If

$$f(x) = g * h = \int_{-\infty}^{\infty} dx' g(x')h(x-x'). \tag{4.55}$$

Then

$$F(k) = G(k)H(k), \tag{4.56}$$

so that

$$f(x) = \mathcal{F}^{-1} G(k)H(k). \tag{4.57}$$

We see that $\gamma_{12}(\tau)$ <u>is a Fourier transform of power spectrum of the source</u>. Explicitly, that is

$$|\Gamma_{12}| = e^{i\alpha_{12}(\tau)} = 4\int_0^{\infty} G_{12}(\omega) e^{-i(\omega-\overline{\omega})\tau} d\omega. \tag{4.58}$$

where

$$\Gamma_{12} = |\Gamma_{12}| e^{i\alpha_{12}(\tau) - i\omega\tau}. \tag{4.59}$$

and

$$G_{12}(\omega) = \lim_{T\to\infty} \frac{1}{2T} V_T(\mathbf{r}_1,\omega) V_T^*(\mathbf{r}_2,\omega) \sim V_1(\omega)V_2^*(\omega). \tag{4.60a}$$

$$V(\omega) = \mathcal{F}(\Psi). \tag{4.60b}$$

$$V_T = \int_{-\infty}^{\infty} \Psi_T^{(r)}(\mathbf{r},t) e^{i\omega t}. \tag{4.60c}$$

$$\Psi_T^{(r)} = \begin{cases} \operatorname{Re}\Psi & |t| \le T \\ 0 & |t| \ge T \end{cases} \tag{4.60d}$$

Reading: These results weren't derived here. For that see §10.3.2 [2].

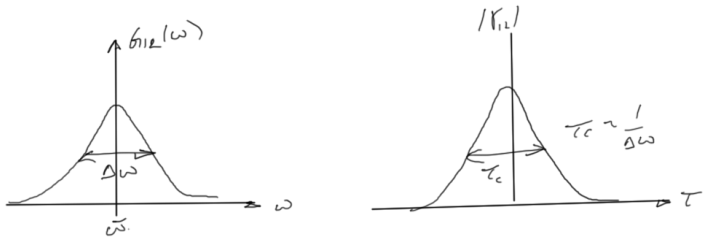

Figure 4.29: Gaussian power spectrum and correlation.

Example. Gaussian (Fourier transform of a Gaussian is a Gaussian)

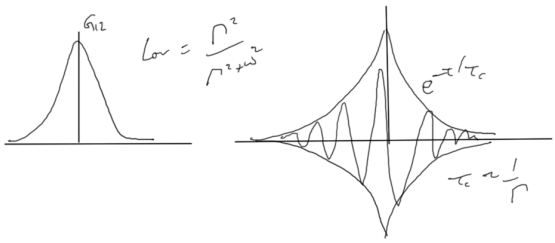

Figure 4.30: Lorentzian power spectrum and correlation.

Example. Lorenzian

4.6 TEMPORAL COHERENCE (CONT.)

For two source interference

$$I = I_1 + I_2 + 2\sqrt{I_1 I_2}\,\mathrm{Re}\,\gamma_{12}. \tag{4.61}$$

We call $I_1 + I_2$ the incoherent sum, and now know that γ_{12} is the Fourier transform of the spectral intensity

$$\gamma_{12} = \mathcal{F}\{I(\omega)\}. \tag{4.62}$$

when

$$\gamma_{12} = \gamma(\tau). \tag{4.63}$$

and

$$\tau = \frac{s_2 - s_1}{c}. \tag{4.64}$$

Example in fig. 4.31. Beyond the coherent time τ_{COH} we have only

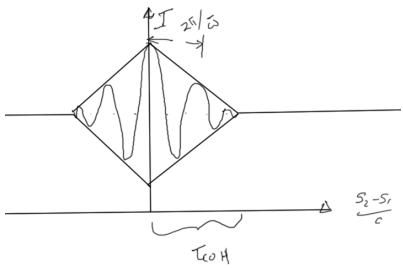

Figure 4.31: Intensity example.

the incoherent intensity, what we'd expect from two flashlights for example. We can write this down in a nice format

$$I = I_1 + I_2 + 2\sqrt{I_1 I_2}|\gamma_{12}| \cos\left(\alpha_{12}(\tau) - \delta\right), \tag{4.65}$$

where $\delta \equiv \overline{\omega}\tau$. Plotting the absolute value $|\gamma_{12}|$ we have something like fig. 4.32. Here α_{12} is the difference in the phase from

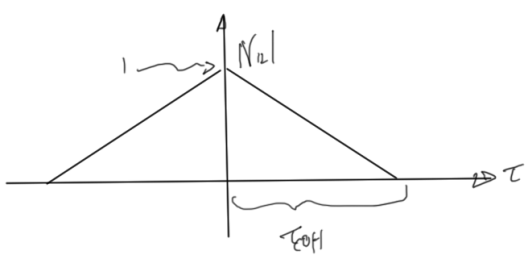

Figure 4.32: Absolute γ_{12}.

the average. As an illustration we may be considering a phase shift at one of the points fig. 4.33. Resulting in a non-zero α_{12} as in fig. 4.34. As opposed to fig. 4.35. where the figure had been drawn with $\alpha_{12} = 0$.

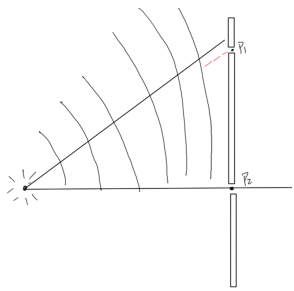

Figure 4.33: α_{12} illustrated.

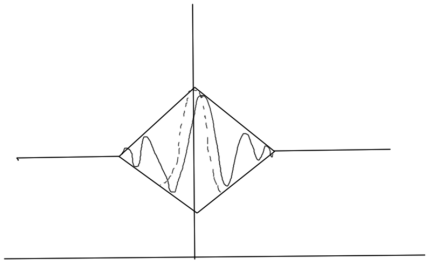

Figure 4.34: Non zero α_{12}.

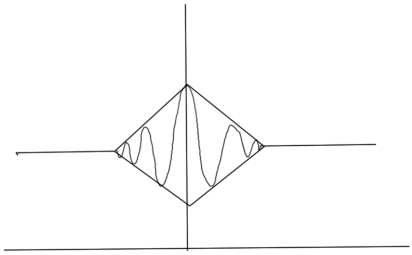

Figure 4.35: Zero α_{12}.

Now suppose we rewrite things as:

$$I = |\gamma_{12}| \overbrace{\boxed{\left(I_1 + I_2 + 2\sqrt{I_1 I_2}\cos\left(\alpha_{12}(\tau) - \delta\right)\right)}}^{I_{\text{coh}}} + (1 - |\gamma_{12}|) \overbrace{\boxed{(I_1 + I_2)}}^{I_{\text{incoh}}}. \tag{4.66}$$

$$\begin{cases} |\gamma_{12}| = 1 & \text{complete coherence} \\ \gamma = 0 & \text{incoherent} \\ 0 < |\gamma| < 1 & \text{partially coherent light} \end{cases} \tag{4.67}$$

The first case $|\gamma_{12}| = 1$ is the easiest case to deal with fig. 4.36. We

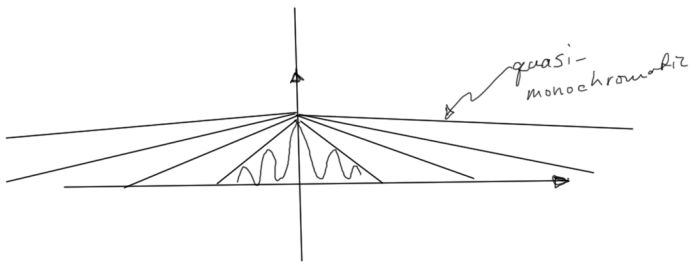

Figure 4.36: Quasi-monochromatic.

can speak of Quasi-monochromatic as the case when

$$\tau \ll \tau_{\text{coh}}, \tag{4.68}$$

so that we are ignoring finite coherence time, $\omega \to \overline{\omega}$

We can get this in the lab, by taking an spectrally distributed source like fig. 4.37 and filtering it as in fig. 4.38. If our original intensity looked like fig. 4.39, perhaps we now have fig. 4.40. We loose some of the maximum possible intensity, but can introduce a lot more fringes. I'm assuming here that the point here is to use one source to explicitly interfere with another for measurement, so that we want interference, and can make this more severe by reducing the spectral width of the source.

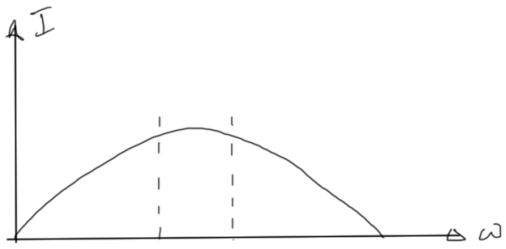

Figure 4.37: Spectrally distributed source.

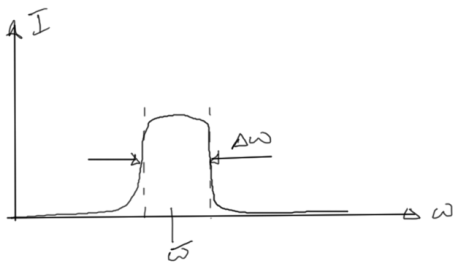

Figure 4.38: Filtered source without spectral distribution.

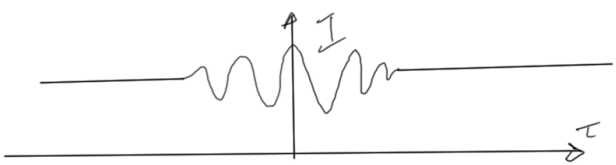

Figure 4.39: Intensity for distributed source.

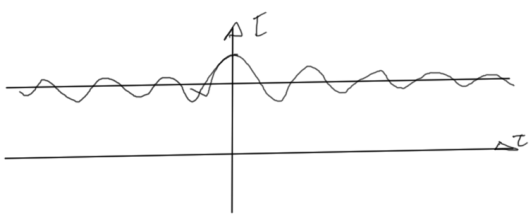

Figure 4.40: Intensity for filtered source.

4.7 SPATIAL COHERENCE.

We'll talk a bit by how a spatially broad source will mess up the fringes we could measure.

Consider two sources with no mutual coherence as in fig. 4.41. Here what is Γ_{12}? Recall that

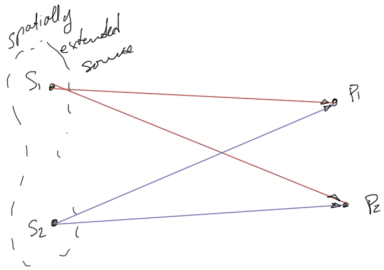

Figure 4.41: Spatially distributed source.

$$\Gamma_{12} = \langle \Psi^*(\mathbf{r}_1)\Psi(\mathbf{r}_2)\rangle \,. \tag{4.69}$$

NOTE: switch of convention here! In eq. (4.30) we used opposite conjugation.

It could be that we've scrambled up any possible fringes. We'll eventually be considering a spatially extended source (i.e. a filament in a light bulb fig. 4.42), and will deal with that by summing over a source distribution, and first need to know how to deal with a pair of sources. If are kilometres away from the light bulb,

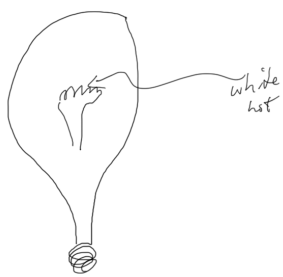

Figure 4.42: Spatially distributed source, only when close up.

the spatial distribution of this source will not matter.

We will find that

$$\Gamma_{12} \to 0, \tag{4.70}$$

over distance

$$\frac{1}{\Delta k} = l_{\text{tc}}, \tag{4.71}$$

or

$$\frac{\lambda}{\Delta\theta} = l_{\text{tc}}. \tag{4.72}$$

i.e. there's only spatial properties here being considered. Writing things out

$$\begin{aligned}
\Gamma_{12} &= \langle \Psi^*(\mathbf{r}_1)\Psi(\mathbf{r}_2)\rangle \\
&= \left\langle \left(\Psi_{1a} + \Psi_{1b}\right)^* \left(\Psi_{2a} + \Psi_{2b}\right)\right\rangle \\
&= \overbrace{\boxed{\langle \Psi_{1a}^* \Psi_{2a}\rangle}}^{\Gamma_{12}^a} + \overbrace{\boxed{\langle \Psi_{1a}^* \Psi_{2b}\rangle}}^{=0} + \overbrace{\boxed{\langle \Psi_{1b}^* \Psi_{2a}\rangle}}^{=0} + \overbrace{\boxed{\langle \Psi_{1b}^* \Psi_{2b}\rangle}}^{\Gamma_{12}^b}.
\end{aligned} \tag{4.73}$$

Here we kill the middle terms because a and b have no phase correlation. Now let's think about what these things look like. We had an example where we had

$$\Gamma_{12}^a = \sqrt{I_1 I_2}|\gamma_{12}^a|e^{-i\omega\tau_a}. \tag{4.74a}$$

$$\Gamma_{12}^b = \sqrt{I_1 I_2}\left|\gamma_{12}^b\right|e^{-i\omega\tau_b}. \tag{4.74b}$$

where

$$\tau_a \equiv \frac{r_{1a} - r_{2a}}{c}. \tag{4.75a}$$

$$\tau_b \equiv \frac{r_{1b} - r_{2b}}{c}. \tag{4.75b}$$

For Quasi-monochromatic sources we assume that we have approximately

$$|\gamma_{12}^a| = \left|\gamma_{12}^b\right| = 1, \tag{4.76}$$

so that

$$\Gamma_{12}^{a} = \sqrt{I_1 I_2} e^{-i\omega\tau_a}. \tag{4.77a}$$

$$\Gamma_{12}^{b} = \sqrt{I_1 I_2} e^{-i\omega\tau_b}. \tag{4.77b}$$

Suppose also that we have equal intensities (we are in the Far field)

$$I_1^a = I_2^a = I_1^b = I_2^b, \tag{4.78}$$

This will be valid when

$$|r| \gg |r_1^a - r_2^a| \quad \text{etc..} \tag{4.79}$$

We are left with

$$\gamma_{12} = \frac{1}{2}\gamma_{12}^a + \frac{1}{2}\gamma_{12}^b = \frac{1}{2}e^{-i\omega\tau_a} + \frac{1}{2}e^{-i\omega\tau_b}. \tag{4.80}$$

Recall that γ_{12} was defined in eq. (4.41). We have

$$\gamma_{12} = \cos\left(\frac{\omega(\tau_a - \tau_b)}{2}\right). \tag{4.81}$$

In absolute value, this is plotted in fig. 4.43.

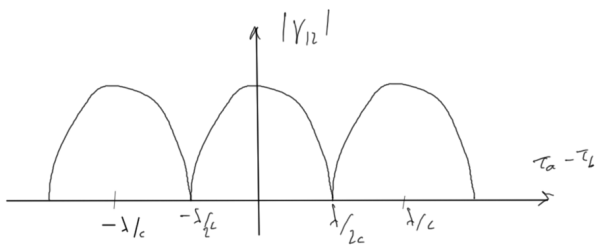

Figure 4.43: $|\gamma_{12}|$.

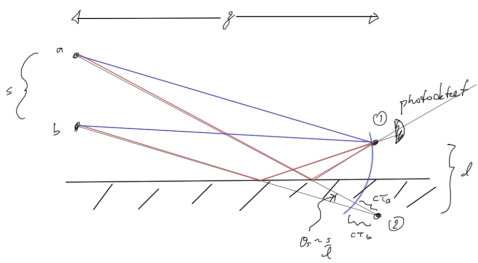

Figure 4.44: Spatial inferometry with Lloyd's mirror.

4.8 SPATIAL COHERENCE (CONT.)

We want to look more at spatial distribution of sources, and the effects of that spread on the coherence. In general we'd have to deal with both spectral width as well as spatial distribution, but here we choose to only dealing with the spatial distribution. Consider a pair of point sources as in fig. 4.44.

$$\Gamma_{12} = \langle \Psi^*(t, \mathbf{r}_1)\Psi(t, \mathbf{r}_2)\rangle \,. \tag{4.82}$$

At the photodetector, we have

$$I = I_1 + I_2 + 2\,\mathrm{Re}\,\Gamma_{12}. \tag{4.83}$$

This last bit $2\,\mathrm{Re}\,\Gamma_{12}$ predicts the fringe.

Last time, as plotted in fig. 4.43, we found

$$|\gamma_{12}| = \left|\cos\left(\frac{\omega(\tau_a - \tau_b)}{2}\right)\right|. \tag{4.84}$$

This absolute value of γ_{12} is telling us what the visibility of the fringes is. Suppose we move around source a so that we are changing $c\tau_a$. This will give us something like fig. 4.45 where the envelope is

$$\cos\left(\frac{\omega(\tau_a - \tau_b)}{2}\right). \tag{4.85}$$

and the fast phase frequencies oscillate with the higher frequency

$$\cos\left(\frac{\omega(\tau_a + \tau_b)}{2}\right). \tag{4.86}$$

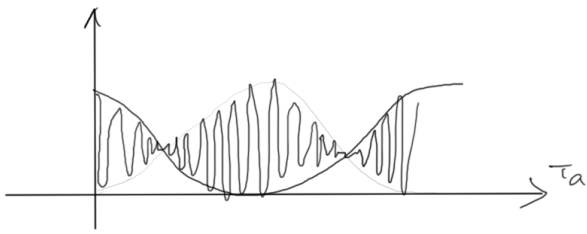

Figure 4.45: Intensity.

To see this we need to calculate this difference in τ's.

$$\tau_a - \tau_b = \frac{r_{1a} - r_{2a} - r_{1b} + r_{2b}}{c} = \text{see Prof's notes} = \frac{ls}{cz} = \frac{l\theta_s}{c}. \tag{4.87}$$

In the last step we've used a small angle approximation

$$\theta_s \approx \frac{s}{z}. \tag{4.88}$$

This is one way to calculate this difference, and we see that in the limit of a single point source, this difference

$$\tau_a - \tau_b \to 0, \tag{4.89}$$

and

$$|\gamma_{12}| \to 1, \tag{4.90}$$

as we expect, so the visibility of the fringe disappears (illustrated with figure?) When

$$\Delta\tau = \frac{\lambda}{2c}. \tag{4.91}$$

when

$$l\theta_s = \frac{\lambda}{2}. \tag{4.92}$$

We can do this in a more general way, using some math we already know if we think about these two sources separated by some vector $\mathbf{r}_s$, going to two points, again with vector separation $\Delta\mathbf{r}$, as in fig. 4.46. We also introduce a vector average $\mathbf{r}_{\text{av}}$ from point a to the

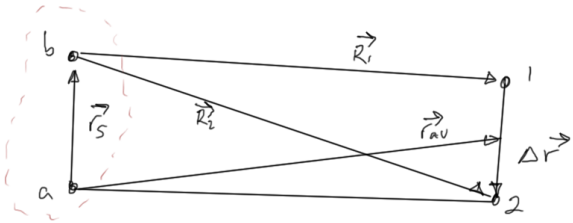

Figure 4.46: Vector spatial coherence diagram.

midpoint of 1 and 2. Note that we are preparing for a setup with an extended source where we'll be integrating over points a so we don't want this midpoint to start from the average of a and b as may be expected.

We find in exercise 4.1

$$k\left(R_1 - R_2\right) \approx -k\left(\mathbf{r}_{\text{av}} - \mathbf{r}_s\right) \cdot \frac{\Delta\mathbf{r}}{r_{\text{av}}}, \tag{4.93}$$

where $\mathbf{R}_1$, and $\mathbf{R}_2$ are the vectors from b to 1 and 2 respectively. This includes an overall phase shift

$$\mathbf{k}_{\text{av}} \cdot \Delta\mathbf{r}, \tag{4.94}$$

and a shift

$$k\frac{\mathbf{r}_s \cdot \Delta\mathbf{r}}{r_{\text{av}}}. \tag{4.95}$$

4.9 WHAT'S SPECIAL ABOUT THE PATHLENGTH DIFFERENCE?

$$\Delta\tau = \tau_a - \tau_b = \frac{\lambda}{2c}. \tag{4.96}$$

Let's consider the fringes that we make, using an interferometer such as Lloyd's mirror, from source a made from these two points fig. 4.47. Could set things up so that the phase of the pairs of contributions are exactly opposite in phase as in fig. 4.48. Our intensities could then add to produce no fringes as in fig. 4.49.

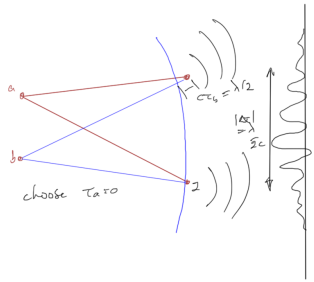

Figure 4.47: Path length differences.

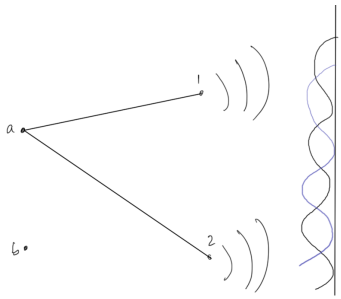

Figure 4.48: Opposing phase contributions eliminating fringes.

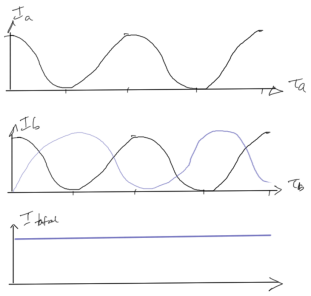

Figure 4.49: Fringe elimination.

4.10 CONTINUUM SPATIAL DISTRIBUTION.

Our small angle source contributions fig. 4.50 fig. 4.51 provide us with one sinusoidal term whereas our larger angular spreads

Figure 4.50: small θ_s.

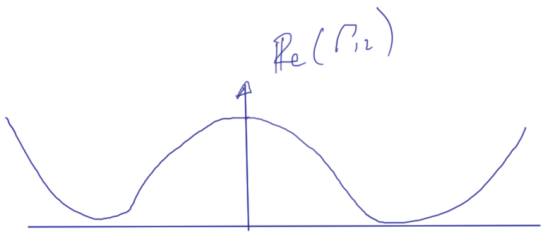

Figure 4.51: One frequency contribution.

fig. 4.52 will give us more terms, none that will reduce the main peak. In the sum fig. 4.53 we may end up with something like fig. 4.54. Our total intensity is

$$\begin{aligned} I_{\text{total}} &= \sum_{\mathbf{k}} \left| \Psi_k(\mathbf{r}_1, t) + e^{i\mathbf{k}\cdot\mathbf{l}} \Psi_k(\mathbf{r}_1, t) \right|^2 \\ &= \sum \left(2 I_{\mathbf{k}} + 2 \operatorname{Re} \left(\Psi_k^* e^{i\mathbf{k}\cdot\mathbf{l}} \Psi_k \right) \right) \\ &= \text{incoherent sum} + 2 \operatorname{Re} \left(\sum_{\mathbf{k}} I_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{l}} \right). \end{aligned} \tag{4.97}$$

Here

$$\sum_{\mathbf{k}} I_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{l}} = \Gamma_{12} = \mathcal{F}(I_{\mathbf{k}}). \tag{4.98}$$

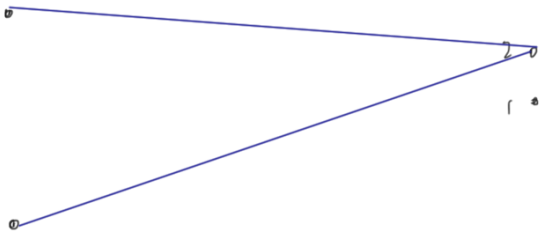

Figure 4.52: larger θ_s.

Figure 4.53: Many contributions.

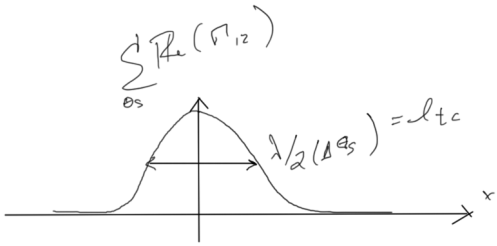

Figure 4.54: Resulting superposition.

This is called the Van Cittert-Zernike Theorem. Like the time domain result, we have something that essentially says that the coherence is a Fourier transform of the distribution.

Our next task will be to extend this result to continuous spatial distributions as in fig. 4.55.

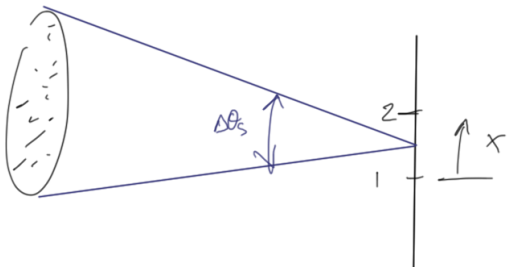

Figure 4.55: Spatial distribution.

4.11 FULL DERIVATION OF THE VAN CITTERT-ZERNIKE THEOREM.

We never did the complete derivation of the Van Cittert-Zernike theorem for spatial mutual coherence in class (or if we did I didn't understand it). There were also aspects of the class notes derivation that I had trouble with. Lets try this from scratch, going through all the details in sequence.

The geometry that we wish to consider is illustrated in fig. 4.56. Our diffraction integral is

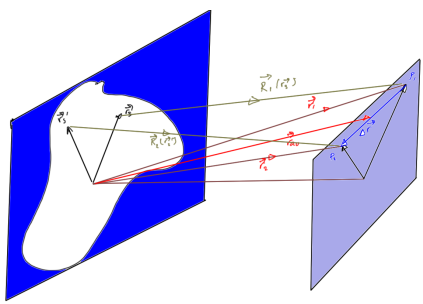

Figure 4.56: Geometry for spatial coherence.

$$\Psi(\mathbf{r}) = \int \Psi_s(\mathbf{r}_s)\frac{e^{ikR}}{i\lambda R}d^2r_s, \tag{4.99}$$

with a correlation function of

$$\begin{aligned}\Gamma_{12} &= \langle \Psi(\mathbf{r}_2, t)\Psi^*(\mathbf{r}_1, t)\rangle \\ &= \left\langle \int d^2r_s d^2r'_s \frac{e^{ikR_2(\mathbf{r}'_s)}}{i\lambda R_2(\mathbf{r}'_s)}\frac{e^{-ikR_1(\mathbf{r}_s)}}{-i\lambda R_1(\mathbf{r}_s)}\Psi_s(\mathbf{r}'_s, t)\Psi^*_s(\mathbf{r}_s, t)\right\rangle.\end{aligned} \tag{4.100}$$

where as in the figure we have

$$\mathbf{R}_2(\mathbf{r}'_s) + \mathbf{r}'_s = \mathbf{r}_2. \tag{4.101a}$$

$$\mathbf{R}_1(\mathbf{r}'_s) + \mathbf{r}_s = \mathbf{r}_1. \tag{4.101b}$$

We can express an assumption that the wave sources at two different points are uncorrelated by writing

$$\left\langle \Psi_s(\mathbf{r}'_s, t)\Psi^*_s(\mathbf{r}_s, t)\right\rangle = I(\mathbf{r}_s)\delta(\mathbf{r}_s - \mathbf{r}'_s). \tag{4.102}$$

This makes some intuitive sense, but still seems sort of like it's been pulled from a magic hat. I guess the idea is that if the source points are the same then the average of the autocorrelation is an integral of the intensity over all time (thus diverging), while for different points expressing no correlation over time.

Substitution of this delta function and integration over the $d^2r'_s$ source coordinates, leaves us

$$\Gamma_{12} = \frac{1}{\lambda^2}\int d^2r_s \frac{e^{ikR_2(\mathbf{r}_s)}}{R_2(\mathbf{r}_s)}\frac{e^{-ikR_1(\mathbf{r}_s)}}{R_1(\mathbf{r}_s)}I(\mathbf{r}_s). \tag{4.103}$$

We loose the dependence of R_1 and R_2 on the pair of source points and reformulate the $R_2 - R_1$ difference in terms of the average distance between $\mathbf{r}_2$ and $\mathbf{r}_1$, plus the incremental distance between these. That is

$$\mathbf{r}_2 = \mathbf{r}_{\text{av}} + \frac{1}{2}\Delta\mathbf{r}. \tag{4.104a}$$

$$\mathbf{r}_1 = \mathbf{r}_{\text{av}} - \frac{1}{2}\Delta\mathbf{r}. \tag{4.104b}$$

so that

$$\mathbf{R}_2 = \mathbf{r}_{\text{av}} + \frac{1}{2}\Delta\mathbf{r} - \mathbf{r}_s. \tag{4.105a}$$

$$\mathbf{R}_1 = \mathbf{r}_{\text{av}} - \frac{1}{2}\Delta\mathbf{r} - \mathbf{r}_s. \tag{4.105b}$$

These have magnitudes

$$\begin{aligned} R_{2,1} &= r_{\text{av}}\left|\hat{\mathbf{r}}_{\text{av}} \pm \frac{1}{2r_{\text{av}}}\Delta\mathbf{r} - \frac{1}{r_{\text{av}}}\mathbf{r}_s\right|^{1/2} \\ &= r_{\text{av}}\sqrt{1 + \left(\pm\frac{1}{2r_{\text{av}}}\Delta\mathbf{r} - \frac{1}{r_{\text{av}}}\mathbf{r}_s\right)^2 + 2\hat{\mathbf{r}}_{\text{av}}\cdot\left(\pm\frac{1}{2r_{\text{av}}}\Delta\mathbf{r} - \frac{1}{r_{\text{av}}}\mathbf{r}_s\right)} \\ &= r_{\text{av}}\sqrt{1 + \frac{(\Delta\mathbf{r})^2}{4r_{\text{av}}^2} + \frac{r_s^2}{r_{\text{av}}^2} \mp \frac{1}{r_{\text{av}}}\Delta\mathbf{r}\cdot\frac{1}{r_{\text{av}}}\mathbf{r}_s \pm \hat{\mathbf{r}}_{\text{av}}\cdot\frac{1}{r_{\text{av}}}\Delta\mathbf{r} - \frac{2}{r_{\text{av}}}\hat{\mathbf{r}}_{\text{av}}\cdot\mathbf{r}_s}. \end{aligned} \tag{4.106}$$

To first order, with the important parts highlighted, this is

Only these terms contribute to a difference

$$R_{2,1} = r_{\text{av}} + \frac{(\Delta\mathbf{r})^2}{8r_{\text{av}}} + \frac{r_s^2}{2r_{\text{av}}}\boxed{\mp\frac{1}{2}\Delta\mathbf{r}\cdot\frac{1}{r_{\text{av}}}\mathbf{r}_s \pm 2\hat{\mathbf{r}}_{\text{av}}\cdot\Delta\mathbf{r}} - \hat{\mathbf{r}}_{\text{av}}\cdot\mathbf{r}_s. \tag{4.107}$$

The difference is

$$R_2 - R_1 = -\Delta\mathbf{r}\cdot\frac{1}{r_{\text{av}}}\mathbf{r}_s + \hat{\mathbf{r}}_{\text{av}}\cdot\Delta\mathbf{r} = \Delta\mathbf{r}\cdot\left(\hat{\mathbf{r}}_{\text{av}} - \frac{1}{r_{\text{av}}}\mathbf{r}_s\right). \tag{4.108}$$

Our autocorrelation is now

$$\Gamma_{12} = \frac{1}{\lambda^2}e^{ik\Delta\mathbf{r}\cdot\hat{\mathbf{r}}_{\text{av}}}\int d^2r_s\frac{I(\mathbf{r}_s)}{R_1(\mathbf{r}_s)R_2(\mathbf{r}_s)}e^{-ik\Delta\mathbf{r}\cdot\mathbf{r}_s/r_{\text{av}}}. \tag{4.109}$$

It's been implied that the integration limits described the aperture. Let's make that explicit with an aperture function $g(\mathbf{r}_s)$ so that we can allow the integration range to go to infinity in both directions without bound. Let's also assume that the distances R_1 and R_2 don't vary much from their averages

$$\overline{R_{2,1}} = \frac{\int d^2r_s g(\mathbf{r}_s)R_{2,1}(\mathbf{r}_s)}{\int d^2r_s g(\mathbf{r}_s)}. \tag{4.110}$$

so that the autocorrelation now takes the form

$$\Gamma_{12} \sim \frac{1}{\lambda^2 \overline{R}_1 \overline{R}_2} e^{ik\Delta\mathbf{r}\cdot\hat{\mathbf{r}}_{\text{av}}} \int d^2 r_s g(\mathbf{r}_s) I(\mathbf{r}_s) e^{-ik\Delta\mathbf{r}\cdot\mathbf{r}_s/r_{\text{av}}}. \tag{4.111}$$

If both of the vectors $\mathbf{r}_s$ and the vector $\Delta\mathbf{r}$ lie in the same plane, then the autocorrelation is found to be the Fourier transform of $g(\mathbf{r}_s)I(\mathbf{r}_s)$ evaluated at $k\Delta\mathbf{r}/r_{\text{av}}$.

4.12 PROBLEMS.

Exercise 4.1 Spatial distribution vector difference.

Referring to fig. 4.46 calculate $R_1 - R_2$.

Answer for Exercise 4.1

From the figure we see that we have

$$\mathbf{r}_s + \mathbf{R}_1 + \frac{1}{2}\Delta\mathbf{r} = \mathbf{r}_{\text{av}}. \tag{4.112a}$$

$$\mathbf{r}_s + \mathbf{R}_2 - \frac{1}{2}\Delta\mathbf{r} = \mathbf{r}_{\text{av}}, \tag{4.112b}$$

or

$$\mathbf{R}_1 = \mathbf{r}_{\text{av}} - \mathbf{r}_s - \frac{1}{2}\Delta\mathbf{r}. \tag{4.113a}$$

$$\mathbf{R}_2 = \mathbf{r}_{\text{av}} - \mathbf{r}_s + \frac{1}{2}\Delta\mathbf{r}. \tag{4.113b}$$

Squaring for the magnitudes, we have

$$\mathbf{R}_1^2 = (\mathbf{r}_{\text{av}} - \mathbf{r}_s)^2 + \left(\frac{1}{2}\Delta\mathbf{r}\right)^2 - \Delta\mathbf{r}\cdot(\mathbf{r}_{\text{av}} - \mathbf{r}_s). \tag{4.114a}$$

$$\mathbf{R}_2^2 = (\mathbf{r}_{\text{av}} - \mathbf{r}_s)^2 + \left(\frac{1}{2}\Delta\mathbf{r}\right)^2 + \Delta\mathbf{r}\cdot(\mathbf{r}_{\text{av}} - \mathbf{r}_s). \tag{4.114b}$$

Assuming that $|\mathbf{r}_{\rm av}| \gg |\mathbf{r}_s|$, so that $|\mathbf{r}_{\rm av} - \mathbf{r}_s| \sim |\mathbf{r}_{\rm av}|$, we have

$$R_{1,2}^2 \sim r_{\rm av}^2 \left(1 + \left(\frac{1}{2r_{\rm av}}\Delta\mathbf{r}\right)^2 \mp \frac{1}{r_{\rm av}^2}\Delta\mathbf{r}\cdot(\mathbf{r}_{\rm av} - \mathbf{r}_s)\,.\right), \tag{4.115}$$

which, to first order, is

$$R_{1,2} \sim r_{\rm av} + \frac{1}{2r_{\rm av}}\left(\frac{1}{2}\Delta\mathbf{r}\right)^2 \mp \frac{1}{2r_{\rm av}}\Delta\mathbf{r}\cdot(\mathbf{r}_{\rm av} - \mathbf{r}_s)\,, \tag{4.116}$$

with a difference of

$$R_1 - R_2 \sim -\frac{\Delta\mathbf{r}}{r_{\rm av}}\cdot(\mathbf{r}_{\rm av} - \mathbf{r}_s)\,. \tag{4.117}$$

Exercise 4.2 Solar interference. (*2012 Ps2, P3*)

Let's consider the prospects for interference fringes using direct sunlight.

a. Consider the sun to be a disc subtending a 0.5 degree diameter. Using the van Cittert-Zernike theorem, find the mutual coherence function on earth from sunlight. How close would two pinholes need to be to see a 50% visibility interference pattern behind them? For this part, make the (wrong) assumption that the sun is a quasimonochromatic source centered at λ =500 nm.

b. Another difficulty with the sun (when using it as a source for interferometry) is that it is spectrally broadband. As with any blackbody, a typical spectral width is $\Delta\omega = k_{\rm B}T/\hbar$, where $T \approx 5000\,$K for the sun. *Estimate* the effect of finite coherence time on fringe visibility, and make a qualitative sketch of the fringe pattern you would expect to observe.

c. In what situations is the spectral width a more severe problem for visibility than the spatial coherence?

Answer for Exercise 4.2

Part a. From the class notes (page 7, 11, 12) we have for the mutual coherence

$$\Gamma_{12} = e^{i\mathbf{k}_{\rm av}\cdot\Delta\mathbf{r}}\frac{1}{\lambda^2\overline{R_1}\,\overline{R_2}}\iint d^2r_s e^{-i\mathbf{k}_s\cdot\Delta\mathbf{r}} I(\mathbf{k}_s). \tag{4.118}$$

where

$$\mathbf{k}_s = k\frac{\mathbf{r}_s}{r_{\text{av}}}. \tag{4.119a}$$

$$\mathbf{k}_{\text{av}} = k\hat{\mathbf{r}}_{\text{av}}. \tag{4.119b}$$

$$\Delta\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2. \tag{4.119c}$$

Let's write for the disk radius R, distance from the disk D, separation of the observation points d. We'll place the observation points in the plane of the disk, symmetrically separated around the normal to the disk from the center setup our coordinates as in fig. 4.57

$$\mathbf{r}_{\text{av}} = D\hat{\mathbf{z}}. \tag{4.120a}$$

$$\mathbf{r}_1 = \mathbf{r}_{\text{av}} + \frac{d}{2}\hat{\mathbf{x}}. \tag{4.120b}$$

$$\mathbf{r}_2 = \mathbf{r}_{\text{av}} - \frac{d}{2}\hat{\mathbf{x}}. \tag{4.120c}$$

$$\Delta\mathbf{r} = (d)\hat{\mathbf{x}}. \tag{4.120d}$$

We have

$$\hat{\mathbf{r}}_{\text{av}} \cdot \Delta\mathbf{r} = \hat{\mathbf{z}} \cdot (d)\hat{\mathbf{x}} = 0, \tag{4.121}$$

so our mutual coherence is reduced to

$$\Gamma_{12} = \cancel{e^{i\mathbf{k}_{\text{av}}\cdot\Delta\mathbf{r}}}\frac{1}{\lambda^2 D^2}\iint d^2 r_s e^{-ik\frac{\mathbf{r}_s}{r_{\text{av}}}\cdot\hat{\mathbf{x}}d} I(\mathbf{k}_s). \tag{4.122}$$

Using an approximation of constant intensity $I(\mathbf{k}_s) = I_0$ over the disk, and employing radial coordinates

$$\mathbf{r}_s = \rho(\cos\theta, \sin\theta). \tag{4.123}$$

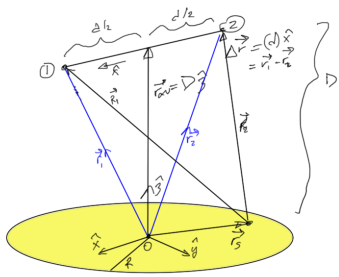

Figure 4.57: Geometry for solar disk interference problem.

we have

$$\Gamma_{12} = \frac{I_0}{\lambda^2 D^2} \int_0^{2\pi} d\theta \int_0^R \rho d\rho e^{-ik\frac{d}{D}\rho\cos\theta}. \tag{4.124}$$

With a substitution $a = -kd/D$, and some supplication to Mathematica (modernOpticsProblemSet2work.cdf), we find for the integral

$$\int_0^{2\pi} d\theta \int_0^R \rho d\rho e^{ia\rho\cos\theta} = \frac{2\pi R J_1(aR)}{a}, \tag{4.125}$$

so that the mutual coherence is

$$\Gamma_{12} = \frac{I_0}{\lambda^2 D^2} \frac{2\pi R J_1(kdR/D)}{kd/D} = \frac{I_0 2\pi R^2}{\lambda^2 D^2} \frac{J_1(kdR/D)}{kdR/D}. \tag{4.126}$$

We note that from the point(s) of observation, the observed angle of the disk is

$$\theta_s \sim \frac{2R}{D}, \tag{4.127}$$

and we also note that

$$k = \frac{2\pi}{\lambda}, \tag{4.128}$$

so our Bessel argument can be rewritten as

$$\frac{kdR}{D} = \frac{2\pi}{\lambda} d \frac{\theta_s}{2} = \frac{\pi d \theta_s}{\lambda}. \tag{4.129}$$

so that our mutual coherence in terms of desired variables is

$$\boxed{\Gamma_{12} = I_0 \frac{\pi\theta_s^2}{2\lambda^2} \frac{J_1(\pi\theta_s d/\lambda)}{\pi\theta_s d/\lambda}} \tag{4.130}$$

with

$$I_1 = I_2 = \Gamma_{12}|_{d=0} = I_0 \frac{\pi\theta_s^2}{2\lambda^2}\frac{1}{2}. \tag{4.131}$$

we have

$$\gamma_{12} = \frac{\Gamma_{12}}{\sqrt{I_1^2}} = \frac{\Gamma_{12}}{I_1} = 2\frac{J_1(\pi\theta_s d/\lambda)}{\pi\theta_s d/\lambda}. \tag{4.132}$$

To calculate the total intensity we have

$$I = 2I_1 + 2\,\mathrm{Re}\,\Gamma_{12} = 2I_1(1 + \mathrm{Re}\,\gamma_{12}). \tag{4.133}$$

so that

$$I_{\max} = 2I_1(1 + |\gamma_{12}|). \tag{4.134a}$$

$$I_{\min} = 2I_1(1 - |\gamma_{12}|). \tag{4.134b}$$

So that for the visibility

$$\mathcal{V} = \frac{I_{\max} - I_{\max}}{I_{\max} + I_{\max}} = \frac{2I_1 2|\gamma_{12}|}{4I_1} = |\gamma_{12}| = 2\left|\frac{J_1(\pi\theta_s d/\lambda)}{\pi\theta_s d/\lambda}\right|. \tag{4.135}$$

The Bessel function ratio that we have in the absolute values here is plotted in fig. 4.58. We find numerically that $J_1(x)/x = 0.25$

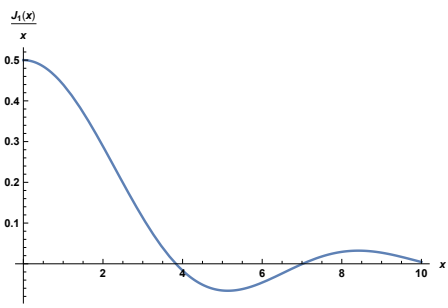

Figure 4.58: Sinc like first order Bessel function.

occurs for $x = 2.22$, so 50 % visibility for the 500 nm average wavelength at 0.5 degrees occurs when

$$d = \frac{2.22\lambda}{\pi\theta_s} = \frac{2.22 \times 500 \times 10^{-9}\mathrm{m}}{\pi\frac{\pi}{180}\frac{1}{2}} = 0.04\mathrm{mm}. \tag{4.136}$$

Part b. Suppose that we have two sources of identical amplitude, each separated from the average frequency by half the spectral width

$$\Psi_1 = \sqrt{I_0} e^{i(\overline{\omega}-\Delta\omega)t}. \tag{4.137a}$$

$$\Psi_2 = \sqrt{I_0} e^{i(\overline{\omega}+\Delta\omega)t}. \tag{4.137b}$$

To compute the correlation of these we compute

$$\Psi_1(t)\Psi_2^*(t+\tau) = I_0 e^{i(\overline{\omega}-\Delta\omega)t - i(\overline{\omega}+\Delta\omega)(t+\tau)} = I_0 e^{-2i\Delta\omega t - i\overline{\omega}\tau}, \tag{4.138}$$

so that

$$\begin{aligned}
\gamma &= \left\langle e^{-2i\Delta\omega t - i\overline{\omega}\tau} \right\rangle \\
&= e^{-i\overline{\omega}\tau} \left\langle e^{-2i\Delta\omega t} \right\rangle \\
&= e^{-i\overline{\omega}\tau} \frac{1}{2\tau_a} \int_{-\tau_a}^{\tau_a} e^{-2i\Delta\omega t} dt \\
&= e^{-i\overline{\omega}\tau} \frac{1}{2\tau_a} \frac{\sin(2\Delta\omega\tau_a)}{\Delta\omega} \\
&= e^{-i\overline{\omega}\tau} \operatorname{sinc}(2\Delta\omega\tau_a).
\end{aligned} \tag{4.139}$$

Our intensity is

$$I = 2I_0 + 2I_0 \operatorname{Re}\gamma = 2I_0 \left(1 + \cos(\overline{\omega}\tau)\right) \operatorname{sinc}(2\Delta\omega\tau_a). \tag{4.140}$$

As the spectral width increases for a fixed period of observation τ_a our intensity dies off from its maximum to the average $2I_0$. This is sketched roughly in fig. 4.59

Part c. With a coherence time inversely proportional to the frequency

$$t_c \sim \frac{1}{\Delta\omega} = \frac{\hbar}{k_B T}, \tag{4.141}$$

a larger spectral width (larger T) will result in a smaller coherence time, and a requirement for faster detection circuitry. This is clearly more of an issue than the spatial coherence when the distance to the object is very far. One such example is the stellar interferometer discussed in [8] §12.4.2 where this coherence time was used to indirectly determine the diameter of a stellar source.

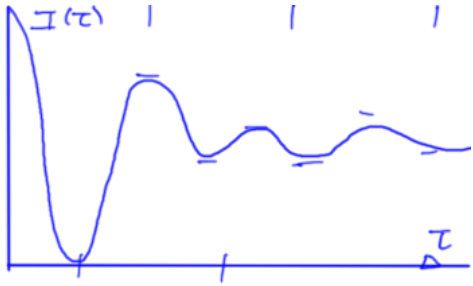

Figure 4.59: Possible fringes from two sources at the boundaries of the spectral width.

Exercise 4.3 Doublets and Combs. (*2012 Ps3, P1*)

For each of the following optical power spectra, **find the intensity output** of a balanced Michelson interferometer $I(\tau)$, where $c\,\tau$ is the difference in path length of the two arms of the interferometer. For both problems, you will find it useful to use the convolution theorem. You can assume that both $I(\tau)$ and $I(\omega)$ are even functions, which simplifies the Fourier math.

a. A frequency "doublet"

$$I(\omega) = \frac{I_0}{2}\frac{\sigma^2}{\sigma^2 + (|\omega| - \omega_1)^2} + \frac{I_0}{2}\frac{\sigma^2}{\sigma^2 + (|\omega| - \omega_2)^2}, \tag{4.142}$$

where σ is the line width, and $\omega_{1,2}$ are peak frequencies. In interpreting the result, you can assume that these are narrow lines with close frequencies, split by some Δ, so that $\omega_{1,2} = \overline{\omega} \pm \Delta/2$, and $\overline{\omega} \gg \Delta \gg \sigma$. In this limit, you should find that the interference has a fast component with frequency $\overline{\omega}$ under an envelope with frequency $\Delta/2$. **Sketch** $I(\tau)$, and **label** the various time scales.

b. A "frequency comb"

$$I(\omega) = \frac{I_0}{N}\sum_{n=0}^{N-1}\frac{\sigma^2}{\sigma^2 + (|\omega| - \omega_n)^2}, \tag{4.143}$$

where $\omega_n = \omega_0 + n\delta$. Here you should find that the correlation is pulsed (!), and that the normalized mutual coherence can be written

$$\Re\{\gamma(\tau)\} = e^{-\sigma|\tau|}\left(\frac{\sin\left(N\pi\tau/\tau_{\text{rep}}\right)}{N\sin\left(\pi\tau/\tau_{\text{rep}}\right)}\right)\cos\overline{\omega}\tau, \quad (4.144)$$

where τ_{rep} is a time between pulses, and $\overline{\omega}$ is an average frequency of the spectrum. **Find** expressions for both $\overline{\omega}$ and the rep rate τ_{rep}. **Plot** the Michelson output for $\sigma = \delta/20$, $\overline{\omega} = 50\delta$, $N = 10$, for τ in the range $0 \to 4\pi/\delta$. (note we're in the limit of small comb spacing and even narrower peaks: $\sigma \ll \delta \ll \overline{\omega}$.)

Answer for Exercise 4.3

Setup, for my own benefit, assembling all required concepts in one place. First consider the average intensity due to the contributions of both path components, with respective pathlengths r_a, and $r_a + c\tau$

$$\begin{aligned} I(\tau) &= \left\langle \left|\Psi(r_a + c\tau) + \Psi(r_a, t)\right|^2 \right\rangle \\ &= \left\langle \left|\Psi(r_a + c\tau)\right|^2 \right\rangle + \left\langle \left|\Psi(r_a)\right|^2 \right\rangle + 2\,\text{Re}\left\langle \Psi(r_a + c\tau)\Psi^*(r_a, t)\right\rangle . \end{aligned} \quad (4.145)$$

We can write this as

$$I(\tau) = 2I_0\left(1 + \text{Re}\,\frac{\Gamma(\tau)}{\Gamma(0)}\right), \quad (4.146)$$

where the pathlength difference of $c\tau$ introduces an autocorrelation for the two paths of

$$\Gamma(\tau) = \left\langle \Psi(r_a + c\tau, t)\Psi^*(r_a, t)\right\rangle . \quad (4.147)$$

To evaluate this for the Michelson setup, lets assume initially a spherical wave packet at the outputs of interferometer paths, where r is the total distance that the incident wavepacket travels after all transmission and reflection

$$\Psi(r, t) = \frac{1}{\sqrt{2\pi}}\int \tilde{\Psi}(\omega)\frac{e^{-i\omega(t-r/c)}}{r}d\omega. \quad (4.148)$$

Somewhat loosely, we can evaluate this average

$$
\begin{aligned}
&\Gamma(\tau) \\
&= \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} dt \frac{1}{2\pi} \int \tilde{\Psi}(\omega) \frac{e^{-i\omega(t-r_a/c-\tau)}}{r_a + c\tau} d\omega \int \tilde{\Psi}^*(\omega') \frac{e^{i\omega'(t-r_a/c)}}{r_a} d\omega' \\
&\sim \lim_{T\to\infty} \frac{1}{4\pi T r_a^2} \int_{-T}^{T} dt \int \tilde{\Psi}(\omega) e^{-i\omega(t-r_a/c-\tau)} d\omega \int \tilde{\Psi}^*(\omega') e^{i\omega'(t-r_a/c)} d\omega'.
\end{aligned}
\tag{4.149}
$$

Here we've assumed that $c\tau \ll r_a$, so that we can pull out the spatial variation of the amplitude. Now lets also assume that the time domain wave forms are bounded for some $T = T_c$, so that

$$
\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} = \frac{1}{2T_c} \int_{-\infty}^{\infty},
\tag{4.150}
$$

allowing us to increase the bounds of the dt integral, and make a delta function identification

$$
\begin{aligned}
\Gamma(\tau) &\sim \frac{1}{2T_c r_a^2} \int d\omega \tilde{\Psi}(\omega) \int d\omega' \tilde{\Psi}^*(\omega') e^{i\omega\tau} \frac{1}{2\pi} \int dt e^{i(\omega'-\omega)(t-r_a/c)} \\
&= \frac{1}{2T_c r_a^2} \int d\omega \tilde{\Psi}(\omega) \int d\omega' \tilde{\Psi}^*(\omega') \delta(\omega' - \omega) e^{i\omega\tau} \\
&= \frac{1}{2T_c r_a^2} \int d\omega \tilde{\Psi}(\omega) \tilde{\Psi}^*(\omega) e^{i\omega\tau} \\
&\sim \mathcal{F}^{-1}\left(|\Psi(\omega)|^2\right),
\end{aligned}
\tag{4.151}
$$

or

$$
\Gamma(\tau) \sim \mathcal{F}^{-1}\left(I(\omega)\right).
\tag{4.152}
$$

With

$$
\gamma(\tau) = \frac{\Gamma(\tau)}{\Gamma(0)},
\tag{4.153}
$$

our intensity eq. (4.145) as a function of the additional pathlength $c\tau$, takes the form

$$
I(\tau) = 2I_0(1 + \operatorname{Re}\gamma(\tau)).
\tag{4.154}
$$

Because there is this constant offset in the time domain intensity, we have to be slightly careful to find this given the frequency domain intensity, and eq. (4.154) shows us exactly how to do this.

Part a. Frequency doublet We are now set to tackle this specific problem. Our first task is the Fourier inversion of the frequency domain intensity. Note that the form of the doublet as given in this problem is not directly invertible without the use of special functions. In general the absolute value of the frequency introduces a discontinuity in the derivative at the origin that makes life "fun". What we can do, however, is assume that $\omega_{1,2} \gg \sigma$ so that we can make the approximation

$$\frac{\sigma^2}{\sigma^2 + (|\omega| - \omega_0)^2} \sim \frac{\sigma^2}{\sigma^2 + (\omega - \omega_0)^2} + \frac{\sigma^2}{\sigma^2 + (\omega + \omega_0)^2}. \tag{4.155}$$

A comparison of these for small ω_0 is shown in fig. 4.60. For large

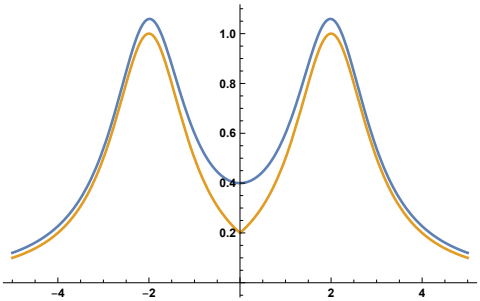

Figure 4.60: Comparison to doubled frequency form.

enough ω_0 there is no visible difference in the two functions as seen in fig. 4.61. To Fourier invert one of these peaked intensity

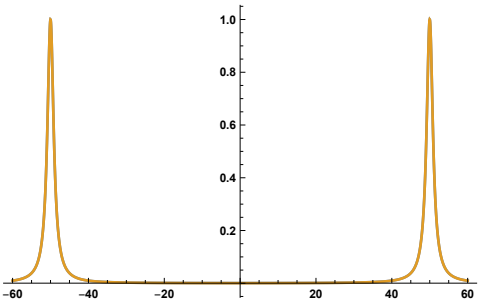

Figure 4.61: More widely separated peak frequencies.

functions

$$f(\omega) = \frac{\sigma^2}{\sigma^2 + (\omega - \omega_0)^2}. \tag{4.156}$$

We first note that

$$\int e^{i\omega\tau} f(\omega - \omega_0) d\omega = \int e^{i(\omega' + \omega_0)\tau} f(\omega') d\omega'. \tag{4.157}$$

so

$$\mathcal{F}^{-1}(f(\omega - \omega_0)) = e^{i\omega_0\tau} \mathcal{F}^{-1}(f(\omega)). \tag{4.158}$$

We have only to consider the Fourier inversion of a function of the form

$$g(\omega) = \frac{\sigma^2}{\sigma^2 + \omega^2}, \tag{4.159}$$

but we recognize this from class, where we looked at the Fourier transform of a unit area symmetric damped exponential ($\sigma > 0$)

$$\begin{aligned}
\mathcal{F}\frac{\sigma}{2} e^{-\sigma|\tau|} &= \frac{\sigma}{2} \int_{-\infty}^{\infty} e^{-i\omega\tau - \sigma|\tau|} dt \\
&= \frac{\sigma}{2} \int_{0}^{\infty} e^{-i\omega\tau - \sigma\tau} dt + \frac{\sigma}{2} \int_{-\infty}^{0} e^{-i\omega\tau + \sigma\tau} dt \\
&= \frac{\sigma}{2} \frac{e^{i\omega\tau - \sigma\tau}}{i\omega - \sigma}\bigg|_{0}^{\infty} + \frac{\sigma}{2} \frac{e^{i\omega\tau + \sigma\tau}}{i\omega + \sigma}\bigg|_{-\infty}^{0} \\
&= \frac{\sigma}{2} \left(\frac{1}{\sigma - i\omega} + \frac{1}{\sigma + i\omega} \right) \\
&= \frac{\sigma}{2} \frac{2\sigma}{\sigma^2 + \omega^2},
\end{aligned} \tag{4.160}$$

so that (up to a potential constant multiplicative factor according to the Fourier transform convention in use)

$$\mathcal{F}^{-1}\left(\frac{\sigma^2}{\sigma^2 + (\omega - \omega_0)^2} \right) = \frac{\sigma}{2} e^{-\sigma|\tau|} e^{i\omega_0\tau}. \tag{4.161}$$

We can now immediately write the Fourier transform pair, finding the mutual coherence from the frequency domain intensity

$$\begin{aligned}
\Gamma(\omega) &\sim \frac{I_0}{2} \sum_{\omega_k = \pm\omega_1, \pm\omega_2} \frac{\sigma^2}{\sigma^2 + (\omega - \omega_k)^2} \\
&\leftrightarrow \frac{I_0}{2} \sigma e^{-\sigma|\tau|} \left(\cos(\omega_1\tau) + \cos(\omega_2\tau) \right).
\end{aligned} \tag{4.162}$$

With $\omega_{1,2} = \overline{\omega} \mp \Delta/2$, we rewrite

$$\begin{aligned}\cos(\omega_1\tau) + \cos(\omega_2\tau) &= \cos\left(\left(\overline{\omega} - \Delta/2\right)\tau\right) + \cos\left(\left(\overline{\omega} - \Delta/2\right)\tau\right) \\ &= 2\cos(\overline{\omega}\tau)\cos(\Delta\tau/2).\end{aligned} \tag{4.163}$$

so that

$$\Gamma(\tau) \sim I_0 \sigma e^{-\sigma|\tau|} \cos(\overline{\omega}\tau)\cos(\Delta\tau/2). \tag{4.164}$$

To find the intensity as a function of the additional path delay τ, we normalize the mutual correlation

$$\gamma(\tau) = \frac{\Gamma(\tau)}{\Gamma(0)} = e^{-\sigma|\tau|}\cos(\overline{\omega}\tau)\cos(\Delta\tau/2), \tag{4.165}$$

From eq. (4.146), we have for the time domain intensity

$$\boxed{I(\tau) = I_0\left(1 + e^{-\sigma|\tau|}\cos(\overline{\omega}\tau)\cos(\Delta\tau/2)\right).} \tag{4.166}$$

This is sketched in fig. 4.62.

Grading note (−1) $\overline{\omega} > \Delta$ (typically) unless if one of the frequencies is a lot larger than the other one!

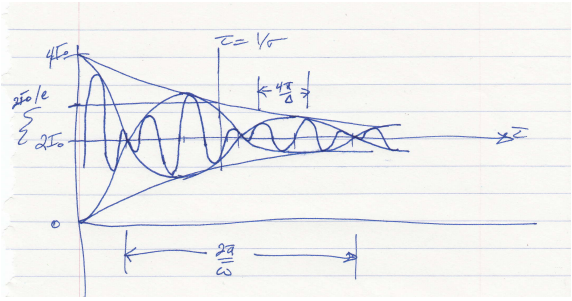

Figure 4.62: Intensity as a function of additional path length.

Part b. Frequency comb For a multiple frequency comb, we will again make a double peak approximation before inverse Fourier transforming

$$
\begin{aligned}
I(\omega) &= \frac{I_0}{N}\sum_{n=0}^{N-1}\frac{\sigma^2}{\sigma^2+(\overline{\omega}-\omega_n)^2} \\
&\sim \frac{I_0}{N}\sum_{n=0}^{N-1}\left(\frac{\sigma^2}{\sigma^2+(\omega-\omega_n)^2}+\frac{\sigma^2}{\sigma^2+(\omega+\omega_n)^2}\right) \\
&\leftrightarrow \frac{I_0}{N}\sum_{n=0}^{N-1}\sigma e^{-\sigma|\tau|}\cos(\omega_n\tau) \\
&\sim \Gamma(\tau).
\end{aligned}
\tag{4.167}
$$

Since

$$
\sum_{n=0}^{N-1}(1) = N. \tag{4.168}
$$

we can switch immediately to the normalized form

$$
\begin{aligned}
\gamma(\tau) &= \frac{\Gamma(\tau)}{\Gamma(0)} \\
&= \frac{1}{N}e^{-\sigma|\tau|}\sum_{n=0}^{N-1}\cos(\omega_n\tau) \\
&= \frac{1}{N}e^{-\sigma|\tau|}\,\mathrm{Re}\left(\sum_{n=0}^{N-1}e^{i\omega_n\tau}\right) \\
&= \frac{1}{N}e^{-\sigma|\tau|}\,\mathrm{Re}\left(\sum_{n=0}^{N-1}e^{i(\omega_0+n\delta)\tau}\right) \\
&= \frac{1}{N}e^{-\sigma|\tau|}\,\mathrm{Re}\left(e^{i\omega_0\tau}\sum_{n=0}^{N-1}e^{in\delta\tau}\right) \\
&= \frac{1}{N}e^{-\sigma|\tau|}\,\mathrm{Re}\left(e^{i\omega_0\tau}\frac{e^{iN\delta\tau}-1}{e^{i\delta\tau}-1}\right) \\
&= \frac{1}{N}e^{-\sigma|\tau|}\,\mathrm{Re}\left(e^{i\omega_0\tau}\frac{e^{iN\delta\tau/2}}{e^{i\delta\tau/2}}\right)\frac{\sin(N\delta\tau/2)}{\sin(\delta\tau/2)},
\end{aligned}
\tag{4.169}
$$

or

$$
\gamma(\tau) = \frac{1}{N}e^{-\sigma|\tau|}\cos((\omega_0+(N-1)\delta/2)\tau)\frac{\sin(N\delta\tau/2)}{\sin(\delta\tau/2)}. \tag{4.170}
$$

We observe that the average of the ω_n frequencies is

$$\begin{aligned}\overline{\omega} &= \frac{1}{N}\sum_{n=0}^{N-1}(\omega_0 + n\delta) \\ &= \omega_0 + \frac{\delta}{N}\sum_{n=0}^{N-1} n \\ &= \omega_0 + \frac{\delta}{N}(N)(N-1)/2 \\ &= \omega_0 + \frac{(N-1)\delta}{2},\end{aligned} \tag{4.171}$$

we have

$$\boxed{\gamma(\tau) = e^{-\sigma|\tau|}\cos(\overline{\omega}\tau)\frac{\sin(N\delta\tau/2)}{N\sin(\delta\tau/2)}} \tag{4.172}$$

With interesting stuff happening every $\delta\tau_{\text{rep}}/2 = \pi$, we have

$$\tau_{\text{rep}} = \frac{2\pi}{\delta}, \tag{4.173}$$

and recover the desired result eq. (4.144). We are asked to plot with

$$\sigma = \frac{\delta}{20} = \frac{\pi}{10\tau_{\text{rep}}}. \tag{4.174a}$$

$$\overline{\omega} = 50\delta = \frac{100\pi}{\tau_{\text{rep}}}. \tag{4.174b}$$

$$\tau \in [0, 4\pi/\delta] = [0, 2\tau_{\text{rep}}]. \tag{4.174c}$$

Non-dimensionalising with $u = \tau/\tau_{\text{rep}}$ and $N = 10$, we plot this in fig. 4.63 over $u \in [0, 2]$

$$\gamma(u) = e^{-\frac{\pi}{10}|u|}\cos(100\pi u)\frac{\sin(10\pi u)}{10\sin(\pi u)}. \tag{4.175}$$

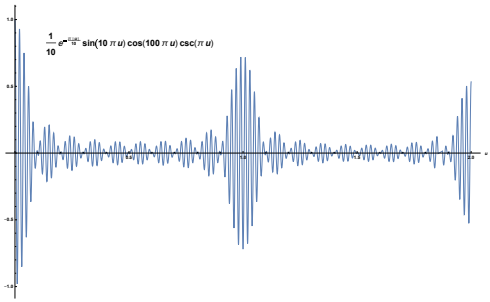

Figure 4.63: 10 frequency input to Michelson interferometer.

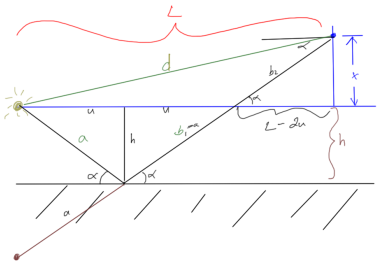

Figure 4.64: Lloyd's mirror.

Exercise 4.4 Lloyd's mirror.

This is a problem that examines the periodicity of interference for monochromatic point source. Consider the interference for the Lloyd's mirror configuration of fig. 4.64.

Answer for Exercise 4.4

We want to consider the pathlength differences along the direct path d, to that of $b = 2a + b_2$. The lengths of these paths are

$$d = \sqrt{L^2 + x^2}. \tag{4.176a}$$

$$b = \sqrt{L^2 + (2h + x)^2}. \tag{4.176b}$$

The pathlength difference, for $x, h \ll L$ is then

$$b - d = L\left(\sqrt{1 + \left(\frac{2h + x}{L}\right)^2} - \sqrt{1 + \frac{x^2}{L^2}}\right) \sim L\frac{1}{2L^2}(4h^2 + 4hx),$$

(4.177)

or

$$b - d = \frac{2h}{L}(h + x). \tag{4.178}$$

So, supposing that we have a spherical wave emitted from the point source, allowing for a different (random) in each direction, the waves that arrive at the observation point, having traveled along paths d and b respectively are

$$\Psi_d = \frac{\Psi_0}{d} e^{ikd - i\omega t + \phi_d(t)}. \tag{4.179a}$$

$$\Psi_b = \frac{\Psi_0}{b} e^{ikb - i\omega t + \phi_b(t)}. \tag{4.179b}$$

Our average intensity, should there be a time delay of τ in the b path, is

$$\begin{aligned} I(\tau) &= \left\langle \left(\Psi_d(t) + \Psi_b(t+\tau)\right)\left(\Psi_d^*(t) + \Psi_b^*(t+\tau)\right)\right\rangle \\ &= \left\langle |\Psi_d(t)|^2 \right\rangle + \left\langle |\Psi_b(t+\tau)|^2 \right\rangle \\ &\quad + 2\,\mathrm{Re}\left\langle \frac{\Psi_0}{d} e^{ikd - i\omega t + \phi_d(t)} \frac{\Psi_0^*}{b} e^{-ikb + i\omega(t+\tau) - \phi_b(t+\tau)} \right\rangle \\ &= I_d + I_b + 2\,\mathrm{Re}\left(\frac{\Psi_0}{d}\frac{\Psi_0^*}{b} e^{i\omega\tau} e^{ik(d-b)} \left\langle e^{i\phi_d(t) - i\phi_b(t+\tau)} \right\rangle \right) \\ &\sim I_d + I_b + 2\,\mathrm{Re}\left(\frac{\Psi_0}{d}\frac{\Psi_0^*}{b} e^{i\omega\tau} e^{ik(d-b)} \left(1 - \frac{\tau}{\tau_0}\right) \right) \Theta(\tau_0 - \tau). \end{aligned} \tag{4.180}$$

The interference term, scaling by requiring a unit value at $\tau = 0$, and writing $y = x + h$ (effectively re positioning our origin) is therefore

$$\gamma(y, \tau) \sim \cos\left(\frac{2kh}{L} y - \omega\tau\right)\left(1 - \frac{\tau}{\tau_0}\right)\Theta(\tau_0 - \tau). \tag{4.181}$$

This has a maximum at $(y, \tau) = (0, 0)$. We've also got a set of level curves, marking the amplitudes of equal magnitude

$$\tau = \frac{2khy}{L\omega} = \frac{2hyd}{L} = \text{constant}. \tag{4.182}$$

This wasn't exactly what I recalled calculating on the midterm, but I'd forgotten the exact midterm question. Now that the midterm is posted, I see that I didn't recall the problem that well. We were asked to calculate the periodicity given a monochromatic point source. From eq. (4.181) we see that we can consider either spatial or temporal periodicity. The spatial periodicity is what probably makes the most sense to consider since we aren't explicitly introducing any delays in this mirror scenario. Maximums repeat every Δx where

$$\frac{2kh\Delta x}{L} = 2\pi. \tag{4.183}$$

That is

$$\Delta x = \frac{\pi L}{kh}. \tag{4.184}$$

Or with $1/k = \lambda/2\pi$ that is

$$\boxed{\Delta x = \frac{L\lambda}{2h}} \tag{4.185}$$

Should we wish to reintroduce the angle α from the figure, we have for the small angle approximation $\alpha \sim 2h/L$, which gives peaks every $\Delta x = \lambda/\alpha$.

Exercise 4.5 Lloyd's mirror. Non monochromatic source.

Suppose we are told to assume that the source had a Gaussian frequency distribution. How do things change?

Answer for Exercise 4.5

Let's play around with evolving things and suppose that we slightly generalize the spherical waves we'd interfered by allowing a superposition of the form

$$\Psi_i = \frac{1}{\sqrt{2\pi} r_i} \int d\omega \tilde{\Psi}_0(\omega) e^{i\omega(r_i/c - t) + \phi_i(t)}. \tag{4.186}$$

Now, if we delay one such wave function in time at the source by time τ, our resultant field is

$$\begin{aligned}\Psi = &\frac{1}{\sqrt{2\pi} r_1} \int d\omega \tilde{\Psi}_0(\omega) e^{i\omega(r_1/c - t) + i\phi_1(t)} \\ &+ \frac{1}{\sqrt{2\pi} r_2} \int d\omega \tilde{\Psi}_0(\omega) e^{i\omega(r_2/c - t - \tau) + i\phi_2(t+\tau)},\end{aligned} \tag{4.187}$$

with average intensity proportional to

$$\begin{aligned} I &= I_1 + I_2 \\ &+ \frac{|\Psi_0|^2}{\pi r_1 r_2} \mathrm{Re}\langle \int d\omega \tilde{\Psi}_0(\omega) e^{i\omega(r_1/c-t)+i\phi_1(t)} \times \\ &\int d\omega' \tilde{\Psi}_0^*(\omega') e^{-i\omega'(r_2/c-t-\tau)-i\phi_2(t+\tau)} \rangle. \end{aligned} \tag{4.188}$$

Let

$$\Psi_0(t) = \frac{1}{\sqrt{2\pi}} \int d\omega \tilde{\Psi}_0(\omega) e^{-i\omega t}, \tag{4.189}$$

so that the interference term is

$$\Gamma(\tau) = \frac{2}{r_1 r_2 T} \int_{-T/2}^{T/2} dt \Psi_0(t - r_1/c) \Psi_0^*(t + \tau - r_2/c) e^{i\phi_1(t) - i\phi_2(t+\tau)}. \tag{4.190}$$

Clearly this will be more tractable if we fold the random phase functions $\phi_i(t)$ into the Fourier integrals [1], as in

$$\Psi_i = \frac{1}{\sqrt{2\pi} r_i} \int d\omega \tilde{\Psi}_0(\omega) e^{i\omega(r_i/c-t)}. \tag{4.191}$$

so that our interference term is reduced to

$$\begin{aligned} \Gamma(\tau) &= \frac{2}{r_1 r_2 T} \int_{-T/2}^{T/2} dt \Psi_0(t - r_1/c) \Psi_0^*(t + \tau - r_2/c) \\ &= \frac{2}{r_1 r_2 T} \int_{-T/2-r_1/c}^{T/2-r_1/c} du \Psi_0(u) \Psi_0^*(u + \tau + (r_1 - r_2)/c). \end{aligned} \tag{4.192}$$

Now let's impose an additional constraint requiring for some finite T that we have for $|u| > T$

$$\Psi_0(u) = 0. \tag{4.193}$$

Ignoring the details about the range restriction we require for τ for now, our interference term will then be proportional to the

1 This is a reasonable construction, and we can easily verify that application of the wave equation $\nabla^2 - (1/c^2)\partial_{tt} = (1/r^2)\partial_r r^2 \partial_r - (1/c^2)\partial_{tt}$ to an integral of the form $\int dk A(k) e^{ik(r-ct)}/r$ gives zero away from the origin as desired.

plain old convolution integral, and we can re express this beastie in terms of assumed transform pairs

$$\tilde{\Psi}_0(\omega) = \frac{1}{\sqrt{2\pi}} \int dt \Psi_0(t) e^{i\omega t}. \tag{4.194a}$$

$$\Psi_0(t) = \frac{1}{\sqrt{2\pi}} \int d\omega \tilde{\Psi}_0(\omega) e^{-i\omega t}. \tag{4.194b}$$

$$\begin{aligned}
\Gamma(\tau) &\sim \int_{-\infty}^{\infty} \Psi_0(u) \Psi_0^*(u + \tau + (r_1 - r_2)/c) \\
&= \frac{1}{2\pi} \int du d\omega \tilde{\Psi}_0(\omega) e^{i\omega u} d\omega' \tilde{\Psi}_0^*(\omega') e^{-i\omega'(u+\tau+(r_1-r_2)/c)} \\
&= \int \delta(\omega - \omega') d\omega \tilde{\Psi}_0(\omega) d\omega' \tilde{\Psi}_0^*(\omega') e^{-i\omega'(\tau+(r_1-r_2)/c)} \\
&= \int d\omega \tilde{\Psi}_0(\omega) \tilde{\Psi}_0^*(\omega) e^{-i\omega(\tau+(r_1-r_2)/c)} \\
&= \int d\omega \left|\tilde{\Psi}_0(\omega)\right|^2 e^{-i\omega(\tau+(r_1-r_2)/c)}.
\end{aligned} \tag{4.195}$$

Again, dropping multiplicative constants, our interference term has the following proportionality

$$\Gamma(\tau) \sim \mathcal{F}\left(\left|\tilde{\Psi}_0(\omega)\right|^2\right)\Big|_{t=\tau+(r_1-r_2)/c}. \tag{4.196}$$

For this problem, we are told that our wave packet has a Gaussian frequency distribution

$$\Psi(\omega) = \frac{1}{\sqrt{2\pi}\Delta\omega} \exp\left(-\frac{(\omega - \omega_0)^2}{2(\Delta\omega)^2}\right). \tag{4.197}$$

We expect this to also be Gaussian in the time domain. Let's perform that Fourier inversion to see what that looks like. Because $k = \omega/c$ isn't constant, we can't just toss in the spatial dependency after the fact, so we hack it in here as a retarded time, adding in the $1/r$ factor required for a spherical wave

$$\begin{aligned}
\Psi(r,t) &= \frac{1}{(\sqrt{2\pi})^2 r \Delta\omega} \int d\omega \exp\left(-\frac{(\omega - \omega_0)^2}{2(\Delta\omega)^2} + i\omega(r/c - t)\right) \\
&= \frac{1}{\sqrt{2\pi} r} \exp\left(-\frac{1}{2}(t - r/c)^2 (\Delta\omega)^2 - i\omega_0(t - r/c)\right).
\end{aligned}$$

(4.198)

Plotted implicitly against the retarded time $t - r/c$ for some non-zero r, we've got a Gaussian envelope, and oscillations within that as in fig. 4.65. We now want our auto-correlation eq. (4.196) for

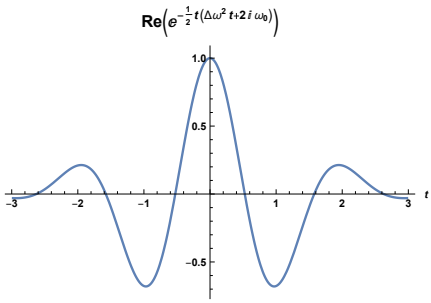

Figure 4.65: Gaussian wave packet.

this Lloyd's configuration. Fourier transforming the square of our frequency spectrum we have

$$\frac{1}{2\pi\Delta\omega}\int d\omega \exp\left(-\frac{(\omega-\omega_0)^2}{(\Delta\omega)^2} - i\omega t\right) = \frac{\exp\left(-\frac{1}{4}t^2(\Delta\omega)^2 - i\omega_0 t\right)}{4\pi^{3/2}(\Delta\omega)^2}. \tag{4.199}$$

Again we have a Gaussian envelope, with oscillations at the average frequency. We recall that for the Lloyd's configuration our path length difference eq. (4.178) was

$$\frac{r_1 - r_2}{c} = \frac{2h}{Lc}(h+x), \tag{4.200}$$

so our mutual correlation is

$$\begin{aligned}
&\Gamma(\tau) \\
&\sim \exp\left(-\frac{1}{4}\left(\tau - \frac{2h}{Lc}(h+x)\right)^2(\Delta\omega)^2 - i\omega_0\left(\tau - \frac{2h}{Lc}(h+x)\right)\right) \\
&\sim \exp\left(-\left(\frac{\tau}{2} - \frac{h}{Lc}(h+x)\right)^2(\Delta\omega)^2 + \left(\frac{h}{Lc}(h+x)\right)^2(\Delta\omega)^2\right) \times \\
&\quad \exp\left(-i\omega_0\left(\tau - \frac{2h}{Lc}(h+x)\right)\right) \\
&= \exp\left((\Delta\omega)^2\left(\frac{\tau h}{Lc}(h+x) - \frac{\tau^2}{4}\right)\right) \times \exp\left(-i\omega_0\left(\tau - \frac{2h}{Lc}(h+x)\right)\right).
\end{aligned}$$

(4.201)

We've got a Gaussian envelope with oscillations at the average frequency as in fig. 4.66. The falloff of the Gaussian will be dom-

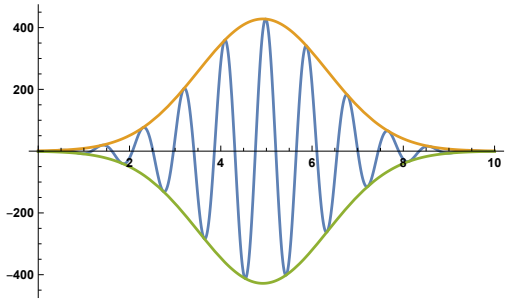

Figure 4.66: Mutual coherence of Gaussian source.

inated by $e^{-(\Delta\omega)^2\tau^2/4}$, so our coherence time, the time for a $1/e$ reduction, is

$$\boxed{\tau_c = \frac{2}{(\Delta\omega)}.} \tag{4.202}$$

While the mutual correlation has a dependence on the path length difference of the Lloyd's configuration, the coherence time is independent of that, and only depends on the width of source spectrum. Is this correct?

Exercise 4.6 Wave functions for Lloyd's mirror configuration.

For a linearly spread source distribution illuminating a Lloyd's mirror configuration, find the wave functions at the observation point.

Answer for Exercise 4.6

Let's re-do the geometrical part of the task we did above, allowing for an additional offset from the point average position of a linear source as in fig. 4.67. We see that the distance for the direct line of sight, and for the bounced rays are respectively

$$d = \sqrt{L^2 + (x - x')^2}. \tag{4.203a}$$

$$b = \sqrt{L^2 + (2h + 2x' + x - x')^2}. \tag{4.203b}$$

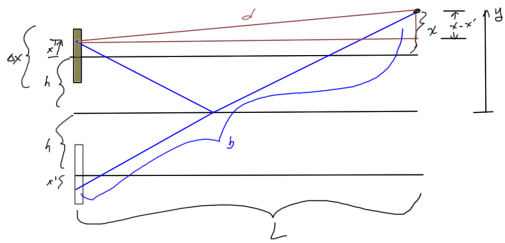

Figure 4.67: Lloyd's mirror configuration for a distributed source.

or with $y = x + h$

$$d = L\sqrt{1 + (y - h - x')^2/L^2}. \tag{4.204a}$$

$$b = L\sqrt{1 + (y + h + x')^2/L^2}, \tag{4.204b}$$

The wave function at the observation point for a monochromatic source is therefore

$$\begin{aligned}\Psi &= \frac{\Psi_s}{i\lambda}\int_{-\Delta x/2}^{\Delta x/2} dx' \left(\frac{e^{ikd}}{d} + \frac{e^{ikb}}{b}\right) \\ &\approx \frac{\Psi_s}{i\lambda L}\int_{-\Delta x/2}^{\Delta x/2} dx' \left(e^{ikL\sqrt{1+(y-h-x')^2/L^2}} + e^{ikL\sqrt{1+(y+h+x')^2/L^2}}\right) \\ &\approx \frac{\Psi_s e^{ikL}}{i\lambda L}\int_{-\Delta x/2}^{\Delta x/2} dx' \left(e^{ik(y-h-x')^2/(2L)} + e^{ik(y+h+x')^2/(2L)}\right).\end{aligned} \tag{4.205}$$

These now have the structure of Fresnel integrals. We make the following change of variables for the respective exponentials

$$\frac{\pi}{2}w^2 = \frac{k(y - h - x')^2}{2L} = \frac{\pi(x' + h - y)^2}{L\lambda}. \tag{4.206a}$$

$$\frac{\pi}{2}w^2 = \frac{k(y + h + x')^2}{2L} = \frac{\pi(x' + h + y)^2}{L\lambda}. \tag{4.206b}$$

We find that our interference wave function is

$$\Psi(y) = \frac{\Psi_s e^{ikL}}{i\sqrt{2\lambda L}} \left(\left(C(w) + iS(w)\right)\Big|_{\sqrt{\frac{2}{L\lambda}}(h-y-\Delta x/2)}^{\sqrt{\frac{2}{L\lambda}}(h-y+\Delta x/2)} + \left(C(w) + iS(w)\right)\Big|_{\sqrt{\frac{2}{L\lambda}}(h+y-\Delta x/2)}^{\sqrt{\frac{2}{L\lambda}}(h+y+\Delta x/2)} \right). \tag{4.207}$$

As a sanity check observe that things look appropriate in the $\Delta x \to 0$ limit, where we have

$$\begin{aligned}
\Psi(y) &\sim e^{ikL} \left(e^{i\frac{\pi}{2}\frac{2}{L\lambda}(h-y)^2} + e^{i\frac{\pi}{2}\frac{2}{L\lambda}(h+y)^2} \right) \\
&= e^{ikL} \left(e^{i\frac{k}{2L}(h-y)^2} + e^{i\frac{k}{2L}(h+y)^2} \right) \\
&= e^{ikL\left(1+\frac{1}{2L^2}(h-y)^2\right)} + e^{ikL\left(1+\frac{1}{2L^2}(h+y)^2\right)} \\
&\sim e^{ikL\sqrt{1+\frac{1}{L^2}(h-y)^2}} + e^{ikL\sqrt{1+\frac{1}{L^2}(h+y)^2}} \\
&= e^{ik\sqrt{L^2+(h-y)^2}} + e^{ik\sqrt{L^2+(h+y)^2}}.
\end{aligned} \tag{4.208}$$

In both the small Δx limit and in terms of the Fresnel sines and cosines we clearly have total constructive interference at the $y = 0$ point where both path length contributions are equal. Can we do a first order expansion of the Fresnel sines and cosines to look at how a finite Δx changes things?

Let's not try that for now. Instead, a more reasonable approach is probably to attempt using the Fraunhofer approximation instead.

Exercise 4.7 Wave functions for Lloyd's mirror configuration.

If the source is spatially spread, how far apart does it have to be for a one half reduction in the fringe visibility?

Answer for Exercise 4.7

This is the precise statement of the problem on the midterm. Let's attempt it using the Fraunhofer diffraction approximation, with coordinates as in

FIXME: F5

We write

$$\mathbf{R} + \mathbf{r}' = \mathbf{r}. \tag{4.209}$$

or

$$\mathbf{R} = \mathbf{r} - \mathbf{r}'. \tag{4.210}$$

with scalar magnitude

$$\begin{aligned} R &= r\left(1 + \frac{r'^2}{r^2} - \frac{2}{r^2}\mathbf{r}\cdot\mathbf{r}'\right)^{1/2} \\ &\sim r + \frac{r'^2}{2r} - \frac{1}{r}\mathbf{r}\cdot\mathbf{r}' \\ &\sim r - \frac{1}{f}(0, f\theta, f)\cdot(0, y', 0) \\ &= r - \theta y'. \end{aligned} \tag{4.211}$$

We can now write the diffraction integral

$$\begin{aligned} \Psi(0, \theta f, f) &\sim \Psi_s \frac{e^{ikf}}{f}\int_A e^{-ik\theta y'}\,dy' \\ &= \Psi_s \frac{e^{ikf}}{f}\int_A e^{-ik_y y'}\,dy'. \end{aligned} \tag{4.212}$$

Here we write $\mathbf{k} = k(\cos\theta, \sin\theta, 1) = (k_x, k_y, k_z)$, after making the small angle approximation $\sin\theta \sim \theta$. We integrate over the ranges $A_+ = [h - \Delta y/2, h + \Delta y/2]$, and $A_- = [-h - \Delta y/2, -h + \Delta y/2]$.

For A_+ we have

$$\begin{aligned} \int e^{-ik_y y'}\,dy' &= \left.\frac{e^{-ik_y y'}}{-ik_y}\right|_{h-\Delta y/2}^{h+\Delta y/2} \\ &= \frac{1}{ik_y}\left(-e^{-ik_y(h+\Delta y/2)} + e^{-ik_y(h-\Delta y/2)}\right) \\ &= \frac{2}{k_y}e^{-kk_y h}\sin(k_y\Delta y/2), \end{aligned} \tag{4.213}$$

For A_- we just flip the sign on h. Adding the two we have

$$\Psi = 2\Psi_s \frac{e^{ikf}}{f}\cos(k_y h)\frac{\sin(k_y\Delta y/2)}{k_y/2}. \tag{4.214}$$

Compare this to our point source treatment, which is

$$\Psi = \Psi_s \frac{e^{ikf}}{f}\left(e^{-ik\theta h} + e^{ik\theta h}\right) = 2\Psi_s \frac{e^{ikf}}{f}\cos(k_y h). \tag{4.215}$$

In particular we note that the intensities of the point and distributed sources in this Lloyd's mirror configuration are respectively

$$I_{\text{point source}} \sim \cos^2(k_y h). \tag{4.216a}$$

$$I_{\text{source distributed over width } \Delta y} \sim \cos^2(k_y h) \frac{\sin^2(k_y \Delta y/2)}{(k_y/2)^2}. \tag{4.216b}$$

We see that increasing Δy will continually decrease the amplitude of the intensity until $k_y \Delta y/2 = \pi/2$. For a 50 % decrease in intensity we want

$$\sin^2\left(k_y \Delta y/2\right) = \frac{1}{2}. \tag{4.217}$$

or

$$k_y \frac{\Delta y}{2} = \frac{\pi}{4}. \tag{4.218}$$

or

$$\Delta y = \frac{\pi \lambda}{2(2\pi)\sin\theta} = \frac{\lambda}{4\sin\theta}. \tag{4.219}$$

For $\theta = \pi/2$, we see that a source spread as small as $\lambda/4$ will decrease the intensity by 50%.

Exercise 4.8 Attempt this again using the mutual coherence.

Published midterm solution uses the results from the notes for mutual coherence γ_{12} due to a distributed source. This looks like how we should have attempted this. Try that way (esp. now that the theorem in question is now understood.)

Answer for Exercise 4.8

It's not immediately clear to me how to apply the Van Cittert-Zernike theorem to the Lloyd's mirror configuration. What two points are of interest? We have intensity at any single point? Do we look at a point on the maximum of a fringe and look at separation from that? Some review finds the answer back in lecture 11, where it was pointed out that we can consider the Lloyd's mirror

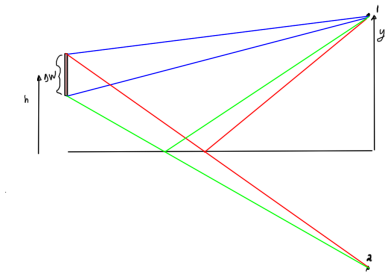

Figure 4.68: Spatial interferometry with Lloyd's mirror.

as the superposition of a real and a virtual observation point as in fig. 4.68. I'd been thinking above of only virtual sources, not virtual observation points. With a virtual observation point in a Lloyd's mirror configuration, we can treat this as if we are looking at the sum of intensities resulting from addition of the wave functions at the points $\pm(y+h)$. This is because the path length at the virtual observation point will be the same of the bounced ray that ends up at the detector (ignoring any phase change that occurs with reflection). Let's setup coordinates as in fig. 4.69. We have

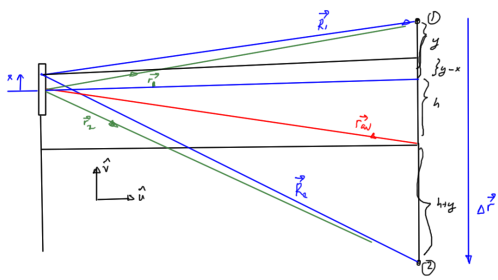

Figure 4.69: Coordinates for Lloyd's mirror spatially distributed interferometry problem.

$$r_1^2 = L^2 + y^2. \tag{4.220a}$$

$$r_2^2 = L^2 + (2h+y)^2. \tag{4.220b}$$

$$R_1^2 = L^2 + (y-x)^2. \tag{4.220c}$$

$$R_2^2 = L^2 + (2h + 2y - x)^2. \tag{4.220d}$$

$$\mathbf{r}_{\text{av}} = L\hat{\mathbf{u}} - h\hat{\mathbf{v}}. \tag{4.220e}$$

$$\Delta\mathbf{r} = -\hat{\mathbf{v}}(2h + 2y). \tag{4.220f}$$

$$\mathbf{r}_s = x\hat{\mathbf{v}}. \tag{4.220g}$$

This gives us

$$\Delta\mathbf{r} \cdot \hat{\mathbf{r}}_{\text{av}} = \frac{2h(h+y)}{\sqrt{h^2 + L^2}}. \tag{4.221a}$$

$$\frac{\Delta\mathbf{r} \cdot \mathbf{r}_s}{r_{\text{av}}} = -\frac{2x(h+y)}{\sqrt{h^2 + L^2}}. \tag{4.221b}$$

For $L \gg h$ as in fig. 4.70, we can write

$$\frac{h+y}{\sqrt{h^2 + L^2}} \approx \frac{h+y}{L} = \tan\theta \sim \theta. \tag{4.222}$$

So that our plug into eq. (4.111) takes the form

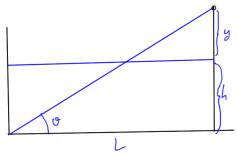

Figure 4.70: Angle from mirror to observation point.

$$\begin{aligned}
\Gamma_{12} &= \frac{e^{2ihk\theta}}{\lambda^2 \overline{R_1}\,\overline{R_2}} \int_{-\Delta w/2}^{\Delta w/2} dx I(x) e^{-2ik\theta x} \\
&\sim \frac{e^{2ihk\theta}}{\lambda^2 L^2} I_0 \left. \frac{e^{-2ik\theta x}}{2ik\theta} \right|_{\Delta w/2}^{-\Delta w/2} \\
&= \frac{e^{2ihk\theta}}{\lambda^2 L^2} I_0 \frac{\sin(k\theta\Delta w)}{k\theta} \\
&= \frac{I_0 \Delta w e^{2ihk\theta}}{\lambda^2 L^2} \operatorname{sinc}(k\theta\Delta w).
\end{aligned} \tag{4.223}$$

Observing that sinc $\to 1$, as $\Delta w \to 0$ we can normalize this as

$$\gamma_{12} = \operatorname{sinc}(k\theta\Delta w). \tag{4.224}$$

We are interested in the visibility

$$\mathcal{V} = |\gamma_{12}|, \tag{4.225}$$

as plotted in fig. 4.71 for $x = k\theta\Delta w$. At what point on that first

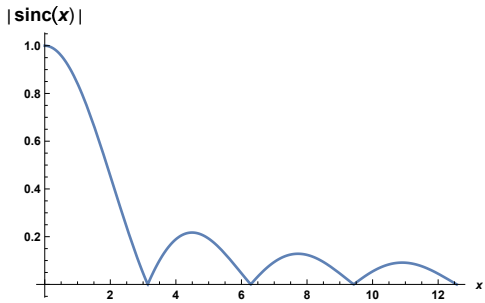

Figure 4.71: Visibility curve for Lloyd's mirror and spatially distributed source.

lobe does the visibility drop to 1/2? With small enough x, where $x \ll \pi/2$ we have

$$|\operatorname{sinc}(x)| \approx \frac{x - \frac{x^3}{3!}}{x} = 1 - \frac{x^2}{6}, \tag{4.226}$$

so we are looking for the value of Δw that satisfies

$$1 - \frac{1}{6}(k\theta\Delta w) = \frac{1}{2}, \tag{4.227}$$

or

$$\boxed{\Delta w = \frac{\sqrt{3}}{k\theta}.} \tag{4.228}$$

5 MULTIPLE INTERFERENCE.

5.1 MULTIPLE INTERFERENCE.

What if

$$\Psi = \Psi_1 + \Psi_2 + \Psi_3 + \Psi_4 + \cdots . \tag{5.1}$$

then

$$\begin{aligned} I &= \left\langle |\Psi|^2 \right\rangle \\ &= \left\langle \left(\sum_i \Psi_i^* \right) \left(\sum_j \Psi_j \right) \right\rangle \\ &= \sum_{i,j} \left\langle \Psi_i^* \Psi_j \right\rangle \\ &= \sum_i \left\langle \Psi_i^* \Psi_i \right\rangle + \sum_{i>j} \left\langle \Psi_i^* \Psi_j + \Psi_j^* \Psi_i \right\rangle \\ &= \text{incoherent sum} + \text{ interference term.} \end{aligned} \tag{5.2}$$

$$\text{incoherent sum} = \sum_i I_i . \tag{5.3a}$$

$$\begin{aligned} \text{interference sum} &= 2\,\mathrm{Re}\left(\sum_{i>j} \left\langle \Psi_i^* \Psi_j \right\rangle \right) \\ &= 2\,\mathrm{Re}\left(\sum_{i>j} \Gamma_{ij} \right) . \end{aligned} \tag{5.3b}$$

We recognize our mutual coherence in the interference term.

Now consider a partially silvered mirror configuration with two mirrors as in fig. 5.1. We are going to ignore the thickness and assume the reflection and transmission coefficients are the same for both surfaces and assume that we have no absorption and

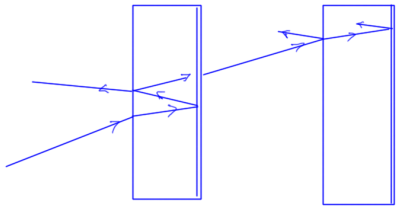

Figure 5.1: Two partially silvered mirror configuration with thickness.

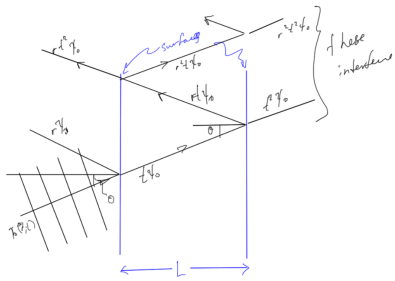

Figure 5.2: Ignoring thickness.

only treat the $n = 1$ everywhere case for now. We'll look at the interference of all the internal reflections on eventual exit from the pair of mirrors as shown in fig. 5.2. We'll want to remember the phase. What is the phase delay between each interfering wave?

We'll find that we get an extra path length of

$$2kL\cos\theta. \tag{5.4}$$

The geometry to consider is fig. 5.3. We see that we have

$$s_1 \cos\theta = L. \tag{5.5a}$$

$$\frac{\Delta y}{2} = s_1 \sin\theta. \tag{5.5b}$$

$$\mathbf{k} = k(\cos\theta, \sin\theta, 0), \tag{5.5c}$$

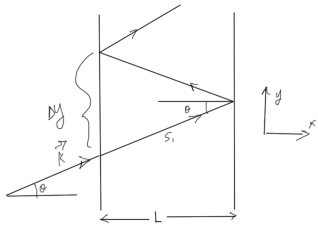

Figure 5.3: Just the geometry of the problem.

So that our phase change to the point A where we have the second internal reflection

$$
\begin{aligned}
\mathbf{k}\cdot\mathbf{x} - \omega t &= k(\cos\theta, \sin\theta, 0)\cdot(0, \Delta y, 0) - \omega\frac{2s_1}{c} \\
&= k\sin\theta\Delta y - \cancel{c}k\frac{2s_1}{\cancel{c}} \\
&= k(\sin\theta\Delta y - 2s_1) \\
&= k(2s_1\sin^2\theta - 2s_1) \\
&= 2ks_1(\sin^2\theta - 1) \\
&= -2ks_1\cos^2\theta \\
&= -2kL\cos\theta.
\end{aligned}
\tag{5.6}
$$

Observe that $\mathbf{k}\cdot\Delta\mathbf{x} - \omega\Delta t$ must be negative for any non-straight line path (in which case it will be zero) between endpoints, provided the media through which the rays travel is of constant index of refraction. In class (and the class notes) there was no such negative sign, and we just considered the absolute difference in phase. This can also be calculated by considering just the contributing portions of the path that lead to interference. Fowles' fig 4.2 marks those as AB, BC, or $s_1\cos(2\theta) + s_1$ in the figure above. The idea is that we consider the second reflection as a generator of plane waves, and those will only start interfering with plane waves at the first transmission (in steady state), after those first transmission waves have traversed that little leg of the path $s_1 - s_1\cos(2\theta)$. This was illustrated in office hours as in fig. 5.4. Of this Prof Thywissen also says: We normally discuss paths as a "time delay". Longer paths have longer delays. Since in the convention that you and I are using, time enters as $e^{-i\omega t}$, this means that multiplying

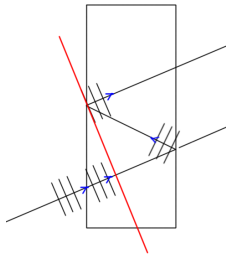

Figure 5.4: Internal interference regions of the path.

by $e^{i\text{phase}}$ gives you an effective time delay of $t = -\text{phase}/\omega$. So, this brings us back to the conclusion that positive path-length giving a negative phase in the exponent is self-consistent.

Anyways, moving on, we get to the point where the wavefunction for transmission is

$$\Psi_{\text{transmission}} = \Psi_0 t^2 + \Psi_0 t^2 \left(r^2 e^{i\delta}\right) + \Psi_0 t^2 \left(r^2 e^{i\delta}\right)^2. \tag{5.7}$$

We lookup (in a spec sheet) the transmission and reflection coefficients and the (associated phase shifts after reflection)

$$r = e^{i\delta_r}\sqrt{R}. \tag{5.8a}$$

$$t = e^{i\delta_t}\sqrt{T}. \tag{5.8b}$$

and get

$$\begin{aligned}\Psi_{\text{transmission}} &= \Psi_0 T e^{2i\delta_t} + \Psi_0 T e^{2i\delta_t} R e^{2i\delta_r + i\delta} + \Psi_0 T e^{2i\delta_t}\left(R e^{2i\delta_r + i\delta}\right)^2 + \cdots \\ &= \Psi_0 t^2 \sum_{n=0}^{\infty}\left(R e^{2i\delta_r + i\delta}\right)^n \\ &= \Psi_0 t^2 \frac{1}{1 - R e^{i\Delta}}.\end{aligned} \tag{5.9}$$

where we've used (for $|a| < 1$)

$$\sum_{n=0}^{\infty} a^n = \frac{1}{1-a}. \tag{5.10}$$

and written

$$\Delta = 2\delta_r + \delta. \tag{5.11}$$

Our measured intensity is

$$\begin{aligned}
I_{\text{trans}} &= \left\langle |\psi_{\text{trans}}|^2 \right\rangle \\
&= I_0 \frac{T^2}{|1 - Re^{i\Delta}|^2} \\
&= I_0 \frac{T^2}{(1 - Re^{i\Delta})(1 - Re^{-i\Delta})} \\
&= I_0 \frac{T^2}{1 + R^2 - 2R\cos\Delta} \\
&= I_0 \frac{T^2}{1 + R^2 - 2R(1 - 2\sin^2(\Delta/2))} \\
&= I_0 \frac{T^2}{(1 - R)^2 + 4R\sin^2(\Delta/2)} \\
&= I_0 \frac{T^2/(1 - R)^2}{1 + (4R/(1 - R)^2)\sin^2(\Delta/2)}.
\end{aligned} \tag{5.12}$$

or

$$I_{\text{trans}} = \frac{I_{\text{max}}}{1 + F\sin^2(\Delta/2)}. \tag{5.13a}$$

$$I_{\text{max}} = \frac{I_0 T^2}{(1 - R)^2}. \tag{5.13b}$$

$$F = \frac{4R}{(1 - R)^2}. \tag{5.13c}$$

This is called Etalon transmission. Plots of I/I_{max} vs. phase shifts for $R \in \{0.1, 0.3, 0.6, 0.97\}$ can be found in fig. 5.5.

5.2 FABRY-PEROT INTERFEROMETRY.

We've got Etalons in real world situations such as light off a CD fig. 5.6. We'd previously considered wavefront splitting fig. 5.7,

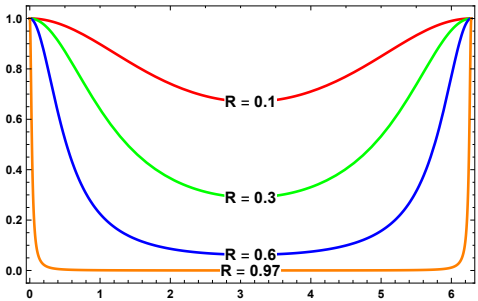

Figure 5.5: Etalon transmission.

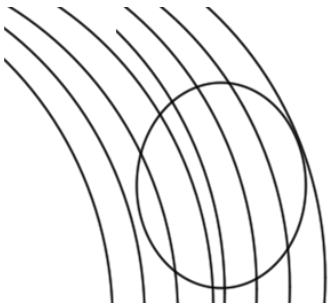

Figure 5.6: Laser on CD.

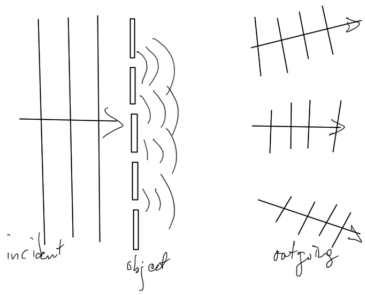

Figure 5.7: Wavefront splitting.

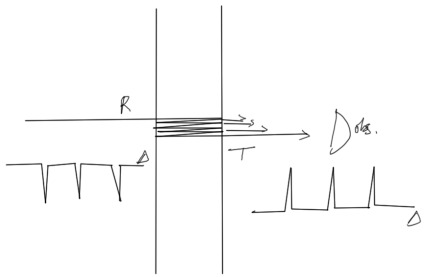

Figure 5.8: Amplitude splitting.

but now wish to consider amplitude splitting fig. 5.8. Last time we found

$$I_t = \frac{I_{\max}}{1 + F \sin^2 \Delta/2}. \tag{5.14a}$$

$$F = \frac{4R}{(1-R)^2}. \tag{5.14b}$$

$$\Delta = 2\delta_r + \delta. \tag{5.14c}$$

$$\delta = 2Lk\cos\theta. \tag{5.14d}$$

We've got sharp peaks at $\Delta = 2\pi m$

How good is an Etalon at resolving frequency?

Suppose we've shined in two beams of the same frequency, and then slowly start changing the frequency of the other beam, until we get to the point where we've got both peaks centered at $2\pi m/\omega_k$ as in fig. 5.9. re-label with

$$2k_1 L = \frac{2\omega_1 L}{c}. \tag{5.15}$$

Relabelling fig. 5.10. We'll consider this "resolved" when the second peak is centered at the point when our first peak has lost half of its intensity as in fig. 5.11. In mathese, this resolution is

$$\frac{I}{I_{\max}} = \frac{1}{2}. \tag{5.16}$$

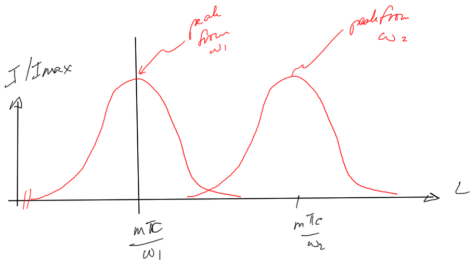

Figure 5.9: Intensity from multiple Etalons.

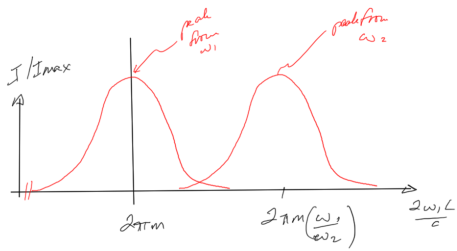

Figure 5.10: Intensity from multiple Etalons, relabeled.

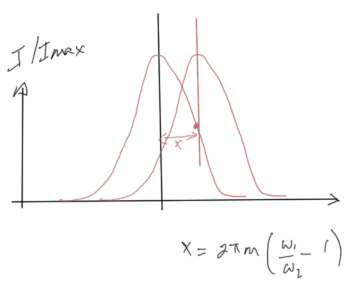

Figure 5.11: Two peaks resolved.

at the peak for ω_2. That is

$$1 + F \sin^2 \frac{\Delta_1}{2} = 2. \tag{5.17}$$

$$\sin \frac{\Delta_1}{2} = \frac{1}{\sqrt{F}}. \tag{5.18}$$

$$x = \frac{2}{\sqrt{F}}. \tag{5.19}$$

$$\Delta_1 = 2\pi m + x. \tag{5.20}$$

Define, the Finesse, as

$$\mathcal{F} = \pi \frac{\sqrt{R}}{1-R} = \frac{\pi}{2}\sqrt{F} \sim \frac{\pi}{T}. \tag{5.21}$$

$$\omega_1 - \omega_2 = \frac{2c}{L\sqrt{F}} = \frac{\pi c}{L\mathcal{F}}. \tag{5.22}$$

$$\frac{\omega_1 - \omega_2}{\overline{\omega}} = \frac{1}{\mathcal{F} m}. \tag{5.23}$$

Roughly speaking $\mathcal{F}$ is an instruction to "buy good mirrors", whereas m means "use a long cavity"

How many reflections?

$$N \sim \frac{1}{T} \sim \mathcal{F}. \tag{5.24}$$

N-wave interference Cavity length is important. Suppose we had fig. 5.12 which gives

$$\Delta = \text{offset} + 2\frac{\overline{\omega}}{c} L = 2\pi(m + j). \tag{5.25}$$

Neglecting the offset so that

$$2\frac{\overline{\omega}}{c} L = 2\pi(m + j). \tag{5.26}$$

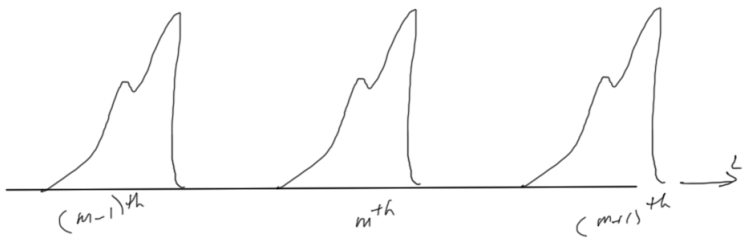

Figure 5.12: Many Etalons.

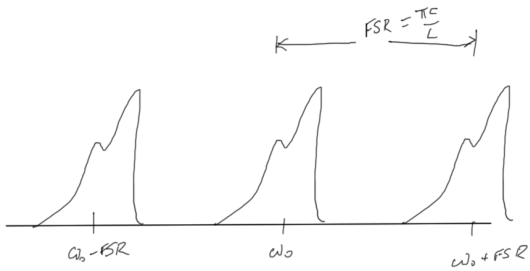

Figure 5.13: Illustrating Free Spectral Resolution.

or

$$\varpi = \frac{\pi c}{L}(m+j) = \omega_0 + j\text{F S R}. \tag{5.27}$$

where F S R is the Free Spectral Range. Re-plotting in fig. 5.13. This is called a Fabry-Perot Spectrometer. These guys, who were first able to achieve a good spectrometer of this sort, achieved $\mathcal{F} \sim 30 - 100$, using $\lambda/100$ flatness, where a typical mirror has $\lambda/10$ flatness!

Reading : §4 of [5].

Additional discussion from last class.

$$\Delta = \frac{2L}{c}\omega + 2\delta_r = 2\pi m. \tag{5.28}$$

Here Δ is the round trip phase. Our resonances are

$$\omega = \frac{c}{2L}2\pi m - \frac{c}{2L}2\delta_r = \text{FSR}m + \text{offset}. \tag{5.29}$$

If FSR = $\pi c/L$.

5.3 FABRY-PEROT ETALON REVIEW.

We've been discussing a Fabry-Perot Etalon as in fig. 5.14. with in-

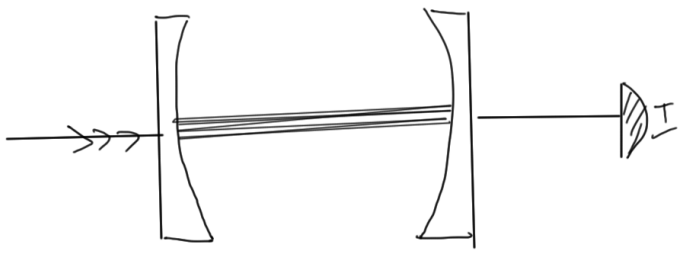

Figure 5.14: Fabry-Perot Etalon.

put that is of a discrete frequency (a spectral line). We'll get something like fig. 5.15. something that is an idealization of fig. 5.16.

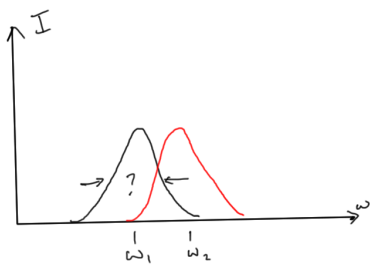

Figure 5.15: Etalon response by frequency.

Since we have widening due to smaller than ideal reflectivity. It's not clear to me what the measurement mechanism here is. We are plotting something against frequency, but only sending in discrete frequencies.

What would make sense to me is to consider the angular dependence of Δ for two sets of frequencies. Plotting that for a narrow range of angles for 500 nm and 650 nm light, $L = 1\,\mathrm{m}$, and $R = 0.97$ we have fig. 5.17, where the first and third peaks are for 500 nm, and the second and fourth peaks for 650 nm. So, if we have a source at a distance, we can expect a different intensity

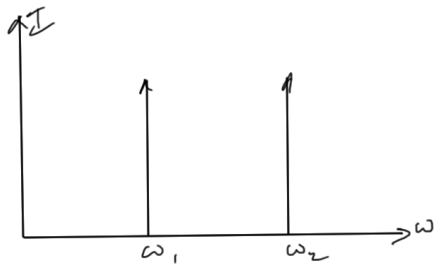

Figure 5.16: Ideal Etalon response.

result at the output for different frequencies, depending on the angle between the source and the device. This I can picture as an experimental setup.

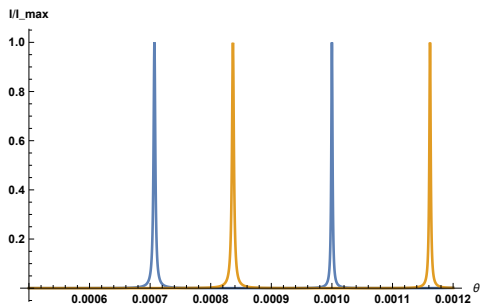

Figure 5.17: Etalon angular dependencies.

Returning to the notes. We found two frequencies $\overline{\omega} \pm \Delta\omega/2$ are resolved when

$$\frac{\Delta\omega}{\overline{\omega}} = \frac{1}{m\mathcal{F}}. \tag{5.30}$$

Here m = order of interference so

$$\Delta = m2\pi. \tag{5.31}$$

$$\mathcal{F} = \pi\frac{\sqrt{R}}{1-R} = \text{Finess of Etalon.} \tag{5.32}$$

Example numbers

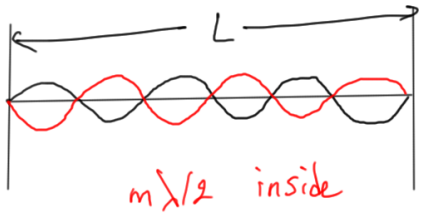

Figure 5.18: Wavelength packing in a cavity.

$$L = 10\text{cm} \rightarrow \text{RST} = 10^{10}\text{s}^{-1}\text{or}1.5\text{GHz}. \tag{5.33a}$$

$$R = 97\% \rightarrow F = 100 \tag{5.33b}$$

$$\bar{\lambda} \sim 0.6\mu m \rightarrow \text{visible light}. \tag{5.33c}$$

$$\bar{\omega} = 3 \times 10^{15}\text{s}^{-1}. \tag{5.33d}$$

$$\bar{\nu} = 500\text{THz}. \tag{5.33e}$$

$$\frac{\Delta\omega}{\bar{\omega}} = 3 \times 10^{-8}. \tag{5.33f}$$

$$m = \frac{\bar{\omega}}{\text{FSR}} = \frac{3 \times 10^{15} s^{-1}}{10^{1}0 s^{-1}} = 3 \times 10^{5}. \tag{5.33g}$$

$\Delta\omega$ is the smallest separation of two frequencies that we can measure.

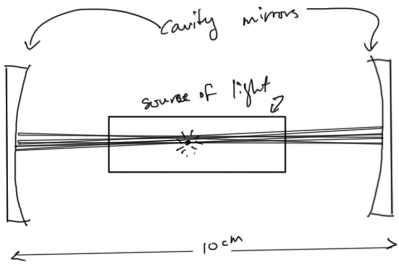

Figure 5.19: Cavity as an oscillator.

5.4 CAVITY (OR ETALON) (FABRY-PEROT) AS AN OSCILLATOR.

Why are we talking so much about a specific interferometer, when this is a class on Advanced Classical Optics. It turns out that the interaction with light in a cavity, as in a large setup fig. 5.19. is basically the same idea as in an implementation of a laser fig. 5.20. If we are saying that something is an oscillator, then we can ask a

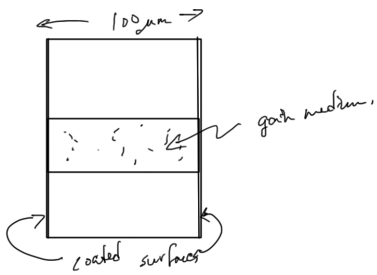

Figure 5.20: Semiconductor cavity.

couple questions:

- What is the resonant frequency?
- What is the alignment?

The resonant frequency occurs every time that we can get an integer number of half wavelengths in the cavity.

We could actually ask what are the resonant frequencies, since we could have a "comb" of resonances fig. 5.21. (transmission of the Etalon: see slides)

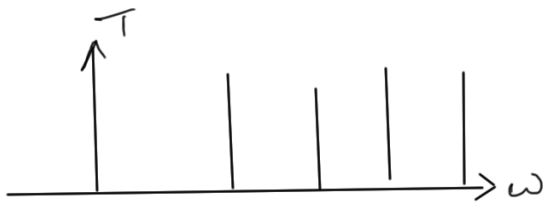

Figure 5.21: Frequency comb.

Answering our question of what are the resonant frequencies, our answer is

$$\omega_m = (\text{offset}) + \text{FSR}m. \tag{5.34}$$

where m is an integer. For the question of line width, consider a Lorenztian

$$\Gamma = \frac{\text{FSR}}{2\mathcal{F}}. \tag{5.35}$$

as in fig. 5.22. We've got

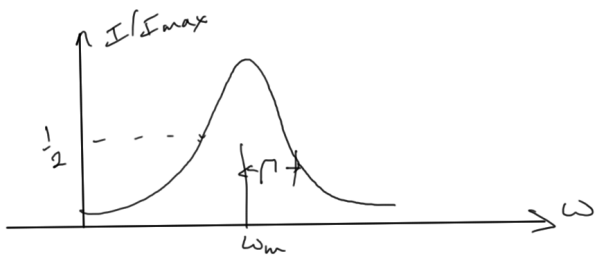

Figure 5.22: Lorentzian.

$$\frac{I}{I_0} = \frac{1}{1 + \frac{4\mathcal{F}^2}{\pi^2}\sin^2(\Delta/2)}. \tag{5.36}$$

Consider the plot of $\sin^2(\Delta/2)$ as in fig. 5.23. Our phase offset from the resonance is

$$\Delta = 2\pi m + \eta. \tag{5.37}$$

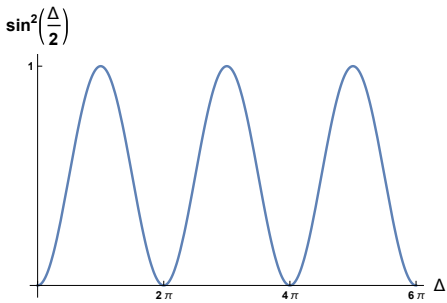

Figure 5.23: Squared sine plot.

$$\eta \ll 1. \tag{5.38}$$

In terms of $\delta = \omega - \omega_m$

$$\eta = \frac{2L}{c}\delta. \tag{5.39}$$

(because $\Delta = \frac{2L}{c}\omega + \text{offset}$), we are left with

$$\frac{I}{I_0} = \frac{1}{1 + \frac{4\mathcal{F}^2}{\pi^2}\left(\frac{\eta}{2}\right)^2} = \frac{1}{1 + \left(\frac{2L\mathcal{F}\delta}{\pi c}\right)^2}. \tag{5.40}$$

or

$$\boxed{\frac{I}{I_0} = \frac{1}{1 + \frac{\delta^2}{\Gamma^2}},} \tag{5.41}$$

if

$$\Gamma = \frac{\pi c}{2L\mathcal{F}} = \frac{\text{FSR}}{2\mathcal{F}}. \tag{5.42}$$

What's the meaning of all of this? It means that the Fabry-Perot oscillator is a device that traps light, and the resonance looks like a Lorentzian.

We need a very high Finess (high reflectivity) to get a good Lorentzian.

Recall that the Lorentzian is a Fourier transform of a damped exponential time domain signal fig. 5.24. Light is trapped in the cavity for a time $\tau \sim 1/\Gamma$.

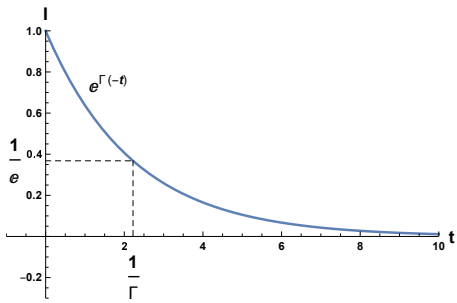

Figure 5.24: Exponential decay.

5.5 DIFFRACTION GRATING INTERFEROMETRY.

We are going to look at the (Fraunhofer) far field of a diffraction grating with N illuminated slits fig. 5.25. Our geometry is

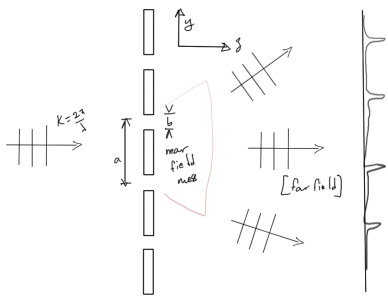

Figure 5.25: Diffraction grating interferometry.

$$\mathbf{R} + \mathbf{r}' = \mathbf{r}. \tag{5.43}$$

$$\mathbf{R}_s + \mathbf{r}' = \mathbf{r}_s. \tag{5.44}$$

for which our path length from $\mathbf{r}'$ to the observation point is

$$|\mathbf{R}| = r\left(1 + \frac{r'^2}{r^2} - 2\frac{\mathbf{r}\cdot\mathbf{r}'}{r^2}\right)^{1/2} \sim r + \frac{r'^2}{2r^2} - \hat{\mathbf{r}}\cdot\mathbf{r}', \tag{5.45}$$

and to first order

$$k|\mathbf{R}| \sim kr - k\hat{\mathbf{r}}\cdot\mathbf{r}' = kr + \mathbf{k}\cdot\mathbf{r}' = kr + ky'\sin\theta. \tag{5.46}$$

Similarly

$$\begin{aligned} k|\mathbf{R}_s| &\sim kr_s - k\hat{\mathbf{r}}\cdot\mathbf{r}' \\ &= kr_s - \mathbf{k}\cdot\mathbf{r}' \\ &= kr_s - k\sin\theta_s y'. \end{aligned} \tag{5.47}$$

With

$$k_y \equiv \frac{2\pi}{\lambda}\sin\theta, \tag{5.48}$$

and

$$k_{y,s} \equiv \frac{2\pi}{\lambda}\sin\theta_s, \tag{5.49}$$

our diffraction integral, in one dimension takes the form

$$\Psi(\mathbf{r}) \sim \int e^{i(k_{y,s}-k_y)y'}dy'. \tag{5.50}$$

or

$$\Psi(\mathbf{r}) \sim \int e^{i\frac{2\pi}{\lambda}(\sin\theta_s-\sin\theta)y'}dy'. \tag{5.51}$$

We'll work with $\theta_s = 0$, normally incident plane waves, for which the diffraction integral reduces to just

$$\Psi(\mathbf{r}) \sim \int e^{-ik_y y'}dy'. \tag{5.52}$$

This is the Fourier transform of the aperture, say $g(y)$ evaluated at $k_y = (2\pi/\lambda)\sin\theta$. Note that $k_y \neq \mathbf{k}\cdot\hat{\mathbf{r}}'$, it is not the projection in the $\hat{\mathbf{y}}$ direction, which is $k\sin\theta_s$. The angle θ here is the angle to the observation point.

We will use the convolution theorem 3.3, constructing the complete aperture as a convolution, with transmission fig. 5.26.

$$f(y) = \int dy' g(y')h(y-y') = g(y)*h(y). \tag{5.53}$$

Consider the convolution with the delta function comb

$$h(y) = \sum_{n=0}^{N-1}\delta(y-na). \tag{5.54}$$

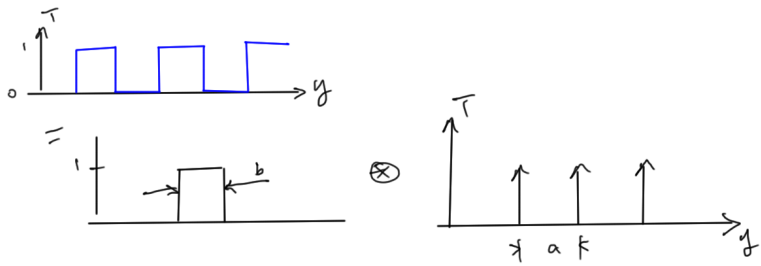

Figure 5.26: Convolution of box with comb.

So that the convolution is

$$\begin{aligned} f(y) &= \int dy' g(y') \sum_{n=0}^{N-1} \delta(y - y' - na) \\ &= \sum_{n=0}^{N-1} \int dy' g(y') \delta(y - y' - na) \\ &= \sum_{n=0}^{N-1} g(y - na). \end{aligned} \tag{5.55}$$

Convolution theorem

$$F(k_y) = H(k_y) G(k_y). \tag{5.56}$$

As mentioned, here we write

$$k_y = \frac{2\pi}{\lambda} \sin\theta. \tag{5.57}$$

(to distinguish from our normal writing of $k = 2\pi/\lambda$)

For a single slit, in the Fraunhofer limit, we compute

$$\begin{aligned} G(k_y) &= \int_{-b/2}^{b/2} e^{-ik_y y} dy \\ &= \left. \frac{e^{-ik_y y}}{-ik_y} \right|_{-b/2}^{b/2} \\ &= \frac{e^{ik_y b/2} - e^{-ik_y b/2}}{2(ik_y)/2} \\ &= \frac{\sin(bk_y/2)}{bk_y/2}. \end{aligned} \tag{5.58}$$

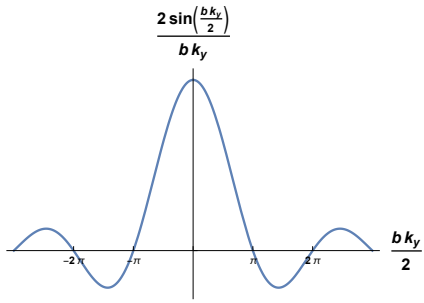

Figure 5.27: Zero of diffraction wavefunction.

as illustrated in fig. 5.27. For the Fourier transform of the delta comb, we have

$$\begin{aligned} H(k_y) &= \int_{-\infty}^{\infty} e^{ik_y y} \sum_{n=0}^{N-1} \delta(y - na) \\ &= \sum_{n=0}^{N-1} \int_{-\infty}^{\infty} e^{ik_y y} \delta(y - na) \\ &= \sum_{n=0}^{N-1} e^{ik_y na} \\ &= \sum_{n=0}^{N-1} \left(e^{ik_y a}\right)^n . \end{aligned} \tag{5.59}$$

Recall that we can sum a finite geometric series, by taking the difference of

$$aS_N = a + a^2 + \cdots a^N. \tag{5.60a}$$

$$S_N = 1 + a + \cdots a^{N-1}. \tag{5.60b}$$

so that

$$(a - 1)S_N = a^N - 1. \tag{5.61}$$

or

$$S_N = \frac{a^N - 1}{a - 1}, \tag{5.62}$$

and we have for our Fourier transform we have

$$\begin{aligned} H(k_y) &= \frac{1 - e^{ik_y aN}}{1 - e^{ik_y a}} \\ &= \frac{e^{ik_y aN/2}}{e^{ik_y a/2}} \frac{e^{-ik_y aN/2} - e^{ik_y aN/2}}{e^{-ik_y a/2} - e^{ik_y a/2}} \\ &= e^{i\gamma(N-1)} \frac{\sin(N\gamma)}{\sin\gamma}. \end{aligned} \tag{5.63}$$

where

$$\gamma = \frac{1}{2} k_y a = \frac{1}{2} ka \sin\theta. \tag{5.64}$$

We want

$$\begin{aligned} I &= \left|F(k_y)\right|^2 \\ &= \left|H(k_y)\right|^2 \left|G(k_y)\right|^2 \\ &= I_0 \left(\frac{\sin\beta}{\beta}\right)^2 \left(\frac{\sin N\gamma}{N\sin\gamma}\right)^2. \end{aligned} \tag{5.65}$$

where

$$\beta = \frac{1}{2} bk_y = \frac{1}{2} bk \sin\theta. \tag{5.66a}$$

$$\gamma = \frac{1}{2} k_y a = \frac{1}{2} ka \sin\theta. \tag{5.66b}$$

Check: $N = 2$

$$\left(\frac{\sin 2\gamma}{2\sin\gamma}\right)^2 = \left(\frac{2\sin\gamma\cos\gamma}{2\sin\gamma}\right)^2 = \cos^2\gamma. \tag{5.67}$$

(Good).

To get a bit of a feeling for what this looks like we can check out a plot, as in fig. 5.28, plotting $n = 5, a = 0.869, b = 0.172, \lambda = 1$ (generated from lecture15figures.nb .) Our far field view was fig. 5.29.

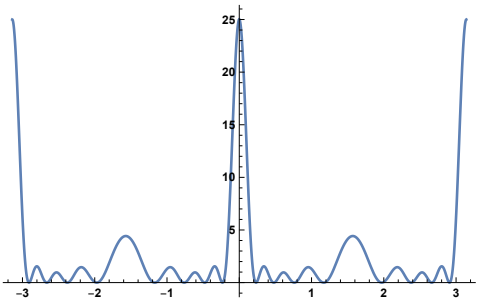

Figure 5.28: A sample intensity pattern for a multiple aperture diffraction grating.

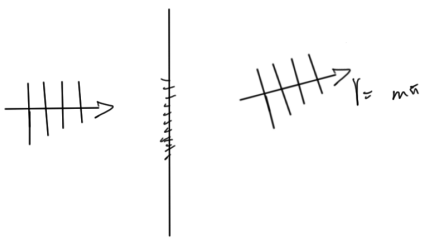

Figure 5.29: Far field view.

Frequency resolution How well can we resolve frequency?

We will write out γ in terms of angular frequency

$$\gamma = \frac{1}{2} a \frac{\omega}{c} \sin\theta = \pi \frac{\omega}{\omega_0} \sin\theta. \tag{5.68}$$

where

$$\omega_0 = \frac{2\pi c}{a}, \tag{5.69}$$

where a is the period of the diffraction grating. The peak is at $\gamma = m\pi$, or

$$\gamma_{\text{peak}} = m\pi = \pi \frac{\omega}{\omega_0} \sin\theta, \tag{5.70}$$

or

$$m = \frac{\omega}{\omega_0} \sin\theta, \tag{5.71}$$

Zeros are at $N\gamma = l\pi$ (provided $l \neq Nm$). Closest zero is

$$l = Nm + 1. \tag{5.72}$$

$$\begin{aligned} \Delta\gamma &= \gamma_{\text{zero}} - \gamma_{\text{peak}} \\ &= \frac{l}{N}\pi - m\pi \\ &= \frac{Nm+1}{N}\pi - m\pi. \end{aligned} \tag{5.73}$$

or

$$\Delta\gamma = \frac{\pi}{N} = \frac{\pi \sin\theta}{\omega_0} \Delta\omega. \tag{5.74}$$

Rework after question: Peaks for $\omega_{1,2}$ are

$$\gamma_{\text{peak},1} = \pi \frac{\omega_1}{\omega_0} \sin\theta = m\pi. \tag{5.75}$$

$$\gamma_{\text{peak},2} = \pi \frac{\omega_2}{\omega_0} \sin\theta = m\pi. \tag{5.76}$$

so that

$$\Delta\gamma = \Delta\omega \frac{\pi}{\omega_0} \sin\theta. \tag{5.77}$$

The peak is resolved when

$$\Delta\gamma = \frac{m\pi}{\omega_1}\Delta\omega = \frac{\pi}{N}, \tag{5.78}$$

as shown in fig. 5.30. Writing $\omega_1 \sim \omega_2 \sim \overline{\omega}$, this is

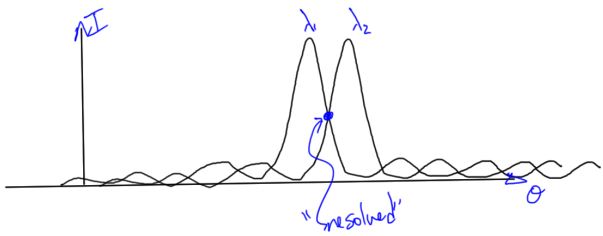

Figure 5.30: Resolution.

$$\boxed{\frac{\Delta\omega}{\overline{\omega}} = \frac{1}{Nm}} \tag{5.79}$$

Observe that this looks like

$$\frac{\Delta\omega}{\overline{\omega}} = \frac{1}{m\mathcal{F}}, \tag{5.80}$$

because $\mathcal{F} \sim$ number of bounces in the Etalon.

But $m \approx 1, 2, 3$, not $m \gg 1$.

5.6 PROBLEMS.

Exercise 5.1 Inside the Fabry Perot. (*2012 Ps3, P2*)

What does the energy density $u(x) \equiv \langle |\Psi(x,t)|^2 \rangle$ look like *inside* the Fabry Perot Etalon? Assume a monochromatic traveling wave $\Psi(x,t) = \sqrt{u_0}\exp(ikx - i\omega t)$ is normally incident on a cavity. The mirrors are two surfaces with transmission $t = \sqrt{T}\exp(i\delta_t)$, reflectivity $r = \sqrt{R}\exp(i\delta_r)$, and $T = 1 - R$. The length of the cavity is L. As in class, the round-trip phase shift will be called Δ.

a. **Show that** $u(x)$ in the cavity can be expressed in the form

$$u = u_0 A \left(1 + R + 2\sqrt{R}\cos[\phi(x)]\right), \tag{5.81}$$

and **give** A and $\phi(x)$ in terms of R, Δ, and k. This should explain why we say the mth order resonance is when there are m standing-wave anti-nodes in the cavity.

b. Peak energy density Find an expression the **peak energy density** $u_{\max}$ in the cavity, and **plot** $u_{\max}/u_0$ versus Δ, for $R = 0.8$. What is the **resonance condition**? **By how much** can $u_{\max}$ exceed u_0? Where is this extra energy density coming from if energy is conserved?

c. Standing waves? Is there **always a standing wave** in the cavity, even off resonance?
Write an expression for the **visibility**, (max-min)/(max+min), of the energy density $u(x)$. One way to measure this visibility would be to look at the optical forces on an atom in the cavity.

d. Compare to class Finally, make sure your expression also makes sense when compared to the I_T we found in class. Since it's only the forward-going component that is transmitted through the last mirror, the intensity outside the cavity is $u_0 AT$. (You don't need to prove this.) **Does this reproduce** I_T**?**

Answer for Exercise 5.1

Part a. Average intensity inside the cavity Referring to fig. 5.31 (where the internal reflections are exaggerated), we want to look at the field after the wave gets to points 1, reflects to 2, reflects again to 3 and so forth. Our electric field at point x is then the sum

$$\begin{aligned}\Psi(x) &= (\Psi_0 t)e^{ikx} &&+(\Psi_0 t)re^{ik(2L-x)}\\ &+(\Psi_0 t)r^2 e^{ik(2L+x)} &&+(\Psi_0 t)r^3 e^{ik(4L-x)}\\ &+(\Psi_0 t)r^4 e^{ik(4L+x)} &&+(\Psi_0 t)r^5 e^{ik(6L-x)}\\ &\ldots\end{aligned} \tag{5.82}$$

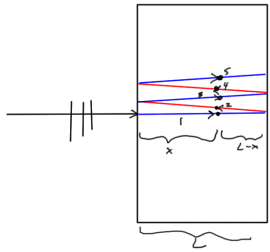

Figure 5.31: Internal Fabry-Perot field geometry.

Factoring out the geometric series we have

$$\begin{aligned}\Psi(x) &= (\Psi_0 t)\left(1 + r^2 e^{2ikL} + r^4 e^{4ikL} + \cdots\right)\left(e^{ikx} + re^{ik(2L-x)}\right) \\ &= (\Psi_0 t)\frac{1}{1 - r^2 e^{2ikL}}\left(e^{ikx} + re^{ik(2L-x)}\right) \\ &= \frac{\Psi_0\sqrt{T}e^{i\delta_t}}{1 - Re^{2ikL+2i\delta_r}}\left(e^{ikx} + \sqrt{R}e^{ik(2L-x)+\delta_r}\right).\end{aligned} \tag{5.83}$$

Introducing a round trip phase

$$\Delta \equiv 2kL + 2\delta_r, \tag{5.84}$$

we have

$$\Psi(x) = \frac{\Psi_0\sqrt{T}e^{i\delta_t}}{1 - Re^{i\Delta}}\left(e^{ikx} + \sqrt{R}e^{-ikx+i\Delta-i\delta_r}\right). \tag{5.85}$$

This has squared magnitude

Wrong?

$$|\Psi(x)|^2 = \frac{u_0 T}{1 + R^2 - 2\cos(\Delta)}\left(1 + R + 2\sqrt{R}\cos\left(2kx - \Delta + \boxed{\delta_r}\right)\right). \tag{5.86}$$

[1]

1 The δ_r above might be wrong, as it was circled by the grader with a question mark. Recalculate.

This has the desired structure of eq. (5.81) with

$$A = \frac{T}{1 + R^2 - 2\cos(\Delta)}. \tag{5.87a}$$

$$\phi(x) = 2kx - \Delta + \delta_r. \tag{5.87b}$$

Part b. Peak energy density From eq. (5.86) we see that maximums and minimums occur respectively whenever

$$2kx - \Delta + \Delta_r = \pi(2m). \tag{5.88a}$$

$$2kx - \Delta + \Delta_r = \pi(2m + 1). \tag{5.88b}$$

At these points $1 + R + 2\sqrt{R}\cos\left(2kx - \Delta + \delta_r\right)$ takes the values

$$1 + R + 2\sqrt{R} = (1 + \sqrt{R})^2. \tag{5.89a}$$

$$1 + R - 2\sqrt{R} = (1 - \sqrt{R})^2. \tag{5.89b}$$

The maximum and minimum energy densities are therefore

$$u_{\max} = \frac{u_0(1 - R)}{1 + R^2 - 2\cos(\Delta)}\left(1 + \sqrt{R}\right)^2. \tag{5.90a}$$

$$u_{\min} = \frac{u_0(1 - R)}{1 + R^2 - 2\cos(\Delta)}\left(1 - \sqrt{R}\right)^2. \tag{5.90b}$$

This maximum is plotted in fig. 5.32.

Grading note: −1 Marked "??". Check against posted solution.

We see the resonance peaks when $1 + R^2 - 2\cos\Delta = 0$. Those points are

$$\Delta_{\text{res}} = \pm\cos^{-1}\left(\frac{1 + R^2}{2}\right). = 4\pi\frac{L}{\lambda_{\text{res}}} + 2\delta_r. \tag{5.91}$$

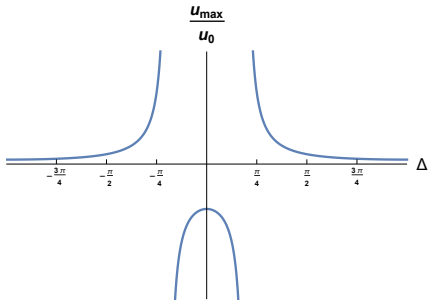

Figure 5.32: peak energy density.

For a given interface phase shift δ_r, and reflectivity R, and cavity width L we see that we have a critical wavelength

$$\lambda_{\text{res}} = \frac{4\pi L}{\pm\cos^{-1}\left(\frac{1+R^2}{2}\right) - 2\delta_r}. \tag{5.92}$$

If large amounts of energy are supplied to the field due to this resonance, I think it would have to come from interactions with the interfaces, thermally cooling the atoms in the mirrors. These thermal effects likely change r as a side effect, producing a feedback effect that would prevent the potential infinite spikes that we see in plot and associated expression of u_{max}.

Grading note: (−2 Check against posted solution.

Part c. Standing waves and visibility From eq. (5.86) we see that our energy density has the form

$$u(x) = \alpha + \beta\cos\left(\frac{4\pi}{\lambda}(x - L) + \delta_r\right). \tag{5.93}$$

We'll have standing waves (possibly phase shifted) only for those input wavelengths that satisfy

$$\frac{4\pi L}{\lambda} = m\pi, \tag{5.94}$$

for integer $m > 0$.

Grading note: −2 Underline portion with question "why?".

From eq. (5.90) we see that our visibility is

$$\mathcal{V} = \frac{1+R+2\sqrt{R}-(1+R-2\sqrt{R})}{1+R+2\sqrt{R}+1+R-2\sqrt{R}} = \frac{2\sqrt{R}}{1+R}. \tag{5.95}$$

Part d. Compare to previously calculated transmitted intensity If we split our wave function into forward $\Psi_+(x)$ and reverse $\Psi_-(x)$ components we have from eq. (5.85)

$$\Psi_+(x) = \frac{\Psi_0\sqrt{T}e^{i\delta_t}}{1-Re^{i\Delta}}e^{ikx}. \tag{5.96a}$$

$$\Psi_-(x) = \frac{\Psi_0\sqrt{T}e^{i\delta_t}}{1-Re^{i\Delta}}\sqrt{R}e^{-ikx+i\Delta-i\delta_r}. \tag{5.96b}$$

The externally transmitted portion of this wave is

$$\begin{aligned}\Psi_T &= t\Psi_+(L)\\ &= \frac{\Psi_0 Te^{2i\delta_t}}{1-Re^{i\Delta}}e^{ikL},\end{aligned} \tag{5.97}$$

which has squared magnitude

$$\begin{aligned}I_T &= |\Psi_T|^2\\ &= \frac{u_0T^2}{|1-Re^{i\Delta}|^2}.\end{aligned} \tag{5.98}$$

Except for the notation change $u_0 \leftrightarrow I_0$, this, as expected, reproduces the result from class.

Exercise 5.2 Spatial coherence, grating. (*2012 Ps3, P3*)

A look inside a grating spectrometer reveals that incident light is passed through a series of slits to increase the transverse spatial coherence. In this problem, we'll try to understand why. For all of the parts below, consider a grating of N slits, periodicity a, and width b.

For parts c and d, neglect the envelope due to a finite slit width b, and consider only the sharp diffraction peaks.

a. Wavelength resolution
For a collimated ($k_{s,y} = 0$), monochromatic source illuminating N slits, diffraction peaks would have an angular width of $\Delta\theta = \lambda/(Na\cos\theta)$, for the first order of diffraction. **rederive this result** for yourself. Show that this gives a wavelength resolution is $\Delta\lambda = \lambda/Nm$

b. Intensity
Next, consider how the output intensity of the grating shifts if the input comes in at an angle θ_s. **Write an expression for $I(\theta_s, \theta)$.** You can also use the variables $k_{s,y} = k\sin\theta_s$ and $k_y = k\sin\theta$, as we did in class.

c. Resolution of spectrometer If the incident beam has an angular spread $\Delta\theta_s$ around normal incidence, what is the **resolution $\Delta\lambda$ of the spectrometer?** Calculate this in the limit of large N, or $N \gg \lambda/a\Delta\theta_s$, where the angular width of the diffracted light is completely determined by the angular width of the incident light.

d. Decreased coherence length at the grating
An alternate view of part c is that by broadening the angular distribution of the source, we also decrease the transverse coherence length at the grating. The number of slits leading to coherent diffraction is reduced to some N_{eff}, which is the number of slits within one coherence length $\ell_{\text{tc}} = \lambda/\Delta\theta_s$. **Sketch a diagram explaining this.** The frequency resolution of the spectrometer is then reduced from λ/N to λ/N_{eff}. (for order $m = 1$) **Compare this to the result you found in part c.**

Answer for Exercise 5.2

Part b. Output intensity given input angle θ_s Let's derive the N slit diffraction wave function and intensity given an off normal input. We'll be able to use this in part a once we do. We'll use a Fraunhofer geometry as in fig. 5.33.

$$\mathbf{R} + \mathbf{r}' = \mathbf{r}. \tag{5.99a}$$

$$\mathbf{R}_s + \mathbf{r}' = \mathbf{r}_s. \tag{5.99b}$$

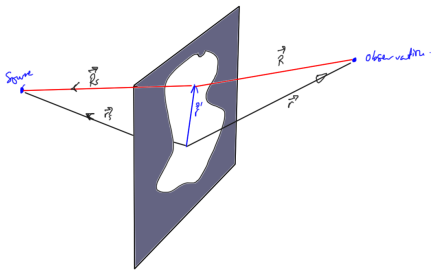

Figure 5.33: Fraunhofer geometry.

for which our path length from $\mathbf{r}'$ to the observation point is

$$\begin{aligned} |\mathbf{R}| &= r\left(1+\frac{r'^2}{r^2}-2\frac{\mathbf{r}\cdot\mathbf{r}'}{r^2}\right)^{1/2} \\ &\sim r+\frac{r'^2}{2r^2}-\hat{\mathbf{r}}\cdot\mathbf{r}', \end{aligned} \tag{5.100}$$

and to first order

$$\begin{aligned} k|\mathbf{R}| &\sim kr-k\hat{\mathbf{r}}\cdot\mathbf{r}' \\ &= kr-ky'\sin\theta. \end{aligned} \tag{5.101}$$

Similarly

$$\begin{aligned} k|\mathbf{R}_s| &\sim kr_s-k\hat{\mathbf{r}}\cdot\mathbf{r}' \\ &= kr_s+\mathbf{k}\cdot\mathbf{r}' \\ &= kr_s+k\sin\theta_s y'. \end{aligned} \tag{5.102}$$

With

$$k_y \equiv \frac{2\pi}{\lambda}\sin\theta, \tag{5.103a}$$

$$k_{y,s} \equiv \frac{2\pi}{\lambda}\sin\theta_s. \tag{5.103b}$$

Our diffraction integral

$$\Psi \sim \int \frac{e^{ik(R+R_s)}}{RR_s}, \tag{5.104}$$

after pulling out and dropping the r and r_s dependent terms, takes the one dimensional form

$$\Psi(\mathbf{r}) \sim \int e^{i(k_{y,s}-k_y)y'}\,dy' = \int e^{i\frac{2\pi}{\lambda}(\sin\theta_s-\sin\theta)y'}\,dy'. \tag{5.105}$$

Let's write

$$\Delta k = k_y - k_{y,s}, \tag{5.106}$$

and evaluate this over intervals $[h+ma, h+ma+b]$, for $m \in [0, N-1]$ as in fig. 5.34. Integrating over the mth slit, we have

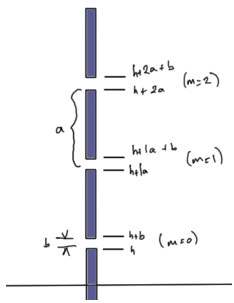

Figure 5.34: N slit geometry.

$$\begin{aligned}
\int_{S_m} e^{-i\Delta k y'}\,dy' &= \int_{h+ma}^{h+ma+b} e^{-i\Delta k y'}\,dy' \\
&= \left.\frac{e^{-i\Delta k y'}}{-i\Delta k}\right|_{h+ma}^{h+ma+b} \\
&= \frac{e^{-i\Delta k(h+ma)}}{-i\Delta k}\left(e^{-i\Delta k b}-1\right) \\
&= \frac{e^{-i\Delta k(h+ma)}}{\Delta k} e^{i\Delta k b/2} 2\sin(\Delta k b/2) \\
&= b e^{-i\Delta k(h+ma)} e^{i\Delta k b/2}\frac{\sin(\Delta k b/2)}{\Delta k b/2}.
\end{aligned} \tag{5.107}$$

Adding all the slit contributions we have

$$\begin{aligned}
\Psi &= b e^{i\Delta k(b/2-h)}\frac{\sin(\Delta k b/2)}{\Delta k b/2}\sum_{m=0}^{N-1} e^{-i\Delta k m a} \\
&= b e^{i\Delta k(b/2-h)}\frac{\sin(\Delta k b/2)}{\Delta k b/2}\frac{1-e^{-i\Delta k a N}}{1-e^{-i\Delta k a N}} \\
&= b e^{i\Delta k(b/2-h)}\frac{\sin(\Delta k b/2)}{\Delta k b/2}\frac{e^{-i\Delta k a N/2}}{e^{-i\Delta k a/2}}\frac{e^{i\Delta k a N/2}-e^{-i\Delta k a N/2}}{e^{i\Delta k a/2}-e^{-i\Delta k a/2}}.
\end{aligned} \tag{5.108}$$

$$\Psi \sim \frac{\sin(\Delta kb/2)}{\Delta kb/2} \frac{\sin(\Delta kaN/2)}{N \sin(\Delta ka/2)}, \tag{5.109}$$

with intensity

$$\boxed{I(\theta_s, \theta) \sim \frac{\sin^2(\Delta kb/2)}{(\Delta kb/2)^2} \frac{\sin^2(\Delta kaN/2)}{N^2 \sin^2(\Delta ka/2)},} \tag{5.110a}$$

$$\Delta k = \frac{2\pi}{\lambda}(\sin\theta - \sin\theta_s). \tag{5.110b}$$

Part a. Peak width and wavelength resolution for normal incidence Here we work with a normal incident $\theta_s = 0$ plane wave source, and write

$$\begin{aligned} \gamma &= \frac{\Delta ka}{2} \\ &= \frac{ka}{2} \sin\theta \\ &= \frac{\pi a}{\lambda} \sin\theta. \end{aligned} \tag{5.111}$$

and seek to understand the characteristics of the Intensity envelope

$$\frac{\sin^2(N\gamma/2)}{N^2 \sin^2(\gamma/2)}. \tag{5.112}$$

To get a feel for what this may look like this is plotted for two wave lengths $\lambda = 3\pi a$, $\lambda' = 4\pi a$, $a = 1$ in fig. 5.35. Observe that this ratio of sines has a unit value for any $\gamma/2 = m\pi$, for integer m since by H'ôpital's rule we have

$$\lim_{\gamma/2 \to m\pi} \frac{\sin(N\gamma/2)}{N \sin(\gamma/2)} = \left. \frac{\cos(N\gamma/2)}{\cos(\gamma/2)} \right|_{\gamma = m\pi} = (-1)^{(N-1)m}. \tag{5.113}$$

So for any $N\gamma/2 = l\pi$, provided $\gamma/2 \neq m\pi$ we have a zero. We find those at

$$N \frac{\pi a}{\lambda} \sin\theta = l\pi, \tag{5.114}$$

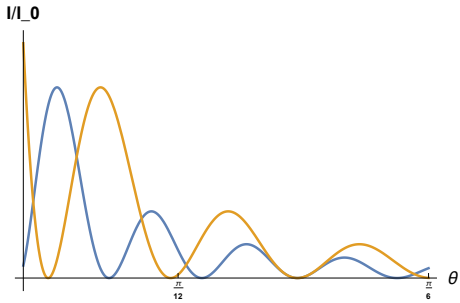

Figure 5.35: Intensity envelope sample plot.

or

$$\sin\theta_l = \frac{l\lambda}{Na}, \tag{5.115}$$

For the distance between zeros past the center $\theta = 0$ lobe for a fixed wavelength, we have

$$\begin{aligned}\sin\theta_{l+1} - \sin\theta_l &= \frac{(l+1)\lambda}{Na} - \frac{l\lambda}{Na} \\ &= \frac{\lambda}{Na}.\end{aligned} \tag{5.116}$$

If $\Delta\theta_l$, or just $\Delta\theta$ (assuming that the peak or zero separation is about the same, although this is artificial in general as we see from the plot), we can compute this by examining the difference

$$\begin{aligned}\sin\theta_{l+1} - \sin\theta_l &\sim \sin(\theta_l + \Delta\theta/2) - \sin(\theta_l + \Delta\theta/2) \\ &= 2\cos\theta_l \sin(\Delta\theta/2) \\ &\sim \Delta\theta\cos\theta_l.\end{aligned} \tag{5.117}$$

This gives us the desired relationship (for the lth zero)

$$\boxed{\Delta\theta_l \sim \frac{\lambda}{Na\cos\theta_l}.} \tag{5.118}$$

Suppose we rather loosely identify this as the peak width, and look at the image around the mth peak, as in fig. 5.36. This is about as close as the wavelengths can be in order that a superposition of the two would be distinguishable as separate (humped near center). That separation of wavelength $\lambda' = \lambda + \Delta\lambda$ is

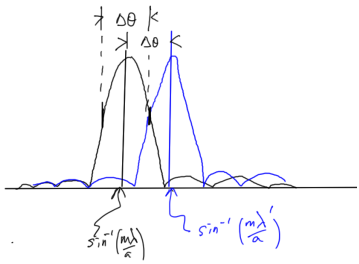

Figure 5.36: Resolvable peak to peak separation.

$$\begin{aligned}\Delta\theta &= \sin^{-1}(m(\lambda+\Delta\lambda)/a) - \sin^{-1}(m\lambda/a) \\ &\sim \frac{d}{d(m\lambda/a)}\left(\sin^{-1}(m\lambda/a)\right)\frac{m\Delta\lambda}{a} \\ &= \frac{1}{\cos\sin^{-1}(m\lambda/a)}\frac{m\Delta\lambda}{a},\end{aligned} \tag{5.119}$$

but we also have

$$\begin{aligned}\Delta\theta &= \frac{\lambda}{Na\cos\theta} \\ &= \frac{\lambda}{Na\cos\sin^{-1}(m\lambda/a)}.\end{aligned} \tag{5.120}$$

Comparing the two

$$\frac{1}{\cancel{\cos\sin^{-1}(m\lambda/a)}}\frac{m\Delta\lambda}{\cancel{a}} = \frac{\lambda}{N\cancel{a}\cancel{\cos\sin^{-1}(m\lambda/a)}}, \tag{5.121}$$

or

$$\boxed{\Delta\lambda = \frac{\lambda}{Nm},} \tag{5.122}$$

which is the wavelength resolution desired.

Part c. Angular spread Neglecting the width of the slits, we've found in eq. (5.110) that our intensity due to a plane wave source and incident light at angle θ_s, we have

$$I \sim \frac{\sin^2(\pi a N(\sin\theta - \sin\theta_s)/\lambda)}{N^2\sin^2(\pi a k(\sin\theta - \sin\theta_s)/\lambda)}. \tag{5.123}$$

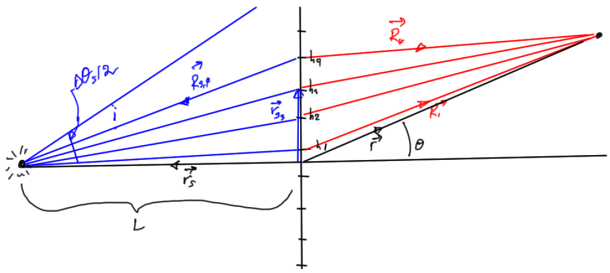

Figure 5.37: Spread source.

However, with an angular source spread, presumably from a point source at some distance L, we have a configuration like fig. 5.37. Our angle of incidence varies with each slit, so this plane wave result doesn't seem applicable. There's no obvious way to get what we want out of this result, so let's start from scratch. Let's assume an even number of slits with a symmetric setup, so that our Fraunhofer geometry is

$$h_m = \pm \left(m - \frac{1}{2} \right) a. \tag{5.124a}$$

$$\mathbf{k}_m = k \frac{L\hat{\mathbf{z}} \pm h_m \hat{\mathbf{y}}}{\sqrt{L^2 + h_m^2}} \sim \frac{k h_m}{L} \hat{\mathbf{y}}. \tag{5.124b}$$

$$\mathbf{r}'_m = \hat{\mathbf{y}} h_m. \tag{5.124c}$$

$$\hat{\mathbf{r}} \cdot \mathbf{r}'_m = \pm h_m \sin\theta. \tag{5.124d}$$

$$\mathbf{k}_m \cdot \mathbf{r}'_m \sim k(\pm h_m)^2 / L. \tag{5.124e}$$

Summing over both positive and negative m our Fraunhofer sum becomes

$$\begin{aligned} \Psi &= 2 \sum_{m=1} e^{ikh_m^2/L} \cos(k h_m \sin\theta) \\ &= 2 \sum_{m=1} e^{i \frac{ka^2}{L} \left(m - \frac{1}{2}\right)^2} \cos\left(ka \left(m - \frac{1}{2} \right) \sin\theta \right). \end{aligned} \tag{5.125}$$

We have two cases to consider. The first is that our the spread completely covers all the slits, in which case the upper bound of the sum above is $m = N/2$. Otherwise, if the slits extend beyond the range of the source spread, we have to sum over an interval where $m_{\max}$ is the largest integer such that

$$\frac{h_{\max}}{L} = a\left(m_{\max} - \frac{1}{2}\right) \leq \tan\left(\frac{\Delta\theta_s}{2}\right). \tag{5.126}$$

or

$$m_{\max} \leq \frac{1}{2} + \frac{L}{a}\tan\left(\frac{\Delta\theta_s}{2}\right) \sim \frac{L\Delta\theta_s}{2a}. \tag{5.127}$$

Let's write M for this sum where $M = N/2$ for a source spread that encompasses than the grating, and $M = m_{\max}$ from eq. (5.127) otherwise.

With $\gamma = ka\sin\theta/2$, our intensity is

$$\begin{aligned}
I &= \Psi\Psi^* \\
&= \operatorname{Re}\Psi\Psi^* \\
&\sim \operatorname{Re}\sum_{m,n=1}^{M} e^{i\frac{ka^2}{L}\left(\left(m-\frac{1}{2}\right)^2-\left(n-\frac{1}{2}\right)^2\right)}\cos\left(2\gamma\left(m-\frac{1}{2}\right)\right)\cos\left(2\gamma\left(n-\frac{1}{2}\right)\right) \\
&\sim \sum_{m,n=1}^{M}\cos\left(\frac{ka^2}{L}\left(\left(m-\frac{1}{2}\right)^2-\left(n-\frac{1}{2}\right)^2\right)\right)\times \\
&\qquad (\cos(2\gamma(m-n))+\cos(2\gamma(m+n-1))) \\
&= \sum_{m,n=1}^{M}\cos\left(\frac{ka^2}{L}(m-n)(m+n-1)\right)\times \\
&\qquad (\cos(2\gamma(m-n))+\cos(2\gamma(m+n-1))).
\end{aligned} \tag{5.128}$$

Do we have any hope whatsoever to evaluate this sum in some sort of closed form? If we are to try it seems clear that we need two sets of change of variables. Should we try $s = m - n$ as in fig. 5.38, we find

$$\begin{aligned}
&\sum_{m,n=1}^{N} f(m-n, m+n-1) \\
&\qquad = \sum_{s=-N+1}^{N-1}\sum_{t=1}^{N-|s|} f(s, |s|+2t-1).
\end{aligned} \tag{5.129}$$

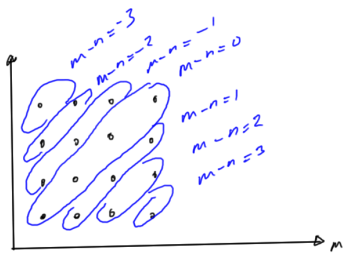

Figure 5.38: Points of constant $m - n$.

With the opposite diagonal orientation, as in fig. 5.39, we find

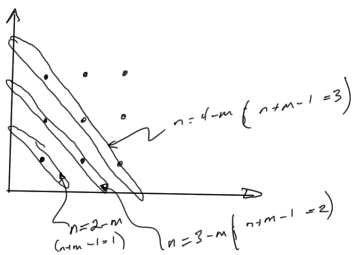

Figure 5.39: Points of constant $m + n$.

$$
\begin{aligned}
&\sum_{m,n=1}^{N} f(m-n, m+n-1) \\
&= \sum_{u=1}^{2N-1} \sum_{t=1}^{N-|u-N|} f(2t-(N-|u-N|)-1, u) \\
&= \sum_{u=1}^{N-1} \sum_{t=1}^{u} f(2t-u-1, u) + \sum_{u=N}^{2N-1} \sum_{t=1}^{2N-u} f(2t-(2N-u)-1, u).
\end{aligned}
\tag{5.130}
$$

The intensity can now be written

$$
\begin{aligned}
I \sim & \sum_{m,n=1}^{M} \cos\left(\frac{ka^2}{L}(m-n)(m+n-1)\right)\cos(2\gamma(m-n)) \\
& + \sum_{m,n=1}^{M} \cos\left(\frac{ka^2}{L}(m-n)(m+n-1)\right)\cos(2\gamma(m+n-1)) \\
= & \sum_{s=-M+1}^{M-1}\sum_{t=1}^{M-|s|} \cos\left(\frac{ka^2}{L}s(2t+|s|-1)\right)\cos(2\gamma s) \\
& + \sum_{s=1}^{2M-1}\sum_{t=1}^{M-|s-M|} \cos\left(\frac{ka^2}{L}s(2t-(M-|s-M|)-1)\right)\cos(2\gamma s)
\end{aligned}
\tag{5.131}
$$

Let's now impose the condition

$$
ka^2 M/L \sim \frac{\pi a \Delta\theta_s}{\lambda} \ll 1, \tag{5.132}
$$

leaving just

$$
I \sim \left(\sum_{s=-M+1}^{M-1} + \sum_{s=1}^{2M-1}\right)\cos(2\gamma s). \tag{5.133}
$$

Using

$$
\sum_{a}^{b} e^{i\alpha m} = e^{i\alpha(b+a)/2}\frac{\sin(\alpha(b-a+1)/2)}{\sin(\alpha/2)}, \tag{5.134}
$$

the intensity sums to

$$
\begin{aligned}
I &\sim (1+\cos(2\gamma M))\frac{\sin(\gamma(2M-1))}{\sin(\gamma)} \\
&\sim (1+\cos(2\gamma M))\frac{\sin(2\gamma M)}{\sin(\gamma)}.
\end{aligned}
\tag{5.135}
$$

Employing half angle formulas and normalizing once more, we get

$$
I \sim \left(\frac{\cos(\gamma M)}{\gamma M}\right)^2 \frac{\sin(\gamma M)}{M\sin(\gamma)}, \tag{5.136}
$$

Since this can be negative, I must have made an error above. My first attempt on paper had this sine ratio squared and $2\gamma M$ instead of γM. I did find an error in that first attempt, and corrected that here, but in order to make progress, let's "average the errors", and assume that the intensity should be

$$I \sim \left(\frac{\cos(2\gamma M)}{2\gamma M}\right)^2 \left(\frac{\sin(2\gamma M)}{2M\sin(\gamma)}\right)^2. \tag{5.137}$$

Specifying that we have more slits than the beam spread so that we use eq. (5.127), and writing $\Delta H = L\Delta\theta_s$ for the total illuminated height of the diffraction device we have

$$\begin{aligned} 2\gamma M &\sim 2\frac{2\pi}{\lambda}\frac{a\sin\theta}{2}\frac{L\Delta\theta_s}{2a} \\ &= \frac{\pi L\sin\theta\Delta\theta_s}{\lambda} \\ &= \frac{\pi\sin\theta\Delta H}{\lambda}, \end{aligned} \tag{5.138}$$

Let's introduce $\Delta\theta_a$ for the angular spread of the diffraction regions separating the slits. Then noting that for the plane wave case where we obtained the wavelength resolution eq. (5.122) from eq. (5.118), we can make the substitution $\Delta H \leftrightarrow Na$, to obtain the wavelength resolution for this spread incident beam case (for the $m = 1$ order peaks)

$$\boxed{\Delta\lambda = \frac{\Delta\theta_a\lambda}{\Delta\theta_s}.} \tag{5.139}$$

Grading notes (−2)

1. The sentence that states "Let's introduce $Delta\theta_a$" was underlined and marked with the question "What is $\Delta\theta_a$?"

2. $m = 1$ was underlined with question: What about general m? I think it was just a scaling by m for the general case, but when I typed up my solution that didn't strike me as important.

3. eq. (5.139) marked with a question mark and note $\Delta\lambda \propto \Delta\theta_s$.

The ratio of the angular spread to the separation spread, takes the place of N in the plane wave case

$$\frac{\Delta\theta_s}{\Delta\theta_a} \leftrightarrow N. \tag{5.140}$$

Part d. Using coherence length If the transverse coherence length is defined as $l_{\text{tc}} = \Delta\theta_s/\lambda$, then this has a clear geometric interpretation shown in fig. 5.40. To match N_{eff} to this we write

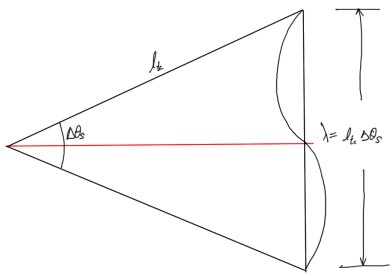

Figure 5.40: Transverse coherence length geometrically.

$$aN_{\text{eff}} = l_{\text{tc}}\Delta\theta_s. \tag{5.141}$$

The angular spread of a single slit is

$$\Delta\theta_a = \frac{a}{l_{\text{tc}}}, \tag{5.142}$$

Grading note: (−1) eq. (5.142) was circled with comment "why?". so we have

$$\begin{aligned}\Delta\lambda &= \frac{\lambda}{N_{\text{eff}}} \\ &= \frac{a\lambda}{l_{\text{tc}}\Delta\theta_s} \\ &= \frac{\Delta\theta_a\lambda}{\Delta\theta_s}.\end{aligned} \tag{5.143}$$

Rather remarkably, considering the fudging that was done in part c, this matches the wavelength resolution result from eq. (5.139).

6 LASERS AND GAUSSIAN BEAMS.

6.1 LASERS.

- 1958 Theory, done by Schawlow (Toronto grad, PhD '49) and Townes.
- 1960 Experiment
- 1960-1980 Reinvention of optics!

Until the laser was invented, the field was mainly engineering (lenses, telescopes, ...), but not much physics.

The basic idea is that we have a cavity as in fig. 6.1. and to

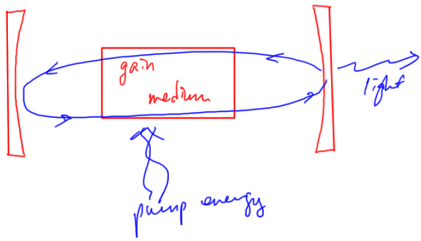

Figure 6.1: Laser cavity.

prevent losses, we have a gain medium (to be defined), and some mechanism to pump in energy (almost always not a thermal source).

Townes and Schawlow were mazer researchers (still important, used in GPS satellites for example as clocks), who realized that the ideas could be carried over into the optical regime.

The history of this actually goes back to Einstein, who in 1917 did a thought experiment with atoms in a box, stimulating radiation, which is bouncing around, exciting the atoms, so that they in turn emit again. We'll consider N atoms in thermal equilibrium with the light in the box as in fig. 6.2. For background on this ma-

terial see a book on thermal or statistical mechanics, such as [11] §7. We expect more in the ground state fig. 6.3.

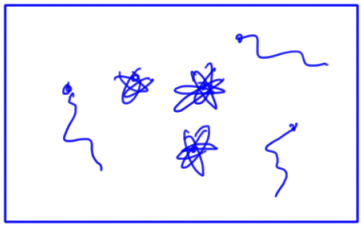

Figure 6.2: Atoms in a box.

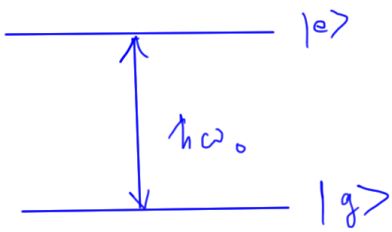

Figure 6.3: Ground and excited state separation.

$$\frac{N_e}{N_g} = \frac{P_e}{P_g} = e^{-(E_e - E_g)/kT}. \tag{6.1}$$

Thermal distribution of radiation: §7 [5]. In summary, that energy density as a function of frequency is

$$u_\omega = \hbar\omega \langle n_\omega \rangle \mathcal{D}(\omega). \tag{6.2a}$$

$$\langle n_\omega \rangle = \frac{1}{e^{\hbar\omega/kT} - 1}. \tag{6.2b}$$

$$\mathcal{D}(\omega) = \frac{\omega^2}{\pi^2 c^3}. \tag{6.2c}$$

How do the number of excited states change, given absorption probability $B_{\text{abs}}u(\omega)$ and stimulated emission probability $B_{\text{se}}u(\omega)$, and spontaneous emission probability A (see figure in the slides)

$$\frac{d}{dt}N_e = -AN_e + B_{\text{abs}}uN_g - B_{\text{se}}uN_e = 0. \tag{6.3a}$$

$$\frac{d}{dt}N_g = -\frac{d}{dt}N_e. \tag{6.3b}$$

Solving for u, with $\dot{N}_e = \dot{N}_g = 0$

$$\begin{aligned} u_\omega &= \frac{AN_e}{B_{\text{abs}}N_g - B_{\text{se}}N_e} \\ &= \frac{A}{B_{\text{abs}}N_g/N_e - B_{\text{se}}} \\ &= \frac{A}{B_{\text{se}}}\frac{1}{B_{\text{abs}}/B_{\text{se}}e^{\hbar\omega/kT} - 1} \\ &= \frac{\hbar\omega^3}{\pi^2c^3}\frac{1}{e^{\hbar\omega/kT} - 1}. \end{aligned} \tag{6.4}$$

We conclude that we must have

$$\frac{A}{B_{\text{se}}} = \frac{\hbar\omega^3}{\pi^2c^3}. \tag{6.5a}$$

$$B_{\text{abs}}/B_{\text{se}} = 1. \tag{6.5b}$$

With $B_{\text{abs}} = B_{\text{se}}$ we don't see stimulated radiation around you because

$$\frac{N_e}{N_g} = e^{-\hbar\omega/kT}. \tag{6.6}$$

Our typical chemical energies are
1 eV = k_B (12000 K)
so that

$$e^{-\hbar\omega/kT} \sim e^{-40}. \tag{6.7}$$

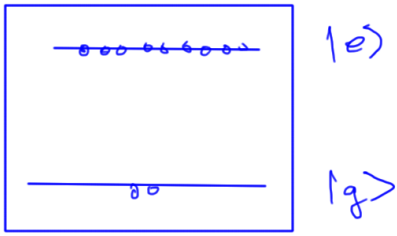

Figure 6.4: Population inversion.

This is very small, and in order to get there we need a population inversion with more atoms in the excited state than in the ground state as in fig. 6.4. With

$$\begin{aligned} \frac{A}{B} &= \frac{\hbar\omega^3}{\pi^2 c^3} \\ &= \hbar\omega\mathcal{D}(\omega) \\ &= \frac{u_\omega}{\langle n_\omega \rangle}. \end{aligned} \tag{6.8}$$

so

$$\frac{Bu}{A} = \langle n_\omega \rangle = \text{mode occupation.} \tag{6.9}$$

(stimulated/spontaneous radiation)

If the mode occupation $\gg 1$, then $Bu > A$, we have more stimulated than spontaneous radiation.

6.2 LASER PUMP RATES.

Referring to the figure "Three-level model of a laser" from the class slides, we want

$$A \gg R, \Gamma_{\text{sp}}, \Gamma_{\text{st}} n. \tag{6.10}$$

$$N_1 \ll N_2 \ll N_0 \approx N. \tag{6.11}$$

Its the N_1, N_2 difference that leads to the population inversion on the $1-2$ excitation.

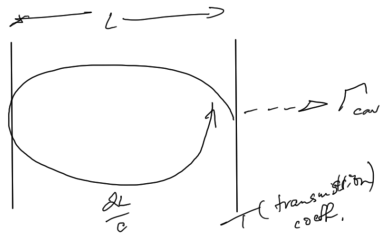

Figure 6.5: Cavity.

We've got something like fig. 6.5.

$$\Gamma_{\text{cav}} = T\frac{c}{2L}. \tag{6.12}$$

where T is the transmission coefficient.

We typically have

$$\Gamma_{\text{sp}} = 10^7 \text{s}^{-1}. \tag{6.13}$$

The spontaneous em rate out of cavity, and

$$\Gamma_{\text{st}} = 1\text{s}^{-1}. \tag{6.14}$$

(the spontaneous emission into cavity mode of interest).

Recall

$$\frac{\text{stimulated}}{\text{spontaneous}} = \frac{Bu_w}{a} = \frac{\Gamma \langle n_w \rangle}{\Gamma}. \tag{6.15}$$

Ignoring the $\Gamma_{\text{st}} n N_1$ transitions, the atomic population is

$$\frac{dN_2}{dt} = NR - N_2\Gamma_{\text{sp}} - N_2\Gamma_{\text{st}} \langle n \rangle . \tag{6.16}$$

and the photon population is

$$\frac{d\langle n \rangle}{dt} = \Gamma_{\text{st}} N_2 (\underbrace{\boxed{1}}_{\text{spontaneous emission into cavity}} + \underbrace{\boxed{\langle n \rangle}}_{\text{stimulated emission}}) - \overbrace{\boxed{\Gamma_{\text{cav}} \langle n \rangle}}^{\text{out coupling}}. \tag{6.17}$$

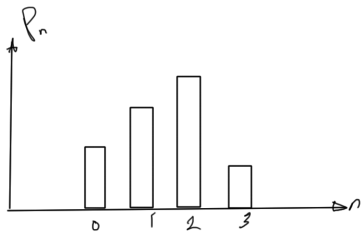

Figure 6.6: Probability distribution.

For an exact treatment we really have a distribution like fig. 6.6.

$$\langle n \rangle = \sum_n n P_n. \tag{6.18a}$$

$$\frac{d\langle n \rangle}{dt} = \sum_n n \frac{dP_n}{dt}. \tag{6.18b}$$

We aren't equipped to do this (covered in PHY2204), however, for the steady state solution, setting $dN_2/dt = 0$ we have for the atomic population

$$N_2 = \frac{NR}{\Gamma_{sp} + \Gamma_{st} \langle n \rangle}. \tag{6.19}$$

$$\frac{NR}{\Gamma_{sp} + \Gamma_{st} \langle n \rangle} \Gamma_{st}(1 + \langle n \rangle) - \Gamma_{cav} \langle n \rangle = 0. \tag{6.20}$$

Plugging into the steady state ($d \langle n \rangle / dt = 0$) photon population equation of eq. (6.17) we have

$$0 = \Gamma_{st} \frac{NR}{\Gamma_{sp} + \Gamma_{st} \langle n \rangle} (1 + \langle n \rangle) - \Gamma_{cav} \langle n \rangle, \tag{6.21}$$

or

$$\begin{aligned} 0 &= \Gamma_{st} NR(1 + \langle n \rangle) - \Gamma_{cav} \langle n \rangle (\Gamma_{sp} + \Gamma_{st}) \langle n \rangle \\ &= - \langle n \rangle^2 \Gamma_{st} \Gamma_{cav} + \langle n \rangle \left(-\Gamma_{cav} \Gamma_{sp} + NR\Gamma_{st} \right) + NR\Gamma_{st}, \end{aligned} \tag{6.22}$$

which after normalization is

$$\langle n\rangle^2 - \left(\frac{NR\Gamma_{st}}{\Gamma_{sp}\Gamma_{cav}} - 1\right)\frac{\Gamma_{sp}}{\Gamma_{st}}\langle n\rangle - \frac{NR}{\Gamma_{cav}} = 0. \tag{6.23}$$

This can be written in a nicer way

$$\langle n\rangle^2 - (C-1)n_s\langle n\rangle - Cn_s = 0. \tag{6.24a}$$

$$C \equiv \frac{NR\Gamma_{st}}{\Gamma_{sp}\Gamma_{cav}}. \tag{6.24b}$$

$$n_s \equiv \frac{\Gamma_{sp}}{\Gamma_{st}} \sim 10^7. \tag{6.24c}$$

Here n_s is called the saturation photon number, and C is called the cooperation parameter, a non-dimensional pump rate. R was the controllable quantity, the voltage of some such knob that we can vary. The rest are fixed.

Our quadratic equation (taking the positive root to avoid non-physical negative photon numbers)

$$\begin{aligned}\langle n\rangle &= \frac{1}{2}(C-1)n_s + \frac{1}{2}\sqrt{(C-1)^2n_s^2 + 4Cn_s} \\ &\approx \frac{1}{2}(C-1)n_s + \frac{1}{2}|C-1|n_s\left(1 + \frac{2C}{(C-1)^2n_s} + \cdots\right) \\ &= \begin{cases}(C-1)n_s & \text{if } C > 1 \\ \frac{C}{1-C} & \text{if } C < 1\end{cases}\end{aligned} \tag{6.25}$$

Our steady state solution is then

$$N_2 = \frac{NR}{\Gamma_{sp} + \Gamma_{st}\langle n\rangle} = \frac{C}{1 + \langle n\rangle / n_s}. \tag{6.26}$$

That's how a laser works, at least from a population point of view.

6.3 GAUSSIAN MODES.

READING: §6.4 - §6.10 [16]. Also Van Driel notes from previous years lectures.

We'll start thinking about transverse modes in a cavity. Our starting point is Maxwell's equations

$$\boldsymbol{\nabla} \times \mathbf{H} = \epsilon \frac{\partial \mathbf{E}}{\partial t}. \tag{6.27a}$$

$$\boldsymbol{\nabla} \times \mathbf{E} = -\mu \frac{\partial \mathbf{H}}{\partial t}. \tag{6.27b}$$

$$\boldsymbol{\nabla} \cdot (\epsilon \mathbf{E}) = 0. \tag{6.27c}$$

$$\boldsymbol{\nabla} \cdot \mathbf{B} = 0. \tag{6.27d}$$

We won't actually need the $\mathbf{B}$ divergence equation, and will be looking for a wave equation where $\epsilon(\mathbf{r})$ varies "slowly".

Reminder

$$n^2 = \frac{\epsilon}{\epsilon_0}. \tag{6.28a}$$

$$v = \frac{c}{n} = \frac{1}{\sqrt{\epsilon\mu}}. \tag{6.28b}$$

$$k = \omega\sqrt{\epsilon\mu} = \frac{\omega n}{c}. \tag{6.28c}$$

This last comes from equating

$$kx - \omega t = k(x - vt) = k\left(x - \frac{v}{c}ct\right) = k\left(x - \frac{c}{n}t\right). \tag{6.29}$$

so that we have $\omega = kc/n$.

Recall the identity for curl of curl eq. (G.2)

$$\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \mathbf{A}) = \boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \mathbf{A}) - \boldsymbol{\nabla}^2 \mathbf{A}. \tag{6.30}$$

and take curls of both sides of the $\mathbf{E}$ curl eq. (6.27b)

$$\begin{aligned}
\boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \mathbf{E}) - \boldsymbol{\nabla}^2 \mathbf{E} &= -\boldsymbol{\nabla} \times \left(\mu \frac{\partial \mathbf{H}}{\partial t}\right) \\
&= -(\boldsymbol{\nabla}\mu) \times \frac{\partial \mathbf{H}}{\partial t} - \mu \boldsymbol{\nabla} \times \frac{\partial \mathbf{H}}{\partial t} \\
&= -(\boldsymbol{\nabla}\mu) \times \frac{\partial \mathbf{H}}{\partial t} - \mu \frac{\partial}{\partial t}\left(\epsilon \frac{\partial \mathbf{E}}{\partial t}\right),
\end{aligned} \tag{6.31}$$

or

$$\nabla \cdot (\epsilon \mathbf{E}) = \epsilon \nabla \cdot \mathbf{E} + (\nabla \epsilon) \cdot \mathbf{E}$$

$$\begin{aligned} \nabla^2 \mathbf{E} - \mu\epsilon \frac{\partial^2 \mathbf{E}}{\partial t^2} &= (\nabla \mu) \times \frac{\partial \mathbf{H}}{\partial t} + \nabla(\boxed{\nabla \cdot \mathbf{E}}) \\ &= (\nabla \mu) \times \frac{\partial \mathbf{H}}{\partial t} + \nabla \left(-\frac{1}{\epsilon} (\nabla \epsilon) \cdot \mathbf{E} \right). \end{aligned} \tag{6.32}$$

So if we assume that $\mu \sim 1$ (or doesn't vary much from that), we have

Neglect this if ϵ varies slowly compared to λ

$$\nabla^2 \mathbf{E} - \mu\epsilon(\mathbf{r}) \frac{\partial^2 \mathbf{E}}{\partial t^2} = \boxed{-\nabla \left(\frac{1}{\epsilon} \mathbf{E} \cdot \nabla \epsilon \right)}. \tag{6.33}$$

We suppose that the time dependence of the electric field is monochromatic, so that

$$\mathbf{E}(\mathbf{r}, t) = \mathbf{E}(\mathbf{r}) e^{-i\omega t}. \tag{6.34}$$

our second time partial is

$$\begin{aligned} \frac{\partial^2}{\partial t^2} \mathbf{E}(\mathbf{r}, t) &= -\mathbf{E}(\mathbf{r}) \omega^2 e^{-i\omega t} \\ &= -\mathbf{E}(\mathbf{r}) \frac{k^2}{\epsilon\mu} e^{-i\omega t} \\ &= -\mathbf{E}(\mathbf{r}) k^2 e^{-i\omega t}. \end{aligned} \tag{6.35}$$

Provided we have

$$\left| \nabla \left(\frac{1}{\epsilon} \mathbf{E} \cdot \nabla \epsilon \right) \right| \ll k^2 \mathbf{E}, \tag{6.36}$$

Our wave equation reduces to

$$\boxed{\nabla^2 \mathbf{E}(\mathbf{r}) + k^2(\mathbf{r}) \mathbf{E}(\mathbf{r}) = 0.} \tag{6.37}$$

Choose $\epsilon(\mathbf{r})$ such that with $k_0 = \omega/c$, we have

$$\boxed{k^2(\mathbf{r}) = k_0^2 - k_0 k_2 r^2.} \tag{6.38}$$

Perhaps this medium looks like fig. 6.7. Let's look for solutions

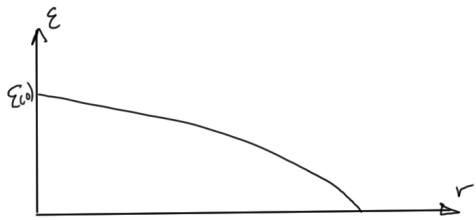

Figure 6.7: Possible ϵ dependence in medium.

of the form

$$\mathbf{E} = \mathbf{E}_0 \boxed{u(r,\theta,z)} e^{ik_0 z}, \tag{6.39}$$

Slowly varying (complex) envelope

where $\mathbf{E}_0$ is a vector with a chosen polarity.

We can now work with a scalar amplitude

$$\Psi(r,\theta,z) = u e^{ik_0 z}. \tag{6.40}$$

Recall that our Laplacian in cylindrical coordinates is

$$\nabla^2 = \overbrace{\boxed{\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r}}}^{\nabla_{\mathrm{T}}^2} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2}. \tag{6.41}$$

We'll look for cylindrical symmetric solutions so that we can ignore the θ dependence in the Laplacian.

$$\begin{aligned}
\frac{\partial^2}{\partial z^2} u e^{ik_0 z} &= \frac{\partial}{\partial z}\left(\frac{\partial u}{\partial z} e^{ik_0 z} + ik_0 u e^{ik_0 z}\right) \\
&= \frac{\partial^2 u}{\partial z^2} e^{ik_0 z} + ik_0 \frac{\partial u}{\partial z} e^{ik_0 z} + ik_0 \frac{\partial u}{\partial z} e^{ik_0 z} - k_0^2 u e^{ik_0 z} \\
&= \left(\frac{\partial^2 u}{\partial z^2} + 2ik_0 \frac{\partial u}{\partial z} - k_0^2 u\right) e^{ik_0 z}.
\end{aligned} \tag{6.42}$$

Noting that

$$\nabla_{\mathrm{T}}^2 u e^{ik_0 z} = \left(\nabla_{\mathrm{T}}^2 u\right) e^{ik_0 z}, \tag{6.43}$$

we can assemble (dropping exponentials)

$$0 = \frac{\partial^2 u}{\partial z^2} + 2ik_0 \frac{\partial u}{\partial z} + \nabla_{\mathrm{T}}^2 u - \cancel{k_0^2 u} + (\cancel{k_0^2} - k_0 k_2 r^2) u. \tag{6.44}$$

This is the paraxial wave equation

$$\boxed{\frac{\partial^2 u}{\partial z^2} + 2ik_0 \frac{\partial u}{\partial z} + \nabla_{\mathrm{T}}^2 u - k_0 k_2 r^2 u = 0} \tag{6.45}$$

If

$$\left| \frac{\partial^2 u}{\partial z^2} \right| \ll k_0 \left| \frac{\partial u}{\partial z} \right|, \tag{6.46}$$

so that u is slowing varying on the wavelength scale, then we can neglect the first term

$$\boxed{2ik_0 \frac{\partial u}{\partial z} + \nabla_{\mathrm{T}}^2 u - k_0 k_2 r^2 u = 0} \tag{6.47}$$

Also note that we didn't need to use cylindrical coordinates here, and could have grouped the transverse Laplacian as just

$$\nabla_{\mathrm{T}}^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}. \tag{6.48}$$

Let's rewrite this in a slightly different order

$$\boxed{\nabla_{\mathrm{T}}^2 u - k_0 k_2 r^2 u = -2ik_0 \frac{\partial u}{\partial z}} \tag{6.49}$$

Observe that this has the same form as the 2D Schrödinger equation

$$\hat{H} = \frac{1}{2m} \hat{p}^2 + \frac{1}{2} m\omega^2 (\hat{x}^2 + \hat{y}^2), \tag{6.50}$$

or in a position basis

$$\begin{aligned} \hat{H} &\to -\frac{\hbar^2}{2m} \overbrace{\boxed{\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right)}}^{\nabla_{\mathrm{T}}^2} + \frac{1}{2} m\omega^2 \overbrace{\boxed{(x^2 + y^2)}}^{r^2} \\ &= i\hbar \frac{\partial}{\partial t}, \end{aligned} \tag{6.51}$$

or

$$-\frac{\hbar^2}{2m}\nabla_{\mathrm{T}}^2\Psi + \frac{1}{2}m\omega^2 r^2\Psi = i\hbar\frac{\partial\Psi}{\partial t}. \tag{6.52}$$

$$\nabla_{\mathrm{T}}^2\Psi - \frac{m^2\omega^2}{\hbar^2}r^2\Psi = -\frac{2mi}{\hbar}\frac{\partial\Psi}{\partial t}. \tag{6.53}$$

We can think of the equivalence in the following form

$$\frac{m^2\omega^2}{\hbar^2} \leftrightarrow k_0 k_2. \tag{6.54a}$$

$$\frac{m}{\hbar}\frac{\partial}{\partial t} \leftrightarrow k_0\frac{\partial}{\partial z}. \tag{6.54b}$$

as illustrated in fig. 6.8. Recall that the first few 1D Quantum SHO

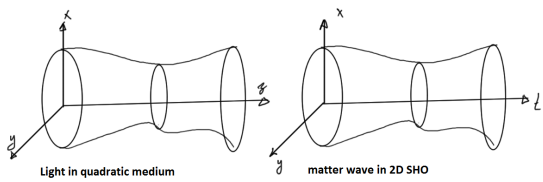

Figure 6.8: Radiation and matter wave equivalence.

are

$$\psi_0(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} e^{-\frac{m\omega x^2}{2\hbar}}. \tag{6.55a}$$

$$\psi_1(x) = \frac{1}{2}\left(\frac{m\omega}{\pi\hbar}\right)^{1/4} e^{-\frac{m\omega x^2}{2\hbar}} 2\sqrt{\frac{m\omega}{\hbar}}x. \tag{6.55b}$$

$$\psi_2(x) = \frac{1}{8}\left(\frac{m\omega}{\pi\hbar}\right)^{1/4} e^{-\frac{m\omega x^2}{2\hbar}} \left(4\left(\sqrt{\frac{m\omega}{\hbar}}x\right)^2 - 2\right). \tag{6.55c}$$

Which look like fig. 6.9, fig. 6.10, and fig. 6.11 respectively.

In 2D our solutions look like fig. 6.12.

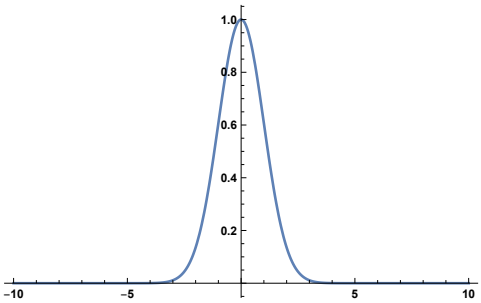

Figure 6.9: First order 1D SHO matter wave.

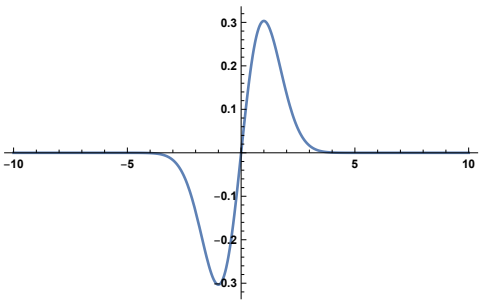

Figure 6.10: Second order 1D SHO matter wave.

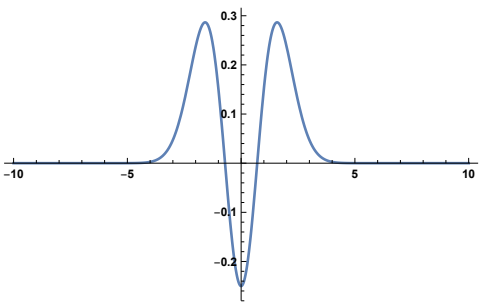

Figure 6.11: Third order 1D SHO matter wave.

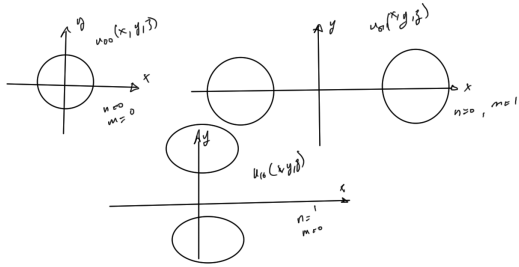

Figure 6.12: 2D SHO solutions.

Formal mapping We have an equivalence

$$z \to \frac{\hbar k_0}{m_{\text{eff}}} t. \tag{6.56}$$

We've seen quantities like $\hbar k/m$ in QM, as velocities. A specific example is the Gaussian with momentum space representation peaked around k_0 with variation Δk, or

$$f(k) \sim \exp\left(-\frac{(k-k_0)^2}{4(\Delta k)^2}\right). \tag{6.57}$$

In §4.4 [3] it is shown that the time evolution of the particle probability with this momentum space distribution has the form

$$|\Psi(x,t)|^2 \sim \exp\left(-\frac{x^2}{2\left((\Delta x)^2 + \hbar^2(\Delta k)^2 \frac{t^2}{m^2}\right)}\right). \tag{6.58}$$

The particle spreads with speed $\hbar \Delta k/m$.

$$m_{\text{eff}} = \frac{\hbar\omega}{c^2}. \tag{6.59}$$

$$m_{\text{eff}} c^2 = \hbar\omega. \tag{6.60}$$

This is a dispersion relation fig. 6.13.

$$\begin{aligned} E &= \hbar\omega \\ &= \hbar c\sqrt{k_x^2 + k_y^2 + k_z^2} \\ &\approx \text{constant} + \frac{1}{2}\frac{\hbar c k_x^2}{k} \\ &= \frac{\hbar^2 k_x^2}{2m_{\text{eff}}}, \end{aligned} \tag{6.61}$$

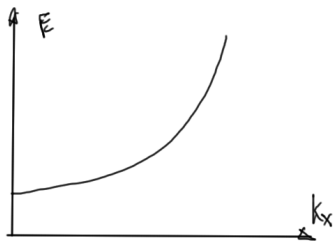

Figure 6.13: Dispersive electric field.

$$k_z \gg k_x, k_y. \tag{6.62}$$

6.4 QM VS. SPATIAL LIGHT EQUATIONS.

That last subsection of class notes wasn't entirely clear to me. Let's see if we can make more sense of things by comparing the Harmonic oscillator equation with this spatial light wave equation.

Classical SHO The classical SHO equation, in Hamiltonian form was

$$H = \frac{1}{2m}p^2 + \frac{1}{2}m\omega^2 x^2, \tag{6.63}$$

where the Hamiltonian equations are found from the canonical transformation $H = p\dot{x} - \mathcal{L}$, or

$$\frac{\partial H}{\partial p} = \dot{x} = \frac{p}{m}. \tag{6.64a}$$

$$\frac{\partial H}{\partial x} = -\dot{p} = m\omega^2 x. \tag{6.64b}$$

That is

$$(m\dot{x})' = -m\omega^2 x, \tag{6.65}$$

or

$$\ddot{x} = -\omega^2 x, \tag{6.66}$$

with solutions

$$x \propto e^{\pm i\omega t}. \tag{6.67}$$

Here $\omega = \sqrt{k/m}$, where k is the spring constant, and m is the mass characterizes the vibrations of the system. This makes me wonder, what is this characteristic angular velocity, for the SHO Schrödinger like form of the paraxial wave equation? Will we have something equivalent to the mass or spring constant in terms of our constants k_0, k_2?

QM SHO The QM form of the SHO was mentioned above in class. Let's put this in a non-dimensional form for comparison to the paraxial wave equation. After observing that

$$\left[\frac{m\omega}{\hbar}\right] = \frac{1}{\mathrm{L}^2}, \tag{6.68}$$

we can non-dimensionalize the QM SHO eq. (6.52) as

$$\frac{\hbar}{m\omega}\boldsymbol{\nabla}_{\mathrm{T}}^2\Psi - \frac{m\omega}{\hbar}r^2\Psi = -\frac{2i}{\omega}\frac{\partial\Psi}{\partial t}. \tag{6.69}$$

In non-dimensionalized form, with $\Psi = U(r)T(t)$ our separation of variables takes the form

$$\frac{1}{U}\left(\frac{\hbar}{m\omega}\boldsymbol{\nabla}_{\mathrm{T}}^2 U - \frac{m\omega}{\hbar}r^2 U\right) = -\frac{2i}{\omega}\frac{T'}{T} = -2\frac{E}{\hbar\omega}. \tag{6.70}$$

If we write our respective non-dimensionalized time and energy and radial distances as

$$\tau = \frac{\omega}{c}(ct). \tag{6.71a}$$

$$\epsilon = \frac{E}{\hbar\omega}. \tag{6.71b}$$

$$\xi = r\sqrt{\frac{m\omega}{\hbar}}. \tag{6.71c}$$

Our SHO now has just the spatial dependence

$$0 = \left(\frac{\partial^2}{\partial\xi^2} + \frac{1}{\xi}\frac{\partial}{\partial\xi} + \left(2\epsilon - \xi^2\right)\right)\Psi(\xi)e^{-i\epsilon\tau}. \tag{6.72}$$

Note that the separation of variables constant was specifically chosen so that we have the conventional time evolution

$$e^{-i\epsilon\tau} = e^{-i\frac{E}{\hbar}t}. \tag{6.73}$$

Spatial SHO for light in media Now let's non-dimensionalize the paraxial equation. Observing that

$$\left[\sqrt{k_0 k_2}\right] = \frac{1}{\mathrm{L}^2}. \tag{6.74a}$$

$$[k_0] = \frac{1}{\mathrm{L}}, \tag{6.74b}$$

we can put the paraxial equation eq. (6.49) in non-dimensionalized form

$$\frac{1}{\sqrt{k_0 k_2}}\nabla_{\mathrm{T}}^2 u - \sqrt{k_0 k_2} r^2 u = -2i\sqrt{\frac{k_0}{k_2}}\frac{\partial u}{\partial z} = -2\epsilon, \tag{6.75}$$

and introduce

$$\tau = \sqrt{\frac{k_2}{k_0}} z. \tag{6.76a}$$

$$\xi = r\left(k_0 k_2\right)^{1/4}. \tag{6.76b}$$

The paraxial light SHO now has just the transverse spatial dependence

$$0 = \left(\frac{\partial^2}{\partial \xi^2} + \frac{1}{\xi}\frac{\partial}{\partial \xi} + \left(2\epsilon - \xi^2\right) \right) u(\xi) e^{-i\epsilon\tau}, \tag{6.77}$$

exactly like our non-dimensionalized QM SHO eq. (6.72). Instead of time evolution of the form $e^{-i(\epsilon/\omega)(\omega/c)(ct)}$ we now have non-transverse z evolution in the form $e^{-i(\epsilon/\sqrt{k_2/k_0})\sqrt{k_2/k_0}z}$.

From this we see that we can make the identifications

$$\frac{\omega_{\text{eff}}}{c} \equiv \sqrt{\frac{k_2}{k_0}}. \tag{6.78a}$$

$$m_{\text{eff}} \equiv \frac{\hbar k_0}{c}, \tag{6.78b}$$

$$t_{\text{eff}} \equiv \frac{z}{c}. \tag{6.78c}$$

We see that our effective "spring constant" is $c\,\hbar k_2$, and our paraxial equation is put into exact correspondence with the QM SHO

$$-\frac{\hbar^2}{2m_{\text{eff}}}\nabla_{\text{T}}^2 u + \frac{1}{2} m_{\text{eff}} \left(\omega_{\text{eff}}\right)^2 r^2 u = i\,\hbar \frac{\partial u}{\partial t_{\text{eff}}}. \tag{6.79}$$

We can also write express our media's spatial dependence in terms of this effective angular velocity

$$k^2(r) = k_0^2 \left(1 - \left(\omega_{\text{eff}}\right)^2 r^2\right). \tag{6.80}$$

6.5 SOLVING THE HOMOGENEOUS PARAXIAL WAVE EQUATION.

We are going to start with the paraxial wave equation for a quadratic index profile

$$\nabla_{\text{T}}^2 u + 2ik\frac{\partial u}{\partial z} - k_0 k_2 r^2 u = 0, \tag{6.81}$$

as derived from the Helmholtz equation, assuming a wave function of the form

$$\Psi = \Psi_0 u(x, y, z) e^{ik_0 z}. \tag{6.82}$$

We now want to try to find approximate solutions for u. We'll ignore the fact that we know the solutions from QM, but use that knowledge try to pick a Gaussian as a trial function

$$u = \exp\left(ip(z) + i\frac{k_0 r^2}{2q(z)}\right). \tag{6.83}$$

This is a strange choice for a trial function, seemingly motivated by trying it at least once and then picking more tractable functions for the exponential argument. Perhaps try later without knowing the answer and see what motivates this strange selection.

With only z dependence in the functions $q = q(z)$, $p = p(z)$, we apply the operator equation to find

$$\begin{aligned}
0 &= \boldsymbol{\nabla}_T^2 u - k_0 k_2 r^2 u + 2ik_0 \frac{\partial u}{\partial z} \\
&= \left(\frac{\partial}{\partial r} + \frac{1}{r}\right)\frac{ik_0 r}{q} u - k_0 k_2 r^2 u + 2ik_0\left(ip' - \frac{ik_0 r^2}{2q^2} q'\right) u \\
&= u\left(\frac{ik_0}{q} - \frac{k_0^2 r^2}{q^2} + \frac{ik_0}{q} - k_0 k_2 r^2 + 2ik_0\left(ip' - \frac{ik_0 r^2}{2q^2} q'\right)\right) \\
&= u r^2 k_0^2 \left(-\frac{1}{q^2} - \frac{k_2}{k_0} + \frac{1}{q^2} q'\right) + 2k_0 u\left(\frac{i}{q} - p'\right).
\end{aligned} \tag{6.84}$$

Requiring equality for $r = 0$ gives us

$$\frac{dp}{dz} = \frac{i}{q}. \tag{6.85}$$

Now killing off the k_2 term (to be revisited in a subsequent lecture), we have

$$\begin{aligned}
0 &= \frac{1}{q^2} - \frac{1}{q^2} q' + \cancel{\frac{k_2}{k_0}} \\
&= \frac{1}{q^2}(1 - q').
\end{aligned} \tag{6.86}$$

so that

$$q = z + \text{constant} \equiv z + q_0 \equiv z - iz_0. \tag{6.87}$$

Integrating for $p(z)$ we have

$$\begin{aligned} p &= i \int \frac{dz}{z+q_0} \\ &= i \ln\left(\frac{z+q_0}{q_0}\right) + p(0), \end{aligned} \tag{6.88}$$

so that

$$\begin{aligned} e^{ip(z)} &= e^{ip(0)} \frac{q_0}{z+q_0} \\ &= e^{ip(0)} \frac{z_0}{iq}. \end{aligned} \tag{6.89}$$

Since $p(0)$ contributes only a constant phase term (that can be incorporated into our multiplicative constant phasor $\mathbf{E}_0$), we can set $p(0) = 1$. This gives us

$$\boxed{u(r,z) = \frac{z_0}{iq} \exp\left(i \frac{k_0 r^2}{2q(z)}\right).} \tag{6.90}$$

At $z = 0$ we have

$$\begin{aligned} u(r,0) &= \exp\left(-\frac{k_0 r^2}{2z_0}\right) \\ &= \exp\left(-\frac{\pi r^2}{\lambda z_0}\right) \\ &\equiv \exp\left(-\frac{r^2}{\omega_0^2}\right). \end{aligned} \tag{6.91}$$

We call w_0 the "waist" or the <u>beam waist</u>, equivalently defining q_0 and z_0 in terms of the beam waist

$$q_0 = -i\frac{w_0^2 k_0}{2} = -i\pi\frac{w_0^2}{\lambda} = -iz_0. \tag{6.92}$$

Inverting this last for z_0 gives

$$\boxed{z_0 = \frac{\pi w_0^2}{\lambda},} \tag{6.93}$$

which is called the <u>Raleigh range</u>.

Real value normalization We've got a bit of a mess of mixed up real and imaginary parts here. We can write this out instead as

$$\boxed{u = \frac{w_0}{w(z)} \exp\left(-\frac{r^2}{w^2(z)} + i\frac{k_0 r^2}{2R(z)} - i\phi(z)\right)} \tag{6.94}$$

where

$$\phi(z) = \operatorname{atan}\left(\frac{z}{z_0}\right). \tag{6.95}$$

$$w^2(z) = w_0^2\left(1 + \frac{z^2}{z_0^2}\right). \tag{6.96}$$

The quantity $w(z)$ is called the beam radius.

$$\frac{1}{R(z)} = \frac{z}{z^2 + z_0^2}. \tag{6.97}$$

The quantity $1/R(z)$ is the phase curvature. Let's verify that this is correct. Starting with the multiplicative term we have

$$\begin{aligned}
\frac{z_0}{iq(z)} &= \frac{z_0}{i(z - iz_0)} \\
&= \frac{z_0}{iz + z_0} \\
&= \frac{1}{1 + iz/z_0} \\
&= \frac{1}{\sqrt{1 + (z/z_0)^2}} e^{-i\operatorname{atan}(z/z_0)} \\
&= \frac{w_0}{w(z)} e^{-i\phi(z)}.
\end{aligned} \tag{6.98}$$

Now for the rest

$$\begin{aligned}
e^{\frac{ik_0 r^2}{2(z - iz_0)}} &= e^{ir^2 \frac{k_0}{z_0} \frac{1}{2(z/z_0 - i)}} \\
&= e^{ir^2 \frac{k_0}{z_0} \frac{z/z_0 + i}{2((z/z_0)^2 + 1)}} \\
&= e^{-r^2 \frac{k_0}{z_0} \frac{1}{2((z/z_0)^2 + 1)}} e^{ir^2 \frac{k_0}{z_0} \frac{z/z_0}{2((z/z_0)^2 + 1)}} \\
&= e^{-r^2 \frac{1}{\omega^2} \frac{1}{(z/z_0)^2 + 1}} e^{ir^2 \frac{k_0}{2} \frac{z}{z^2 + z_0^2}} \\
&= e^{-\frac{k_0 r^2}{\omega^2(z)}} e^{i\frac{r^2}{2R(z)}}.
\end{aligned} \tag{6.99}$$

This demonstrates the claimed identity eq. (6.94).

Plotting the envelope The real portion of the exponential determines the envelope. It's pointed out in the handouts that the constant surfaces

$$-k_0 \frac{r^2}{w_0^2\left(1+\frac{z^2}{z_0^2}\right)} = -k_0 C^2 = \text{constant}, \tag{6.100}$$

or

$$\frac{x^2}{w_0^2 C^2} + \frac{y^2}{w_0^2 C^2} - \frac{z^2}{z_0^2} = 1. \tag{6.101}$$

With $u = x/w_0 C, v = y/w_0 C, w = z/z_0$, this is plotted in fig. 6.14. Playing around with this a bit gaussianBeamHandoutNotes.nb shows that increasing w_0 reduces the pinch off in the center, and increasing z_0 narrows the beam.

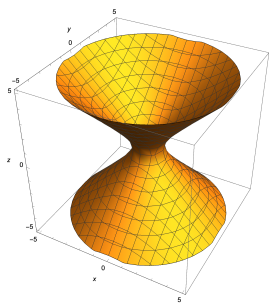

Figure 6.14: Gaussian beam envelope.

Some observations For $z \gg z_0$, $R \to z$, $w(z) \to \frac{w_0 z}{z_0}$. We see that

$$u e^{ik_0 z} \to \frac{z_0}{z} \exp\left(ik_0 z - \frac{r^2}{w^2} + i\frac{k_0}{2z} r^2\right). \tag{6.102}$$

Compare to wave emitted by a point source as in fig. 6.15.

$$\Psi \sim \frac{1}{R} e^{ikR}. \tag{6.103}$$

but

$$R^2 = z^2 + r^2. \tag{6.104}$$

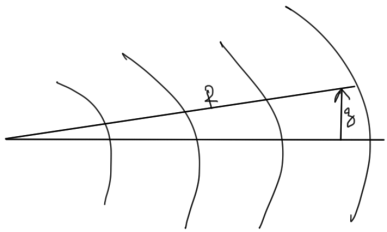

Figure 6.15: Point source.

Taylor expanding to first order

$$R \approx z + \frac{1}{2z} r^2. \tag{6.105}$$

we've got

$$\Psi \sim \frac{1}{z} \exp\left(ikz + ik\frac{r^2}{2z} \right). \tag{6.106}$$

So if we are looking at a point source slightly off axis, we have what looks like a Gaussian beam.

It looks like the Gaussian beam has an additional damping factor (with radius) that the point source does not.

Waist angular dependence Waist angular dependence is roughly illustrated in fig. 6.16. With

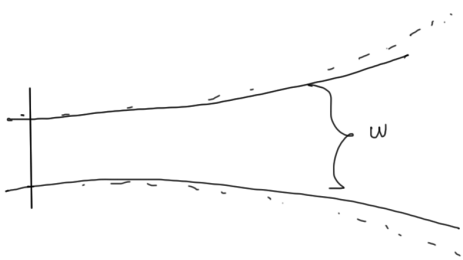

Figure 6.16: Waist angular dependence.

$$\begin{aligned} w(z) &= w_0\sqrt{1+\frac{z^2}{z_0^2}} \\ &\approx \frac{w_0}{z_0}z \\ &= \Theta_{\text{div}}z. \end{aligned} \tag{6.107}$$

Using eq. (6.93) we can write this as

$$\begin{aligned} \Theta_{\text{div}} &= \frac{w_0}{\pi w_0^2}\lambda \\ &= \frac{1}{\pi w_0}\lambda. \end{aligned} \tag{6.108}$$

6.6 GUOY PHASE SHIFTS, HIGHER ORDER MODES.

Review We'd found for the first order Gaussian beam

$$u_{00} = \frac{w_0}{w(z)}\exp\left(-\frac{r^2}{w^2(z)}+\frac{ik_0r^2}{R(z)}-i\phi(z)\right). \tag{6.109}$$

$$w^2(z) = w_0^2\left(1+\frac{z^2}{z_0^2}\right). \tag{6.110}$$

$$z_0 = \frac{\pi w_0^2}{\lambda}. \tag{6.111}$$

$$\frac{1}{R(z)} = \frac{z}{z^2+z_0^2}. \tag{6.112}$$

$$\phi(z) = \operatorname{atan}\left(\frac{z}{z_0}\right). \tag{6.113}$$

Our complete wave was

$$\Psi(x,y,z) = \Psi_0 u(x,y,z)e^{ik_0z-i\omega t}. \tag{6.114}$$

In particular, along the z-axis, where $\phi(0) = 0$ and $r^2 = 0$, we have

$$\Psi(0,0,z) = \Psi_0\frac{w_0}{w(z)}e^{ik_0z-i\omega t}. \tag{6.115}$$

Guoy phase shift Guoy's work was circa 1890. Was that experimental or theoretical work? Some references can be found in [15].

Let's consider the phase velocity at $z \ll z_0$.

We want to consider the constant curves

$$-i\phi(z) + ik_0 z - i\omega t = i\text{constant}. \tag{6.116}$$

See [13] for a quick and nicely written reminder of why we are looking at constant phase fronts.

Taking time derivatives of ϕ, using chain rule and eq. (6.186), we have

$$\frac{d\phi}{dt} = \frac{dz}{dt}\frac{d\phi}{dz} = \frac{dz}{dt}\frac{z_0}{z^2 + z_0^2}. \tag{6.117}$$

The time derivative of eq. (6.116) is then

$$-i\frac{dz}{dt}\frac{z_0}{z^2 + z_0^2} + ik_0\frac{dz}{dt} - i\omega = 0. \tag{6.118}$$

Solve for the phase velocity $V_{\text{ph}} = dz/dt$, we find

$$\boxed{\begin{aligned} \frac{dz}{dt} &= V_{\text{ph}} = \frac{\omega}{k_{\text{eff}}} \\ k_{\text{eff}} &= k_0 - \frac{z_0}{z^2 + z_0^2} \end{aligned}} \tag{6.119}$$

We can write this as

$$\frac{k_{\text{eff}}}{k_0} = 1 - \frac{2}{k_0^2\omega_0^2}\frac{1}{1 + (z/z_0)^2}, \tag{6.120}$$

We have $k_{\text{eff}} > 0$ provided $(z/z_0)^2 > 2/(k_0\omega_0)^2 - 1$. We have k_{eff} take its maximum of k_0 as $z \to \infty$, and takes its minimum value at $z = 0$ of

$$k_{\text{eff}} = k_0 - \frac{2}{k_0\omega_0^2}. \tag{6.121}$$

We plot k_{eff}/k_0 as a function of z/z_0 with $2/(k_0\omega_0^2) = 1/10$ in fig. 6.17. Since $k_{\text{eff}} < k_0$ for all z we have $V_{\text{ph}} > \omega/k_0$, or

$$V_{\text{ph}} > c. \tag{6.122}$$

Our phase velocity always exceeds the speed of light.

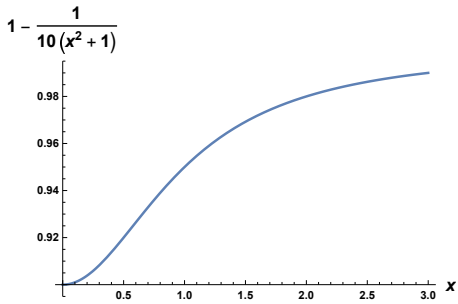

Figure 6.17: Effective Phase velocity.

Cavity

- Mode to be a solution of cavity. Mirror "undoes" propagation.
- Round trip phase shift is 2π(integer) for resonance.

fig. 6.18.

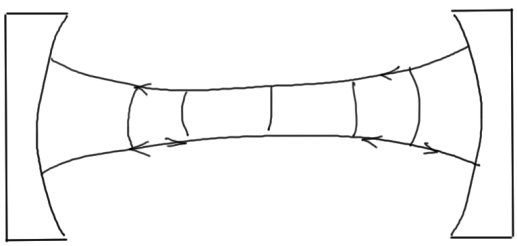

Figure 6.18: Gaussian mode confined in cavity by a set of mirrors.

Higher order modes

$$\begin{aligned} u_{lm}(x,y,z) \\ \sim \frac{w_0}{w(z)} \exp\left(-\frac{r^2}{w^2(z)} + \frac{ik_0 r^2}{R(z)} - i(m+l+1)\phi(z)\right) \times \\ H_l\left(\frac{\sqrt{2}x}{w(z)}\right) H_m\left(\frac{\sqrt{2}y}{w(z)}\right). \end{aligned} \tag{6.123}$$

$$
\begin{aligned}
H_0(x) &= 1 \\
H_1(x) &= 2x \\
H_2(x) &= 4x^2 - 1
\end{aligned}
\tag{6.124}
$$

Now get

$$
k_{\text{eff}} = k_0 - (m + l + 1)\frac{z_0}{z^2 + z_0^2}. \tag{6.125}
$$

Beam parameter We want to look at how the Gaussian beam interacts with mirrors, as in fig. 6.19, to get an idea of how the beam will behave in a cavity (without starting over at the Helmholtz equation). Bringing back in our q notation It turns out that we can

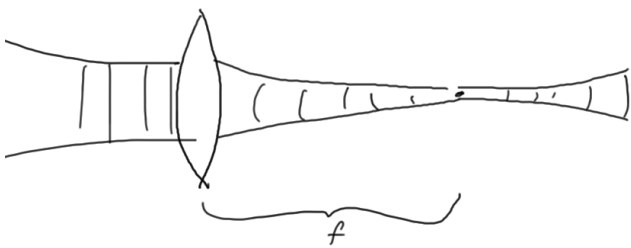

Figure 6.19: Gaussian modes confined to mirror cavity.

consider an equivalent system of mirrors in series as in fig. 6.20.

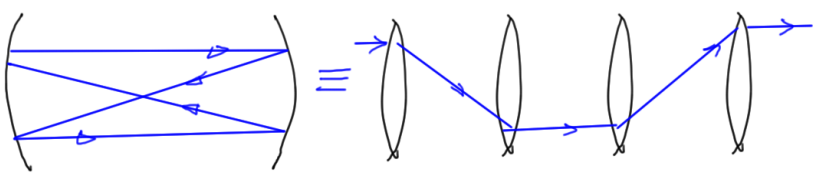

Figure 6.20: Equivalent to mirror cavity.

$$
\frac{1}{q(z)} = \frac{1}{R(z)} + i\frac{\lambda}{\pi w^2(z)}. \tag{6.126}
$$

where Re $\frac{1}{q(z)}$ gives curvature, and Im $\frac{1}{q(z)}$ gives beam radius.

Now

$$u_{lm} = \frac{c_{lm}}{w(z)} H_l H_m e^{\frac{ik_0 r^2}{2q}} e^{-i(l+m+1)\phi}. \tag{6.127}$$

$$u_{00} = \frac{w_0}{w(z)} e^{\frac{ik_0 r^2}{2q}} e^{-i\phi}. \tag{6.128}$$

Found in uniform medium

$$q(z) = z - iz_0. \tag{6.129}$$

Know that if q_1 at some position z_1 then at z_2

$$q_2 = q_1 + (z_2 - z_1). \tag{6.130}$$

Möbius Transform

$$q' = \frac{Aq + B}{Cq + D}. \tag{6.131}$$

where coefficients same as we used in geometric optics

i.e. A, B, C, D transformation, such as that of a lens:

$$M = \begin{bmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{bmatrix} \tag{6.132}$$

($A = 1$, $B = 0$, $C = -1/f$, $D = 1$).

This happens to be (not to be proven) that this is exactly how a Gaussian lens behaves when it encounters a lens/mirror/...

For a lens interaction we have

$$q' = \frac{(1)q + (0)}{(-1/f)q + (1)} = \frac{1}{-\frac{1}{f} + \frac{1}{q}}. \tag{6.133}$$

Example 6.1: Check for free propagation

$$M = \begin{bmatrix} 1 & L \\ 0 & 1 \end{bmatrix} \tag{6.134}$$

$$q' = \frac{(1)q + (L)}{(0)q + (1)} = q + L. \tag{6.135}$$

which is what we know from eq. (6.130).

Example 6.2: With lens transfer matrix

For the lens transformation of eq. (6.133) we have

$$\frac{1}{q'} = \frac{1}{q} - \frac{1}{f}. \tag{6.136}$$

so that

$$u = e^{\frac{ik_0 r^2}{2q}} \to u e^{-\frac{ik_0 r^2}{2f}}. \tag{6.137}$$

Structure of Möbius transformation for rays The Van Driel notes make a nice observation about the relation between the Möbius transformation and the matrix transformation. For

$$\begin{bmatrix} y' \\ \alpha' \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} y \\ \alpha \end{bmatrix} \tag{6.138}$$

We can relate y'/α' and y/α as follows

$$y' = Ay + B\alpha. \tag{6.139a}$$

$$\alpha' = Cy + D\alpha. \tag{6.139b}$$

So that

$$\frac{y'}{\alpha'} = \frac{Ay + B\alpha}{Cy + D\alpha}. \tag{6.140}$$

This takes some of the mystery about the equivalence out of the picture since we see the structure of the Möbius transformation in this ratio, even for plain old rays. It would still be nice to see a proof of exactly how that applies to the Gaussian beams.

Cavity stability See: slides.

The criteria is

$$0 < g_1 g_2 < 1. \tag{6.141}$$

where we introduce dimensionless quantities

$$g_{1,2} = 1 - \frac{L}{R_{1,2}}. \tag{6.142}$$

6.7 SPECTRAL LINE WIDTH (COHERENCE TIME) OF LASER.

READING: [7] Appendix C. The spectral linewidth (or coherence time) of a laser is called the Schawlow-Townes limit.

Laser depicted in fig. 6.21.

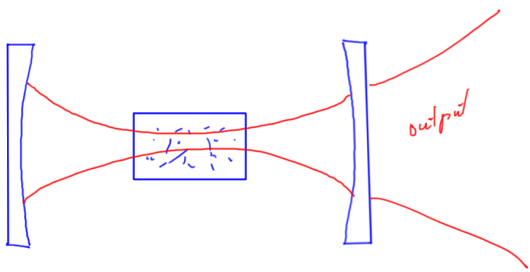

Figure 6.21: Laser cavity.

$$\text{rate of emission into laser mode} = \Gamma_{st}(\underbrace{1}_{\text{spontaneous}} + \underbrace{\langle n \rangle}_{\text{stimulated}})N_2. \tag{6.143}$$

Reminder:

$$\tau_c = \text{coherence time}. \tag{6.144}$$

$$\Gamma_{st} = \frac{1}{\tau_c} = \text{line width}. \tag{6.145}$$

$$c\tau_c = l_c = \text{longitudinal coherence length.} \tag{6.146}$$

Phase of a laser, if monochromatic as in fig. 6.22. goes like

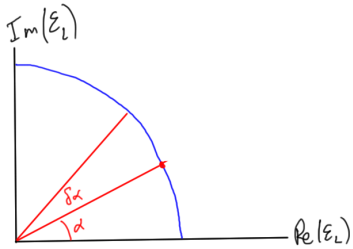

Figure 6.22: Field associated with ground state.

$$\Psi \sim e^{-i\omega_0 t}. \tag{6.147}$$

If not monochromatic fig. 6.23, define

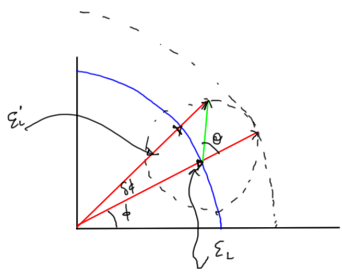

Figure 6.23: Random walk in phase.

$$\boldsymbol{\mathcal{E}}_{\mathrm{L}} = E_{\mathrm{L}} e^{i\phi(t)}. \tag{6.148}$$

(underscore L stands for laser).

$$\Psi = \boldsymbol{\mathcal{E}}_{\mathrm{L}} e^{-i\omega t} \times \text{spatial mode.} \tag{6.149}$$

Random relative phase θ. Two components of $\boldsymbol{\mathcal{E}}_{\mathrm{sp}}$

1. Parallel to $\boldsymbol{\mathcal{E}}_{\mathrm{L}}$ changes amplitude, taken out by steady state operation of laser.

2. perpendicular: changes ϕ by $\delta\phi$.

Considering the average process we have random phase changes, with no effective change in magnitude, or

$$\mathcal{E}'_{\mathrm{L}} \sim \mathcal{E}_{\mathrm{L}} e^{i\delta\phi}. \tag{6.150}$$

This is a random walk fig. 6.24 in phase! Average step:

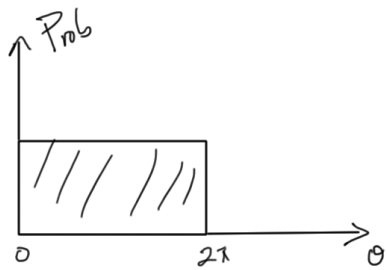

Figure 6.24: Angle is a random variable.

$$\langle \delta\phi \rangle = \frac{E_{\mathrm{sp}}}{E_{\mathrm{L}}} \overbrace{\boxed{\langle \cos\phi \rangle}}^{=0}. \tag{6.151}$$

However, the RMS deviation

$$\begin{aligned} \sigma_\phi &= \sqrt{\langle (\delta\phi)^2 \rangle - \langle \delta\phi \rangle^2} \\ &= \sqrt{\frac{E_{\mathrm{sp}}^2}{E_{\mathrm{L}}^2} \overbrace{\boxed{\left\langle \cos^\theta \right\rangle}}^{1/2} - 0} \\ &= \frac{E_{\mathrm{sp}}}{\sqrt{2} E_{\mathrm{L}}}. \end{aligned} \tag{6.152}$$

After N events

$$\sigma_\phi = \frac{E_{\mathrm{sp}}}{\sqrt{2} E_{\mathrm{L}}} \sqrt{N}. \tag{6.153}$$

The number of spontaneous events is given by the emission rate of the laser mode, and is

$$\text{number of spontaneous events} = T\Gamma_{\mathrm{st}} N_2. \tag{6.154}$$

Also

$$\langle n \rangle = \frac{E_L^2}{E_{sp}^2}. \tag{6.155}$$

After time $T = \tau_c$, $\sigma_\phi = 1$, or

$$\frac{E_{sp}}{\sqrt{2}E_L}\sqrt{\tau_c \Gamma_{st} N_2} = 1. \tag{6.156}$$

Referring to fig. 6.25 we have

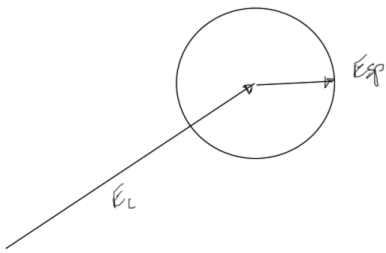

Figure 6.25: Change in field due to spontaneous emission.

$$\frac{E_{sp}}{E_L} = \frac{1}{\sqrt{\langle n \rangle}}. \tag{6.157}$$

or

$$\frac{1}{\sqrt{2}\sqrt{\langle n \rangle}}\sqrt{\tau_c \Gamma_{st} N_2} = 1, \tag{6.158}$$

so

$$\tau_c = \left(\frac{N_2 \Gamma_{st}}{2\langle n \rangle}\right)^{-1}. \tag{6.159}$$

$$\Gamma_{st} = \frac{1}{\tau_c} = \frac{N_2 \Gamma_{st}}{2\langle n \rangle}. \tag{6.160}$$

Energy balance

$$N_2 = \frac{\Gamma_{cav}}{\Gamma_{st}}. \tag{6.161}$$

so

$$\tau_c = \left(\frac{\Gamma_{\text{cav}}}{2\langle n\rangle}\right)^{-1}. \tag{6.162a}$$

$$\Gamma_{\text{st}} = \frac{1}{\tau_c} = \frac{\Gamma_{\text{cav}}}{2\langle n\rangle}. \tag{6.162b}$$

With power output

$$P = \hbar\omega\langle n\rangle\Gamma_{\text{cav}}. \tag{6.163}$$

$$\Gamma_s t = \frac{\Gamma_{\text{cav}}\hbar\omega}{P}. \tag{6.164}$$

$$\Gamma_{\text{cav}} = (\text{Trans})\frac{c}{2L}. \tag{6.165}$$

where L is the cavity length.

Example 6.3: Some numbers

$$\begin{aligned} P &\sim 1\text{mW} \\ L &\sim 1\text{m} \\ T &\sim 2\% \\ \lambda &\sim 500\text{nm} \\ \langle n\rangle &= 8\times 10^8 \\ \Gamma_s t &= 4\times 10^{-3}\text{s}^-1 \end{aligned} \tag{6.166}$$

or about $1m$Hz.

Example 6.4: Some numbers for Diode laser

$$\begin{aligned} L &\sim 300\mu m \\ T &\sim 10\% \end{aligned} \tag{6.167}$$

6.8 NUMBER OF PHOTONS PER FREE SPACE MODE.

Laser is fundamentally characterized by a large number of photons per free space mode. Possible because photons are bosons! Laser light has temporal and spatial coherence.

Review: in a cavity

- Laser light: $\langle n \rangle \sim 10^7 - 10^8$ above threshold fig. 6.26.
- Thermal light: $\langle n \rangle = \frac{1}{e^{\hbar\omega/k_B T} - 1}$.

Example 6.5: Some numbers

$$\langle n \rangle \sim 1. \tag{6.168}$$

$$\hbar\omega \sim k_B T \ln 2. \tag{6.169}$$

$$300\text{K} : \lambda > 70\mu\text{m}. \tag{6.170}$$

$$5000\text{K} : \lambda > 4\mu\text{m}. \tag{6.171}$$

3D Free space mode. Consider "elementary pencil" of light as in fig. 6.27, chosen to be diffraction limited. Here

$$(\Delta x \Delta k_x)_{\min} \sim 1 \tag{6.172}$$

$$(\Delta y \Delta k_y)_{\min} \sim 1 \tag{6.173}$$

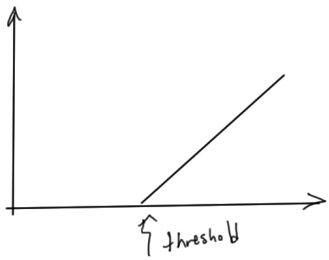

Figure 6.26: Threshold.

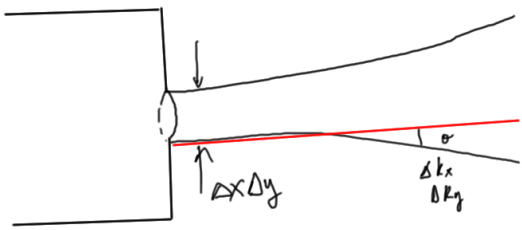

Figure 6.27: Pencil of light.

Figure 6.28: Pulse train.

where 1 here means a constant of order 1.
Along 2: consider pulse train as in fig. 6.28. where $\Delta z = c\tau_0$

$$\Psi(t) = \begin{cases} e^{-i\omega_0 t} & t \in [-\tau_0/2, \tau_0/2] \\ 0 & \text{otherwise} \end{cases} \tag{6.174}$$

What is Δk_z? Making a paraxial approximation

$$\Delta k_z \approx \Delta k = \frac{\Delta\omega}{c}. \tag{6.175}$$

We can compute the Fourier transform as depicted in fig. 6.29.

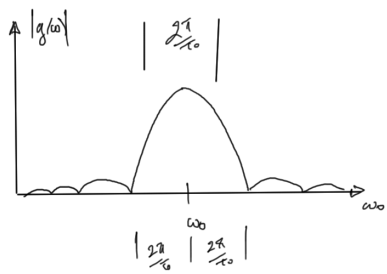

Figure 6.29: Fourier transform of pulse train.

$$g(\omega) = \int e^{i\omega t}\Psi(t)dt = z\frac{\sin\left((\omega-\omega_0)\tau_0/2\right)}{\omega-\omega_0}. \tag{6.176}$$

So

$$\Delta z \Delta k_z = (c\tau_0)\left(\frac{2\pi}{c\tau_0}\right) = 2\pi. \tag{6.177}$$

Now consider source that has width Γ. Imagine emission of pulses of length $\tau_0 = 2\pi/\Gamma$ fig. 6.30.

Laser light :

$$\frac{\text{number of photons}}{\text{free space mode}} = \frac{P/\hbar\omega}{\Gamma_{\text{laser}}} \sim \frac{\Gamma_{\text{cav}}\langle n\rangle}{\Gamma_{\text{cav}}/\langle n\rangle} \sim \langle n\rangle^2. \tag{6.178}$$

Justifying this last operation fig. 6.31.

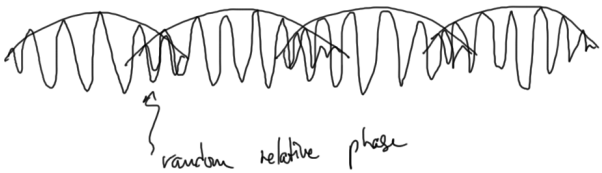

Figure 6.30: Overlapping pulses.

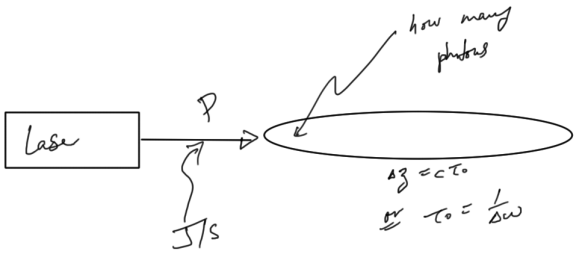

Figure 6.31: Why this division?

6.9 PROBLEMS.

Exercise 6.1 arctan derivative.

Calculate

$$\frac{d}{dx}\operatorname{atan}(x). \tag{6.179}$$

Answer for Exercise 6.1

with

$$f(x) = \operatorname{atan}(x), \tag{6.180}$$

We have

$$\tan f = x. \tag{6.181}$$

Taking derivatives

$$\frac{f'}{\cos^2(f)} = 1, \tag{6.182}$$

or

$$f' = \cos^2(f). \tag{6.183}$$

We've also got

$$x^2 = \tan^2 f = \frac{1 - \cos^2 f}{\cos^2 f}. \tag{6.184}$$

or

$$f' = \cos^2 f = \frac{1}{1 + x^2}. \tag{6.185}$$

Our $\phi(z)$ derivative then follows

$$\begin{aligned} \frac{d}{dz} \operatorname{atan}(z/z_0) &= \frac{d}{du} \operatorname{atan}(u) \bigg|_{u=z/z_0} \frac{d(z/z_0)}{dz} \\ &= \frac{1}{1 + (z/z_0)^2} \frac{1}{z_0} \\ &= \frac{z_0}{z_0^2 + z^2}. \end{aligned} \tag{6.186}$$

Exercise 6.2 Derive expression for $1/q$.

We have an expression for $1/q$ in terms of the radius of curvature and waist functions in the handouts. Derive this.

Answer for Exercise 6.2

$$\begin{aligned} \frac{1}{q} &= \frac{1}{z - iz_0} \\ &= \frac{z + iz_0}{z^2 + z_0^2} \\ &= \frac{1}{R} + \frac{i}{z_0(1 + z^2/z_0^2)} \\ &= \frac{1}{R} + \frac{iw_0^2}{z_0 w^2(z)} \\ &= \frac{1}{R} + \frac{i\lambda z_0}{z_0 \pi w^2(z)}. \end{aligned} \tag{6.187}$$

or

$$\boxed{\frac{1}{q} = \frac{1}{R} + \frac{i\lambda}{\pi w^2(z)}.} \tag{6.188}$$

Exercise 6.3 Paraxial wave equation, Fresnel form.

In some supplementary class notes, it is stated that

$$h(x,y,z) = \frac{1}{z} e^{ikz} e^{ik(x^2+y^2)/2z}. \tag{6.189}$$

is an exact solution to the paraxial wave equation

$$\nabla_{\mathrm{T}}^2 u + 2ik \frac{\partial u}{\partial z} = 0. \tag{6.190}$$

From our lectures, this doesn't seem possible, since we found that this Fresnel like function was an approximation to the u_{00} function for large z. Calculate this directly and verify this suspicion.

Answer for Exercise 6.3

Let's first apply the ∂_{xx} portion of the transverse Laplacian. We find

$$\begin{aligned}
\frac{\partial^2 h}{\partial x^2} &= \frac{\partial}{\partial x}\frac{\partial}{\partial x}\left(\frac{1}{z} e^{ikz} e^{ik(x^2+y^2)/2z}\right) \\
&= \frac{1}{z} e^{ikz} e^{iky^2/2z} \frac{\partial}{\partial x}\frac{\partial}{\partial x}\left(e^{ikx^2/2z}\right) \\
&= \frac{1}{z} e^{ikz} e^{iky^2/2z} \frac{\partial}{\partial x}\left(\frac{ikx}{z} e^{ikx^2/2z}\right) \\
&= \frac{1}{z} e^{ikz} e^{iky^2/2z} \left(\frac{ik}{z} - \frac{k^2x^2}{z^2}\right) e^{ikx^2/2z} \\
&= \left(\frac{ik}{z} - \frac{k^2x^2}{z^2}\right) h.
\end{aligned} \tag{6.191}$$

This gives us, for $r^2 = x^2 + y^2$

$$\nabla_{\mathrm{T}}^2 h = \left(\frac{2ik}{z} - \frac{k^2r^2}{z^2}\right) h. \tag{6.192}$$

For the first partial with respect to z we find

$$\begin{aligned}
\frac{\partial h}{\partial z} &= -\frac{1}{z^2} e^{ikz} e^{ikr^2/2z} + \frac{1}{z}\left(ik - \frac{ikr^2}{2z^2}\right) e^{ikz} e^{ikr^2/2z} \\
&= \left(-\frac{1}{z} + ik - \frac{ikr^2}{2z^2}\right) h.
\end{aligned} \tag{6.193}$$

Putting things together we have

$$\left(\nabla_{\mathrm{T}}^2 + 2ik\frac{\partial}{\partial z}\right) h = \left(\cancel{\frac{2ik}{z}} - \cancel{\frac{k^2 r^2}{z^2}} + 2ik\left(\cancel{\frac{1}{z}} + ik - \cancel{\frac{ikr^2}{2z^2}}\right)\right) h \tag{6.194}$$
$$= -2k^2 h \neq 0.$$

However, since $h \to 0$ as $z \to \infty$, this does at least give zero in the far z limit.

Exercise 6.4 Gaussian beam and lens. (*2012 Ps4, P1*)

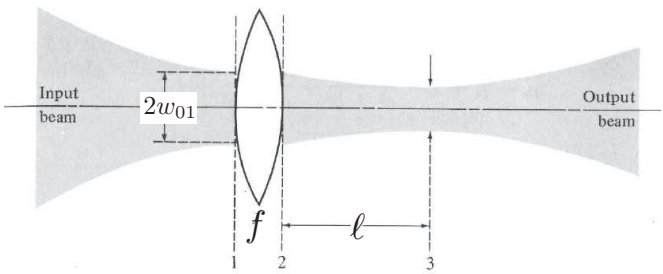

Figure 6.32: Gaussian beam.

Consider a Gaussian beam whose waist is w_{01} and placed at a lens of focal length f. This lens makes a focus some distance ℓ away.

a. Find the distance Find the distance ℓ, in terms of w_{01}, f, and λ. In what limit does the lens create a focus at $\ell = f$, as geometric optics would have predicted? Interpret this physically.

b. new beam waist Find the new beam waist w_{03}. In the same limit where $\ell \approx f$, show that the waist is $w_{03} \approx \lambda f / \pi w_{01}$.

Answer for Exercise 6.4

Part a. The effective focal distance Following the argument in [16].

The geometry of the waist is determined from the imaginary portion of the exponential argument. We need to find break down q'' into real (the transformed radius of curvature) and imaginary parts, as in

$$\frac{1}{q''} = \frac{1}{R''} + \frac{i\lambda}{\pi\left(w''(z)\right)^2} = \frac{1}{q' + l}. \tag{6.195}$$

We saw in part b (done in opposite order) that we had

$$q' = \frac{qf}{f-q}. \tag{6.196}$$

Picking a fixed value of z (say $z = 0$), that is

$$q' = \frac{-iz_0 f}{f-(-iz_0)} = \frac{-iz_0 f}{f+iz_0}, \tag{6.197}$$

so that our (inverse) post propagation transformation of q is

$$\begin{aligned}
\frac{1}{q''} &= \frac{1}{\frac{-iz_0 f}{f+iz_0}+l}\frac{f+iz_0}{-iz_0 f+l(f+iz_0)} \\
&= \frac{f+iz_0}{iz_0(l-f)+lf} \\
&= \frac{(f+iz_0)(-iz_0(l-f)+lf)}{z_0^2(l-f)^2+l^2f^2} \\
&= \frac{lf^2+z_0^2(l-f)}{z_0^2(l-f)^2+l^2f^2} + iz_0\frac{lf-l+f}{z_0^2(l-f)^2+l^2f^2}.
\end{aligned} \tag{6.198}$$

If we consider this far enough away that the beam is planar, with an infinite radius of curvature, then we require the numerator of the $1/R''$ expression above to be zero. That is

$$l(f^2+z_0^2) = z_0^2 f, \tag{6.199}$$

or

$$l = \frac{f}{1+\frac{f^2}{z_0^2}}. \tag{6.200}$$

with $z_0 = \pi w_0^2/\lambda$ this is

$$\boxed{l = \frac{f}{1+\frac{f^2\lambda^2}{\left(\pi w_{01}^2\right)^2}}.} \tag{6.201}$$

We have $l \approx f$ when

$$\boxed{\lambda \ll \frac{\pi w_{01}^2}{f}.} \tag{6.202}$$

Observe that when this condition is met, our radius of curvature is no longer infinite, and we can't consider the beam to be plane wave like. That radius of curvature is instead, exactly the focal length

$$\frac{1}{R''} = \frac{lf^2}{l^2 f^2} = \frac{1}{f}. \tag{6.203}$$

Grading remarks: -2 . Two remarks here, what was "so?" after "the focal length" above, and the other was "the ray optics result is valid for plane waves and therefore for large R!!"

Part b. The waist Observe that the we can write the waist function as a function of q

$$\begin{aligned} w^2(z) &= w_0^2 \left(1 + \frac{z^2}{z_0^2}\right) \\ &= \frac{w_0^2}{z_0^2}(z_0^2 + z^2) \\ &= \frac{w_0^2}{z_0^2}|q(z)|^2. \end{aligned} \tag{6.204}$$

Also observe that $w(0) = w(0)$, so if we are looking for the beam waist after a geometric (Möbius) transformation on q, we can find the new beam waist, looking at the $z = 0$ value.

After just the lens After transmission through the lens, with ABCD matrix

$$M = \begin{bmatrix} 1 & 0 \\ -1/f & 1 \end{bmatrix} \tag{6.205}$$

$$q' = \frac{q + 0}{-q/f + 1}, \tag{6.206}$$

or

$$\begin{aligned} \frac{1}{q'} &= \frac{-q/f + 1}{q} \\ &= \frac{1}{q} - \frac{1}{f}. \end{aligned} \tag{6.207}$$

At $z = 0$ we have

$$\begin{aligned}\left|\frac{1}{q'}\right|^2 &= \left|\frac{1}{-iz_0} - \frac{1}{f}\right| \\ &= \frac{1}{z_0^2} + \frac{1}{f^2}.\end{aligned} \tag{6.208}$$

So that

$$\begin{aligned}w^2(0) &\to \frac{w_{01}^2}{z_0^2}\left(\frac{1}{z_0^2} + \frac{1}{f^2}\right)^{-1} \\ &= \frac{w_{01}^2}{z_0^2}\frac{f^2 z_0^2}{f^2 + z_0^2} \\ &= \frac{w_{01}^2}{1 + z_0^2/f^2}.\end{aligned} \tag{6.209}$$

We see that the beam waist after transmission through the lens is reduced by a factor of

$$\frac{w_{02}}{w_{01}} = \frac{1}{\sqrt{1 + z_0^2/f^2}}. \tag{6.210}$$

After the lens and the free propagation. With transmission through the air and the lens, our free propagation matrix is

$$M_2 = \begin{bmatrix} 1 & l \\ 0 & 1 \end{bmatrix} \tag{6.211}$$

$$\begin{aligned}q &\to \frac{q' + l}{0q' + 1} \\ &= q' + l \\ &= \frac{q}{-q/f + 1} + l,\end{aligned} \tag{6.212}$$

so that the waist function is transformed as

$$w^2(z) \to \frac{w_{01}^2}{z_0^2}\left|\frac{q}{-q/f + 1} + l\right|^2, \tag{6.213}$$

and our waist goes as

$$\begin{aligned} w_{01}^2 &\to \frac{w_{01}^2}{z_0^2}\left|\frac{-iz_0 f}{iz_0+f}+l\right|^2 \\ &= \frac{w_{01}^2}{z_0^2}\left|\frac{-iz_0 f\left(-iz_0+f\right)}{z_0^2+f^2}+l\right|^2 \\ &= \frac{w_{01}^2}{z_0^2}\left|-\frac{z_0^2 f}{z_0^2+f^2}+l-\frac{iz_0 f^2}{z_0^2+f^2}\right|^2. \end{aligned} \tag{6.214}$$

This gives us

$$\boxed{\frac{w_{03}^2}{w_{01}^2} = \frac{\left(z_0^2\left(l-f\right)+f^2 l\right)^2+\left(f^2 z_0\right)^2}{z_0^2\left(z_0^2+f^2\right)^2}.} \tag{6.215}$$

Grading note: -2 "I'm confused how can transmission through air change the waist?" See in the posted solution that the waist is calculated as the point where $\mathrm{Re}(1/q) = 0$. That choice isn't obvious to me. I think some more thought about the geometry of these solutions is required to really get this.

When $l = f$ the square root simplifies nicely, leaving

$$\frac{w_{03}^2}{w_{01}^2} = \frac{f^6+f^4 z_0^2}{z_0^2\left(z_0^2+f^2\right)^2} = \frac{f^4}{z_0^2\left(z_0^2+f^2\right)}, \tag{6.216}$$

or

$$\begin{aligned} w_{03} &= \frac{w_{01} f^2}{z_0\sqrt{z_0^2+f^2}} \\ &= \frac{w_{01} f}{z_0\sqrt{z_0^2/f^2+1}} \\ &= \frac{w_{01} f\lambda}{\pi w_{01}^2\sqrt{z_0^2/f^2+1}} \\ &= \frac{f\lambda}{\pi w_{01}}\frac{1}{\sqrt{z_0^2/f^2+1}}. \end{aligned} \tag{6.217}$$

When $f \gg z_0$ we have the desired result

$$\boxed{w_{03} \approx \frac{f\lambda}{\pi w_{01}}} \tag{6.218}$$

Grading note: The $f \gg z_0$ was underlined with the question "where did you get this from?" The solution points out that this is the Fraunhofer condition and gives some interpretation.

Exercise 6.5 Stability in a cavity. (*2012 Ps4, P2*)

Using ABCD matrices, derive the condition for stability of a Gaussian beam in a cavity.

a. Assuming symmetric beams. Follow Fowles §10.5 to find Eq. (10.32).
b. Stability criterion. Fowles derives the stability criterion from the eigen*values*. Find the rays that are eigen*vectors* of a single-pass ray matrix for the planar case ($L/R = 0$). Comment on why they are the only reasonable choice.
c. Unequal mirror radii. Allowing for unequal mirror radii, derive Fowles Eq. (10.33).

Answer for Exercise 6.5

Part a. Assuming symmetric beams Given the Möbius transform relationship the Gaussian beams, we can first consider the stability conditions for powers of the transfer matrix itself to not diverge. This follows [5], filling in some additional details.

The matrix for a single pass of free propagation through distance d, and then reflection off of a curved mirror with focus f is the composition

$$M = \begin{bmatrix} 1 & 0 \\ -1/f & 1 \end{bmatrix} \begin{bmatrix} 1 & d \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & d \\ -1/f & -d/f+1 \end{bmatrix}. \tag{6.219}$$

Each pass of propagation and reflection adds another power of M to the matrix for which we will base the final Möbius transformation on. We'll want to perform a diagonalization to simplify that matrix exponentiation (computation of M^n, so that if

$$ME = ED, \tag{6.220}$$

or

$$M = EDE^{-1}. \tag{6.221}$$

Here D is a diagonal matrix (with the eigenvalues on the diagonal), and E is the change of basis matrix to make that similarity transformation.

We can express the final matrix transformation after n reflections directly

$$M^n = ED^nE^{-1}. \tag{6.222}$$

We'll need the eigenvalues first. Our characteristic equation is

$$\begin{aligned} 0 &= |M - \lambda I| \\ &= \begin{vmatrix} 1-\lambda & d \\ -1/f & 1-d/f-\lambda \end{vmatrix} \\ &= 1 - \cancel{d/f} - \lambda - \lambda(1-d/f-\lambda) + \cancel{d/f} \\ &= \lambda^2 - 2\lambda\left(1 - \frac{d}{2f}\right) + 1. \end{aligned} \tag{6.223}$$

Following Fowles, we write

$$\alpha = 1 - \frac{d}{2f}, \tag{6.224}$$

So that the characteristic equation is

$$\begin{aligned} 0 &= \lambda^2 - 2\lambda\alpha + 1 \\ &= (\lambda - \alpha)^2 + 1 - \alpha^2. \end{aligned} \tag{6.225}$$

Note that this corrects a sign error in the text. Solving for λ we have

$$\lambda_\pm = \alpha \pm \sqrt{\alpha^2 - 1}. \tag{6.226}$$

Observe that these satisfy our expectation that $\lambda_+\lambda_- = 1$, so we can write these as

$$\lambda_+ = \lambda \tag{6.227}$$

$$\lambda_- = 1/\lambda, \tag{6.228}$$

for some value λ. Our ABCD matrix for n sets of propagate-and-reflect is

$$M^n = E \begin{bmatrix} \lambda^n & 0 \\ 0 & \frac{1}{\lambda^n} \end{bmatrix} E^{-1}. \tag{6.229}$$

If $E = [e_{ij}]$ and $E^{-1} = [f_{ij}]$, then the product takes the value

$$\begin{aligned} ED^nE^{-1} &= [e_{ik}\lambda_k^n\delta_{km}f_{mj}] \\ &= [e_{im}\lambda_m^n f_{mj}] \\ &= [e_{i1}\lambda^n f_{1j} + e_{i2}\frac{1}{\lambda^n}f_{2j}], \end{aligned} \tag{6.230}$$

or

$$M^n = \lambda^n[e_{i1}f_{1j}] + \frac{1}{\lambda^n}[e_{i2}f_{2j}]. \tag{6.231}$$

So, if λ is real and greater than 1, the ABCD matrix will start to grow without bound.

To consider the bounding behavior of λ^n, lets follow Fowles, and separate the eigenvalues into real and complex as follows

$$\lambda = \alpha \pm \begin{cases} \sqrt{\alpha^2 - 1} & \text{if } |\alpha| > 1 \\ i\sqrt{1-\alpha^2} & \text{if } |\alpha| < 1 \end{cases} \tag{6.232}$$

When $|\alpha| = 1$, we have a double eigenvalue with value α and may not be able to find a spanning set of eigenvectors. For the $|\alpha| < 1$ case, we can introduce ϕ such that $\cos\phi = \alpha$, allowing us to write

$$\lambda_\pm = e^{\pm i\phi}. \tag{6.233}$$

What can we say about the real valued eigenvalue case? We know that from $\det M = 1$ both real valued eigenvalues must have matching signs. We can also see from

$$\lambda_\pm = |\alpha|\left(\operatorname{sgn}(\alpha) \pm \sqrt{1 - \frac{1}{\alpha^2}}\right), \tag{6.234}$$

that if these are positive, then one of the eigenvalues is greater than 1, and if negative, at least one is less than -1. If we pick λ as the eigenvalue for which $|\lambda| > 1$, then λ^n will clearly diverge as n grows, and we the ABCD matrix of eq. (6.231) becomes unstable.

This can also be expressed in terms of rays. Given a ray with an eigenvector decomposition

$$\mathbf{x} = a\mathbf{e}_1 + b\mathbf{e}_2, \tag{6.235}$$

where

$$E = [\mathbf{e}_1 \mathbf{e}_2], \tag{6.236}$$

then it follows from eq. (6.231) that

$$M^n \mathbf{x} = a\lambda^n \mathbf{e}_1 + b\frac{1}{\lambda^n}\mathbf{e}_2. \tag{6.237}$$

Both the position and the angle increase without bound, leading the ray out of the cavity if $|\alpha| > 1$.

I'd avoided the "suppose that the ray is an eigenvector" argument from Fowles initially above because the eigenvectors in the complex case are also complex, and that didn't seem physically realistic. It also wasn't clear to me what $\lambda^n \mathbf{x}$ was if $\mathbf{x}$ was a ray and λ was complex (i.e. Eq. (10.32) in the text). However, what resolves this conundrum is the realization that the projection constants a and b above are also complex in this case. So while, for complex eigenvectors we won't ever have a ray that is given exactly by an eigenvector, we can still form a superposition of the two. The result, and the transformation of eq. (6.237), is still necessarily real valued.

The stability criteria Our last task is to express the stability condition $|\alpha| < 1$ in terms of d, f and r (the last for the optical resonator with equal curvature mirrors).

This stability criterion is

$$|\alpha| = \left|1 - \frac{d}{2f}\right| < 1. \tag{6.238}$$

Let's consider this in two separate cases.

1. The condition $\alpha > 0$ can be written

$$1 - \frac{d}{2f} > 0, \tag{6.239}$$

or

$$1 > \frac{d}{2f}, \tag{6.240}$$

or

$$d < 2f. \tag{6.241}$$

For these positive values of α we have

$$1 - \frac{d}{2f} < 1. \tag{6.242}$$

or

$$\frac{d}{2f} > 0, \tag{6.243}$$

which just means that d is positive. This shows that $d \in (0, 2f)$ results in a stable system.

2. The condition $\alpha < 0$ can be written

$$1 - \frac{d}{2f} < 0. \tag{6.244}$$

or

$$1 < \frac{d}{2f}. \tag{6.245}$$

or

$$2f < d. \tag{6.246}$$

The stability criteria for such values of d and f is

$$|\alpha| = -1 + \frac{d}{2f} < 1, \tag{6.247}$$

or

$$d < 4f. \tag{6.248}$$

This shows that $d \in (2f, 4f)$ also results in a stable system.

With both sets of ranges for d mutually exclusive, and the point $d = 2f$ also stable (doubled eigenvalue with value 1), we can form the union of the two, and have a stable system provided

$$d \in (0, 4f). \tag{6.249}$$

This was for a set of lenses set distance d apart. Noting from the text that "a curved mirror of radius r is optically equivalent to a lens of focal length $f = r/2$", we have for the optical resonator

$$\boxed{d \in (0, 2r),} \tag{6.250}$$

the desired result.

Part b. Stability criterion A system with plane mirrors is characterized by $f = \infty$ or $1/f = 0$. Our ABCD matrix is then just

$$M = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & d \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & d \\ 0 & 1 \end{bmatrix}. \tag{6.251}$$

This is a system that has only a single eigenvalue of 1 and is already in Jordan canonical form, so it cannot be diagonalized any further. Our eigenvector can be found by inspection

$$\mathbf{e} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \tag{6.252}$$

since this clearly satisfies $M\mathbf{e} = \mathbf{e}$. Any vector that is a multiple of this is also an eigenvector for this system

$$\mathbf{x} = \begin{bmatrix} y \\ 0 \end{bmatrix}. \tag{6.253}$$

These are the only stable rays, which makes intuitive sense. Only rays that are parallel to the pair of mirrors will not diverge.

Grading remark: -1 parallel was circled, with "How's that possible?". Look at the posted solutions for how to interpret this.

Part c. Unequal mirror radii We now wish to look at the characteristic equation for a round trip through the two sets of geometric elements. That is

$$\begin{aligned} M &= M_2 M_1 \\ &= \begin{bmatrix} 1 & d \\ -1/f_2 & -d/f_2 + 1 \end{bmatrix} \begin{bmatrix} 1 & d \\ -1/f_1 & -d/f_1 + 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & d \\ 2(1-\alpha_2) & 2\alpha_2 - 1 \end{bmatrix} \begin{bmatrix} 1 & d \\ 2(1-\alpha_1) & 2\alpha_1 - 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 - \frac{d}{f_1} & d\left(2 - \frac{d}{f_1}\right) \\ -\frac{d}{f_2} - \frac{d}{f_1} + \frac{d^2}{f_1 f_2} & -\frac{d}{f_2} + \left(1 - \frac{d}{f_2}\right)\left(1 - \frac{d}{f_1}\right) \end{bmatrix}, \end{aligned} \tag{6.254}$$

or

$$\begin{bmatrix} 2\alpha_1 - 1 & d\alpha_1 \\ \frac{2}{d}\left(\alpha_1 + \alpha_2 - 2 + 2(\alpha_1 - 1)(\alpha_2 - 1)\right) & 2(\alpha_2 - 1) + (2\alpha_1 - 1)(2\alpha_2 - 1) \end{bmatrix}$$

Direct naive computation of $0 = \det(M - \lambda I)$ is not pleasant, and gives a big ass messy expression that's hard to make heads or tails of. If we consider the 2 dimensional eigenvalue problem more generally, some simplifications for this problem can be made first. With

$$M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}, \tag{6.255}$$

our characteristic equation, again the $\det M = 1$ constraint for this problem where all the propagation is free

$$\begin{aligned} 0 &= (A - \lambda)(D - \lambda) - BC \\ &= \lambda^2 - \lambda(A + D) + AD - BC \\ &= \lambda^2 - \lambda \operatorname{tr} M + 1 \\ &= \left(\lambda - \frac{\operatorname{tr} M}{2}\right)^2 + 1 - \left(\frac{\operatorname{tr} M}{2}\right)^2. \end{aligned} \tag{6.256}$$

This gives us

$$\lambda = \frac{\operatorname{tr} M}{2} \pm \sqrt{\left(\frac{\operatorname{tr} M}{2}\right)^2 - 1}. \tag{6.257}$$

Our stability criteria is

$$\left|\frac{\operatorname{tr} M}{2}\right| < 1, \tag{6.258}$$

or

$$-1 < \frac{\operatorname{tr} M}{2} < 1. \tag{6.259}$$

This inequality can be first shifted by 1 to put it in the $[0, 2]$ range, then divided through by 2 to put it in the $[0, 1]$ range desired, so that we have stable beam trajectories when

$$0 < \frac{\operatorname{tr} M + 2}{4} < 1. \tag{6.260}$$

Now lets apply this to our ABCD matrix

$$\begin{aligned} \frac{\operatorname{tr} M + 2}{4} &= \frac{1}{4}\left(2\alpha_1 - 1 + 2(\alpha_2 - 1) + (2\alpha_1 - 1)(2\alpha_2 - 1) + 2\right) \\ &= \frac{1}{4}\left(2\alpha_1 - 1 + 2\alpha_2 - 2 + 4\alpha_1\alpha_2 + 1 - 2\alpha_1 - 2\alpha_2 + 2\right) \\ &= \alpha_1\alpha_2. \end{aligned} \tag{6.261}$$

This gives us the desired stability constraint

$$\boxed{0 < \alpha_1\alpha_2 < 1} \tag{6.262}$$

Relating the ray stability and Gaussian beam stability This question was posed in terms of Gaussian beams, which transform according to the Möbius transformation, whereas we've considered only ray transformations. We can relate the two considering the transformation of the waist at the $z = 0$ point in the center of the cavity. Recall that we have

$$\begin{aligned}\mathrm{Im}\left(\frac{1}{q}\right) &= \mathrm{Im}\left(\frac{1}{z - iz_0}\right)\\ &= \frac{z_0}{z^2 + z_0^2}.\end{aligned} \tag{6.263}$$

In particular at $z = 0$, this is

$$\frac{1}{z_0} = \frac{\lambda}{\pi w^2(0)}. \tag{6.264}$$

Now let's look at the Möbius transformation at the pinch of the waist ($z = 0$)

$$\begin{aligned}\frac{1}{q'(0)} &= \frac{Cq(0) + D}{Aq(0) + B}\\ &= \frac{-iCz_0 + D}{-iAz_0 + B}\\ &= \frac{(-iCz_0 + D)(iAz_0 + B)}{A^2z_0^2 + B^2}\\ &= \frac{(-iCz_0 + D)(iAz_0 + B)}{A^2z_0^2 + B^2}.\end{aligned} \tag{6.265}$$

From this we have

$$\begin{aligned}\mathrm{Im}\left(\frac{1}{q'(0)}\right) &= \frac{z_0(AD - BC)}{A^2z_0^2 + B^2}\\ &= \frac{z_0}{A^2z_0^2 + B^2}.\end{aligned} \tag{6.266}$$

Comparing this to eq. (6.264) we see that our waist at $z = 0$ transforms as

$$\frac{\pi(w'(0))^2}{\lambda} = \frac{A^2z_0^2 + B^2}{z_0}. \tag{6.267}$$

Both A and B will be linear functions of the eigenvalues, so we see that in the real eigenvalue case, where the repeated powers of one of these eigenvalues will necessarily diverge, so will the Gaussian beam waist.

Exercise 6.6 Symmetric cavity. (*2012 Ps4, P3*)

Consider the lowest-order mode (u_{00}) of a symmetric cavity with length L and mirror radius $R = R_1 = R_2$. Let's choose the mirror radius to match the wavefront curvature of the Gaussian beam.

a. Beam parameter Find the beam parameter z_0 in terms of L and R. You should find that for the particular case of a confocal cavity, $z_0 = L/2 = R/2$.

b. Beam waist Find the beam waist for a general cavity, in terms of L, R, and λ. Do you notice anything strange for the concentric cavity, $R = L/2$? For the confocal case, you should find that $w_0^2 = L/k_0$.

c. Harmonic frequency in cavity Following the analogy to the 2D harmonic oscillator constructed in class, see if you can show that a confocal cavity provides an effective harmonic frequency of $\omega_{\text{osc}} = 2c/L$. *(Two hints: 1, Look up the form of the ground state of the harmonic oscillator and compare to our Gaussian mode at its waist; and 2, You'll need to use the effective mass we derived in class, $m_{\text{eff}} = \hbar k_0/c$.)*

Answer for Exercise 6.6

Part a. Beam parameter For the right, positive curvature mirror, we have

$$\frac{1}{R(z)} = \frac{z}{z^2 + z_0^2}. \tag{6.268}$$

With $|R_1| = |R_2| = R$, we have a symmetrical setup, so that the z_2 of Van Driel's Eq. (10.3.2) is just $L/2$. That gives us at the boundary

$$\frac{1}{R} = \frac{L/2}{(L/2)^2 + z_0^2}, \tag{6.269}$$

which, after rearrangement, is

$$z_0^2 = \frac{RL}{2} - \left(\frac{L}{2}\right)^2 = \frac{L}{2}\left(R - \frac{L}{2}\right), \tag{6.270}$$

or

$$\boxed{z_0 = \sqrt{\frac{L}{2}\left(R - \frac{L}{2}\right)}} \tag{6.271}$$

Definitions and nice illustrations of the different cavity types can be found in [14]. For the confocal cavity defined by $L = R$, we have

$$z_0 = \sqrt{\frac{L}{2}\left(L - \frac{L}{2}\right)} = \frac{L}{2}, \tag{6.272}$$

as we are to show.

Part b. Beam waist We defined the Raleigh range in terms of the waist as

$$z_0 = \frac{\pi w_0^2}{\lambda}, \tag{6.273}$$

which we can invert as

$$w_0^2 = \frac{\lambda z_0}{\pi} = \frac{\lambda}{\pi}\sqrt{\frac{L}{2}\left(R - \frac{L}{2}\right)}. \tag{6.274}$$

For the concentric cavity where $R = L/2$ we have

$$z_0 = \sqrt{\frac{L}{2}\left(\frac{L}{2} - \frac{L}{2}\right)} = 0, \tag{6.275}$$

so the waist is also zero there. When we derived the u_{00} solution to the Paraxial wave equation we'd required $z_0 \neq 0$ since we offset $q = z - iz_0$ to remove the singularity. This suggests that a different approach is required for the concentric boundary value constraints. Intuitively it's perhaps not unreasonable to expect that we'll have severe pinch off in the center for this configuration (as we would have for rays).

For the confocal configuration of part a we have

$$w_0^2 = \frac{\lambda L}{2\pi} = \frac{L}{k_0}, \tag{6.276}$$

as we were to show.

Part c. Harmonic frequency in cavity From [1] we find for the ground state of the 2D harmonic oscillator

$$E = \hbar\omega\left(0+0+\frac{2}{2}\right) = \hbar\omega. \tag{6.277}$$

and after normalization and adding in the time dependence we find for the ground state wave function

$$\Psi_{00}(r,t) = \sqrt{\frac{m\omega}{\pi\hbar}}e^{-\frac{m\omega r^2}{2\hbar}-i\omega t}. \tag{6.278}$$

For the lowest order Gaussian mode at its waist we have for the envelope of u_{00}

$$|u_{00}| = e^{-r^2/w_0^2}. \tag{6.279}$$

For the confocal cavity, recalling eq. (6.276), we make the identification

$$\frac{1}{w_0^2} = \frac{k_0}{L} \leftrightarrow \frac{m_{\text{eff}}\omega_{\text{osc}}}{2\hbar} = \frac{\hbar k_0\omega_{\text{osc}}}{2\hbar c}. \tag{6.280}$$

or

$$\boxed{\omega_{\text{osc}} = \frac{2c}{L}} \tag{6.281}$$

Exercise 6.7 Möbius vs. matrix transformations. (*2012 Ps4, P4*)

Using the Möbius transform $q' = (Aq+B)/(Cq+D)$, show that transformation using $\{A_1, B_1, C_1, D_1\}$ followed by $\{A_2, B_2, C_2, D_2\}$ is equivalent to transformation using the elements of the single matrix $M = M_2M_1$, where

$$M_1 = \begin{pmatrix} A_1 & B_1 \\ C_1 & D_1 \end{pmatrix} \quad \text{and} \quad M_2 = \begin{pmatrix} A_2 & B_2 \\ C_2 & D_2 \end{pmatrix}.$$

Answer for Exercise 6.7

Proceeding directly with the double application of the Möbius transform, we have

$$\begin{aligned} q'' &= \frac{A_2 q' + B_2}{C_2 q' + D_2} \\ &= \frac{A_2\left(\frac{A_1 q + B_1}{C_1 q + D_1}\right) + B_2}{C_2\left(\frac{A_1 q + B_1}{C_1 q + D_1}\right) + D_2} \\ &= \frac{A_2(A_1 q + B_1) + B_2(C_1 q + D_1)}{C_2(A_1 q + B_1) + D_2(C_1 q + D_1)} \\ &= \frac{(A_2 A_1 + B_2 C_1)q + A_2 B_1 + B_2 D_1}{(C_2 A_1 + D_2 C_1)q + C_2 B_1 + D_2 D_1}. \end{aligned} \tag{6.282}$$

Now compare to the double matrix product transformation

$$\begin{aligned} M &= M_2 M_1 \\ &= \begin{bmatrix} A_2 & B_2 \\ C_2 & D_2 \end{bmatrix} \begin{bmatrix} A_1 & B_1 \\ C_1 & D_1 \end{bmatrix} \\ &= \begin{bmatrix} A_2 A_1 + B_2 C_1 & A_2 B_1 + B_2 D_1 \\ C_2 A_1 + D_2 C_1 & C_2 B_1 + D_2 D_1 \end{bmatrix}. \end{aligned} \tag{6.283}$$

Writing out the transformation this way we find

$$q \to \frac{(A_2 A_1 + B_2 C_1)q + A_2 B_1 + B_2 D_1}{(C_2 A_1 + D_2 C_1)q + C_2 B_1 + D_2 D_1}, \tag{6.284}$$

exactly as we found with double application of the Möbius transformation.

Exercise 6.8 Gaussian beam. (*2010 final exam question 5*)

a. Spot size
A Gaussian beam with wavelength 0.8μm has its minimum waist of 0.5mm located in the middle of a parallel glass plate of thickness 5cm and refractive index 1.5. The axis of the beam is perpendicular to the surfaces of the glass plate. The beam emerges from the glass plate and strikes a mirror at normal incidence 10cm away. When the beam passes back through its original location what is its spot size?

b. Angular divergence
When the beam emerges from the plate again after re-passing its beam waist, what is its angular divergence?

Answer for Exercise 6.8

Part a. Spot size Our optical system and beam has the following configuration fig. 6.33. Going from the glass to the air we have

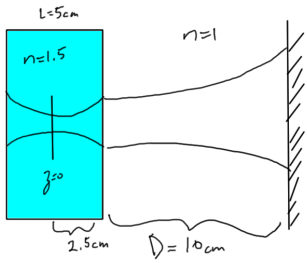

Figure 6.33: Gaussian beam through glass then air.

$n \sin\theta_i = \sin\theta_t$, or in the paraxial approximation

$$\begin{bmatrix} y_t \\ \theta_t \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & n \end{bmatrix} \begin{bmatrix} y_i \\ \theta_i \end{bmatrix} \tag{6.285}$$

The geometric optics for the round trip is

$$\begin{aligned} M &= \begin{bmatrix} 1 & L/2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1/n \end{bmatrix} \begin{bmatrix} 1 & 2D \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & n \end{bmatrix} \begin{bmatrix} 1 & L/2 \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & L/2n \\ 0 & 1/n \end{bmatrix} \begin{bmatrix} 1 & 2Dn \\ 0 & n \end{bmatrix} \begin{bmatrix} 1 & L/2 \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & L/2n \\ 0 & 1/n \end{bmatrix} \begin{bmatrix} 1 & L/2+2Dn \\ 0 & n \end{bmatrix} \\ &= \begin{bmatrix} 1 & L+2Dn \\ 0 & 1 \end{bmatrix}. \end{aligned} \tag{6.286}$$

Our Möbius transformation, in meters is

$$\begin{aligned} q' &= q + 2Dn \\ &= q + 2(0.1)1.5 \\ &= q + 0.3. \end{aligned} \tag{6.287}$$

In the inverse, this has the form

$$\begin{aligned}\frac{1}{q+a} &= \frac{1}{z - iz_0 + a} \\ &= \frac{z + iz_0 + a}{(z+a)^2 + z_0^2} \\ &= \frac{z+a}{(z+a)^2 + z_0^2} + i\frac{z_0}{(z+a)^2 + z_0^2}.\end{aligned} \tag{6.288}$$

Our waist $z = 0$ was originally

$$\frac{\lambda_0}{\pi n w_0^2} = \frac{z_0}{z_0^2} = \frac{1}{z_0}, \tag{6.289}$$

or

$$z_0 = \frac{\pi n w_0^2}{\lambda_0}. \tag{6.290}$$

Assuming that we are given the wavelength within the glass $\lambda = \lambda_0/n = 0.8\mu\text{m}$ (and not the free propagation wavelength outside of the glass), then we have in meters

$$z_0 = \frac{\pi(0.0005)^2}{0.8 \times 10^{-6}} \approx 0.98. \tag{6.291}$$

Our new waist is

$$w_0'^2 = \frac{\pi}{\lambda}\frac{(0-0.3)^2 + z_0^2}{z_0} \approx 0.52\text{mm}. \tag{6.292}$$

Our waist widens slightly from the original after the round trip.

Part b. Angular divergence To examine the beam characteristics after it continues through and out of the glass again, we have to apply another geometric transformation

$$\begin{aligned}M' &= \begin{bmatrix}1 & 0\\ 0 & n\end{bmatrix}\begin{bmatrix}1 & L/2\\ 0 & 1\end{bmatrix} \\ &= \begin{bmatrix}1 & L/2\\ 0 & n\end{bmatrix}.\end{aligned} \tag{6.293}$$

Our Möbius transform is

$$\begin{aligned} q'' &= \frac{q' + L/2}{n} \\ &= \frac{q + 2Dn + L/2}{n}, \end{aligned} \tag{6.294}$$

We are looking at how our initial waist, found at $z = 0$ transformed, so we have

$$\begin{aligned} \frac{1}{q''(0)} &= \frac{n}{-iz_0 + 2Dn + L/2} \\ &= \frac{n(iz_0 + 2Dn + L/2)}{z_0^2 + (2Dn + L/2)^2}. \end{aligned} \tag{6.295}$$

Our new waist is found outside the glass where $n = 1$

$$\begin{aligned} \mathrm{Im}\left(\frac{1}{q''(0)}\right) &= \frac{\lambda_0}{n_{\mathrm{air}} \pi w'^2(0)} \\ &= \frac{\lambda_0}{\pi w'^2(0)} \\ &= \frac{n_{\mathrm{glass}} \lambda}{\pi w'^2(0)}. \end{aligned} \tag{6.296}$$

$$w'^2(0) = \frac{n_{\mathrm{glass}} \lambda}{\pi} \frac{z_0^2 + (2Dn_{\mathrm{glass}} + L/2)^2}{n_{\mathrm{glass}} z_0}. \tag{6.297}$$

With the divergence angle having the value

$$
\begin{aligned}
\Theta &\sim \frac{\sqrt{2}w_0'}{z_0'} \\
&= \sqrt{2}\sqrt{\frac{\lambda}{\pi}\frac{z_0^2 + (2Dn_{\text{glass}} + L/2)^2}{z_0}}\frac{n_{\text{glass}}z_0}{z_0^2 + (2Dn_{\text{glass}} + L/2)^2} \\
&= \sqrt{\frac{2\lambda z_0}{\pi}\frac{n_{\text{glass}}^2}{z_0^2 + (2Dn_{\text{glass}} + L/2)^2}} \\
&= \sqrt{\frac{2\lambda\frac{\pi n_{\text{glass}}w_0^2}{\lambda}}{\pi}\frac{n_{\text{glass}}^2}{\left(\frac{\pi n_{\text{glass}}w_0^2}{\lambda}\right)^2 + (2Dn_{\text{glass}} + L/2)^2}} \\
&= \sqrt{\frac{2n_{\text{glass}}^3 w_0^2}{\left(\frac{\pi n_{\text{glass}}w_0^2}{\lambda}\right)^2 + (2Dn_{\text{glass}} + L/2)^2}} \\
&= \sqrt{\frac{2n_{\text{glass}}w_0^2}{\frac{\pi^2 w_0^4}{\lambda^2} + \left(2D + \frac{L}{2n_{\text{glass}}^2}\right)^2}}.
\end{aligned}
\tag{6.298}
$$

Plugging in the numbers, this is 8.6×10^{-4} radians or 0.05 seconds. That's a small seeming number, but still results in a 1 cm spread after only 5.8 meters.

A FRESNEL EQUATIONS, MIXED POLARIZATION.

A.1 MOTIVATION.

In [8] we have a derivation of the Fresnel equations for the TE and TM polarization modes. Can we do this for an arbitrary polarization angles?

A.2 SETUP.

The task at hand is to find evaluate the boundary value constraints. Following the interface plane conventions of [6], and his notation that is

$$\epsilon_1(\mathbf{E}_i + \mathbf{E}_r)_z = \epsilon_2(\mathbf{E}_t)_z. \tag{A.1a}$$

$$(\mathbf{B}_i + \mathbf{B}_r)_z = (\mathbf{B}_t)_z. \tag{A.1b}$$

$$(\mathbf{E}_i + \mathbf{E}_r)_{x,y} = (\mathbf{E}_t)_{x,y}. \tag{A.1c}$$

$$\frac{1}{\mu_1}(\mathbf{B}_i + \mathbf{B}_r)_{x,y} = \frac{1}{\mu_2}(\mathbf{B}_t)_{x,y}. \tag{A.1d}$$

I'll work here with a phasor representation directly and not bother with taking real parts, or using tilde notation to mark the vectors as complex.

Our complex magnetic field phasors are related to the electric fields with

$$\mathbf{B} = \frac{1}{v}\hat{\mathbf{k}} \times \mathbf{E}. \tag{A.2}$$

Referring to fig. A.1 shows the geometrical task to tackle, since we've got to express all the various unit vectors algebraically. I'll

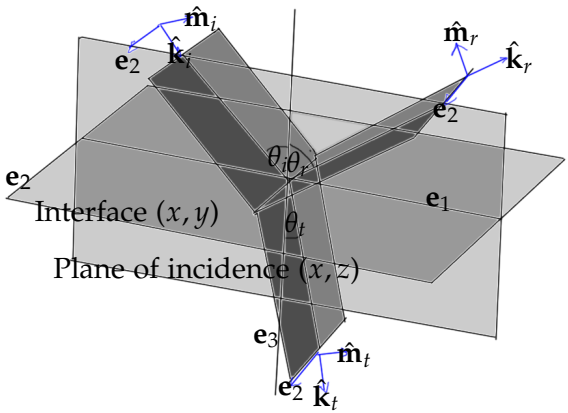

Figure A.1: Reflection and transmission of light at an interface

use Geometric Algebra here to do that for its compact expression of rotations. With

$$j = \mathbf{e}_3\mathbf{e}_1, \tag{A.3}$$

we can express each of the k vector directions by inspection. Those are

$$\hat{\mathbf{k}}_i = \mathbf{e}_3 e^{j\theta_i} = \mathbf{e}_3 \cos\theta_i + \mathbf{e}_1 \sin\theta_i. \tag{A.4a}$$

$$\hat{\mathbf{k}}_r = -\mathbf{e}_3 e^{-j\theta_r} = -\mathbf{e}_3 \cos\theta_r + \mathbf{e}_1 \sin\theta_r. \tag{A.4b}$$

$$\hat{\mathbf{k}}_t = \mathbf{e}_3 e^{j\theta_t} = \mathbf{e}_3 \cos\theta_t + \mathbf{e}_1 \sin\theta_t. \tag{A.4c}$$

Similarly, the perpendiculars $\hat{\mathbf{m}}_p = \mathbf{e}_2 \times \hat{\mathbf{k}}_p$ are

$$\hat{\mathbf{m}}_i = \mathbf{e}_1 e^{j\theta_i} = \mathbf{e}_1 \cos\theta_i - \mathbf{e}_3 \sin\theta_i = \mathbf{e}_3 j e^{j\theta_i}. \tag{A.5a}$$

$$\hat{\mathbf{m}}_r = -\mathbf{e}_1 e^{-j\theta_r} = -\mathbf{e}_1 \cos\theta_r - \mathbf{e}_3 \sin\theta_r = -\mathbf{e}_3 j e^{-j\theta_r}. \tag{A.5b}$$

$$\hat{\mathbf{m}}_t = \mathbf{e}_1 e^{j\theta_t} = \mathbf{e}_1 \cos\theta_t - \mathbf{e}_3 \sin\theta_t = \mathbf{e}_3 j e^{j\theta_t}. \tag{A.5c}$$

In [6] problem 9.14 we had to show that the polarization angles for normal incident ($\mathbf{E} \parallel \mathbf{e}_1$) must be the same due to the boundary constraints. Can we also tackle that problem for both this more general angle of incidence and a general polarization? Let's try so, allowing temporarily for different polarizations of the reflected and transmitted components of the light, calling those polarization angles ϕ_i, ϕ_r, and ϕ_t respectively. Let's set the $\phi_i = 0$ polarization aligned such that $\mathbf{E}_i$, $\mathbf{B}_i$ are aligned with the $\mathbf{e}_2$ and $-\hat{\mathbf{m}}_i$ directions respectively, so that the generally polarized phasors are

$$\begin{bmatrix} \mathbf{E}_p \\ \mathbf{B}_p \end{bmatrix} = \begin{bmatrix} \mathbf{e}_2 \\ -\hat{\mathbf{m}}_p \end{bmatrix} e^{\hat{\mathbf{m}}_p \mathbf{e}_2 \phi_p}. \tag{A.6}$$

We are now set to at least express our boundary value constraints

$$\epsilon_1 \left(\mathbf{e}_2 E_i e^{\hat{\mathbf{m}}_i \mathbf{e}_2 \phi_i} + \mathbf{e}_2 E_r e^{\hat{\mathbf{m}}_r \mathbf{e}_2 \phi_r} \right) \cdot \mathbf{e}_3 = \epsilon_2 \left(\mathbf{e}_2 E_t e^{\hat{\mathbf{m}}_t \mathbf{e}_2 \phi_t} \right) \cdot \mathbf{e}_3. \tag{A.7a}$$

$$\frac{1}{v_1} \left(-\hat{\mathbf{m}}_i E_i e^{\hat{\mathbf{m}}_i \mathbf{e}_2 \phi_i} - \hat{\mathbf{m}}_r E_r e^{\hat{\mathbf{m}}_r \mathbf{e}_2 \phi_r} \right) \cdot \mathbf{e}_3 = \frac{1}{v_2} \left(-\hat{\mathbf{m}}_t E_t e^{\hat{\mathbf{m}}_t \mathbf{e}_2 \phi_t} \right) \cdot \mathbf{e}_3. \tag{A.7b}$$

$$\left(\mathbf{e}_2 E_i e^{\hat{\mathbf{m}}_i \mathbf{e}_2 \phi_i} + \mathbf{e}_2 E_r e^{\hat{\mathbf{m}}_r \mathbf{e}_2 \phi_r} \right) \wedge \mathbf{e}_3 = \left(\mathbf{e}_2 E_t e^{\hat{\mathbf{m}}_t \mathbf{e}_2 \phi_t} \right) \wedge \mathbf{e}_3. \tag{A.7c}$$

$$\frac{1}{\mu_1 v_1} \left(-\hat{\mathbf{m}}_i E_i e^{\hat{\mathbf{m}}_i \mathbf{e}_2 \phi_i} - \hat{\mathbf{m}}_r E_r e^{\hat{\mathbf{m}}_r \mathbf{e}_2 \phi_r} \right) \wedge \mathbf{e}_3 = \frac{1}{\mu_2 v_2} \left(-\hat{\mathbf{m}}_t E_t e^{\hat{\mathbf{m}}_t \mathbf{e}_2 \phi_t} \right) \wedge \mathbf{e}_3. \tag{A.7d}$$

A.3 SOLVING FOR THE FRESNEL EQUATIONS.

Let's try this in a couple of steps. First with polarization angles set so that one of the fields lies in the plane of the interface (with

both variations), and then attempt the general case, first posing the problem in the traditional way to see what equations fall out, and then using superposition.

Before doing so, let's introduce a bit of notation to be used throughout. When we wish to refer to all the fields or angles, for example, $\mathbf{E}_i, \mathbf{E}_r, \mathbf{E}_t$ then we'll write $\mathbf{E}_p$ where $p \in \{i, r, t\}$. Similarly, to refer to just the incident and transmitted components (or angles) we'll use $\mathbf{E}_q$ where $q \in \{i, t\}$. Following [6] we'll also write

$$\boxed{\beta = \frac{\mu_1 v_1}{\mu_2 v_2}} \tag{A.8a}$$

$$\boxed{\alpha = \frac{\cos\theta_t}{\cos\theta_i},} \tag{A.8b}$$

.

Exercise A.1 Sanity check. Verify for $\mathbf{E}$ parallel to the interface.

Answer for Exercise A.1

For the $\mathbf{E}_p \parallel \mathbf{e}_2$ polarization ($\phi_i = \phi_r = \phi_t$) our phasors are

$$\mathbf{E}_p = \mathbf{e}_2 E_p. \tag{A.9a}$$

$$\mathbf{B}_p = -\frac{1}{v_p}\hat{\mathbf{m}}_p E_p. \tag{A.9b}$$

Our boundary value constraints then become

$$\epsilon_1 \left(\mathbf{e}_2 E_i + \mathbf{e}_2 E_r\right) \cdot \mathbf{e}_3 = \epsilon_2 \left(\mathbf{e}_2 E_t\right) \cdot \mathbf{e}_3. \tag{A.10a}$$

$$\frac{1}{v_1}\left(\hat{\mathbf{m}}_i E_i + \hat{\mathbf{m}}_r E_r\right) \cdot \mathbf{e}_3 = \frac{1}{v_2}\left(\hat{\mathbf{m}}_t E_t\right) \cdot \mathbf{e}_3. \tag{A.10b}$$

$$\left(\mathbf{e}_2 E_i + \mathbf{e}_2 E_r\right) \wedge \mathbf{e}_3 = \left(\mathbf{e}_2 E_t\right) \wedge \mathbf{e}_3. \tag{A.10c}$$

$$\frac{1}{\mu_1 v_1}\left(\hat{\mathbf{m}}_i E_i + \hat{\mathbf{m}}_r E_r\right) \wedge \mathbf{e}_3 = \frac{1}{\mu_2 v_2}\left(\hat{\mathbf{m}}_t E_t\right) \wedge \mathbf{e}_3. \tag{A.10d}$$

With $\hat{\mathbf{m}}_p$ substitution this is

$$\epsilon_1 \langle \mathbf{e}_3 \left(\mathbf{e}_2 E_i + \mathbf{e}_2 E_r \right) \rangle = \epsilon_2 \langle \mathbf{e}_3 \left(\mathbf{e}_2 E_t \right) \rangle . \tag{A.11a}$$

$$\frac{1}{v_1} \left\langle \mathbf{e}_3 \left(\mathbf{e}_1 e^{j\theta_i} E_i - \mathbf{e}_1 e^{-j\theta_r} E_r \right) \right\rangle = \frac{1}{v_2} \left\langle \mathbf{e}_3 \left(\mathbf{e}_1 e^{j\theta_t} E_t \right) \right\rangle . \tag{A.11b}$$

$$\langle \mathbf{e}_3 \left(\mathbf{e}_2 E_i + \mathbf{e}_2 E_r \right) \rangle_2 = \langle \mathbf{e}_3 \left(\mathbf{e}_2 E_t \right) \rangle_2 . \tag{A.11c}$$

$$\frac{1}{\mu_1 v_1} \left\langle \mathbf{e}_3 \left(\mathbf{e}_1 e^{j\theta_i} E_i - \mathbf{e}_1 e^{-j\theta_r} E_r \right) \right\rangle_2 = \frac{1}{\mu_2 v_2} \left\langle \mathbf{e}_3 \left(\mathbf{e}_1 e^{j\theta_t} E_t \right) \right\rangle_2 . \tag{A.11d}$$

Evaluating the grade selections we have a separation into an analogue of real and imaginary parts for

$$0 = 0 . \tag{A.12a}$$

$$\frac{1}{v_1} \left(-\sin\theta_i E_i - \sin\theta_r E_r \right) = \frac{1}{v_2} \left(-\sin\theta_t E_t \right) . \tag{A.12b}$$

$$E_i + E_r = E_t . \tag{A.12c}$$

$$\frac{1}{\mu_1 v_1} \left(\cos\theta_i E_i - \cos\theta_r E_r \right) = \frac{1}{\mu_2 v_2} \left(\cos\theta_t E_t \right) . \tag{A.12d}$$

With $\theta_i = \theta_r$ and $\sin\theta_t / \sin\theta_i = n_1 / n_2$ eq. (A.12b) becomes

$$E_i + E_r = \frac{n_1 v_1}{n_2 v_2} E_t = \frac{v_2 v_1}{v_1 v_2} E_t = E_t , \tag{A.13}$$

so that we find eq. (A.12b) and eq. (A.12c) are dependent. We are left with a pair of equations

$$E_i + E_r = E_t . \tag{A.14a}$$

$$E_i - E_r = \frac{\mu_1 v_1}{\mu_2 v_2}\frac{\cos\theta_t}{\cos\theta_i}E_t, \tag{A.14b}$$

Adding and subtracting we have

$$2E_i = \left(1 + \frac{\mu_1 v_1}{\mu_2 v_2}\frac{\cos\theta_t}{\cos\theta_i}\right)E_t. \tag{A.15a}$$

$$2E_r = \left(1 - \frac{\mu_1 v_1}{\mu_2 v_2}\frac{\cos\theta_t}{\cos\theta_i}\right)E_t, \tag{A.15b}$$

with a final rearrangement to yield

$$\frac{E_t}{E_i} = \frac{2\mu_2 v_2 \cos\theta_i}{\mu_2 v_2 \cos\theta_i + \mu_1 v_1 \cos\theta_t}. \tag{A.16a}$$

$$\frac{E_r}{E_i} = \frac{\mu_2 v_2 \cos\theta_i - \mu_1 v_1 \cos\theta_t}{\mu_2 v_2 \cos\theta_i + \mu_1 v_1 \cos\theta_t}. \tag{A.16b}$$

The ratio of field strengths for **E** parallel to the interface, using notation eq. (A.8), is

$$\boxed{\frac{E_t}{E_i} = \frac{2}{1+\alpha\beta}} \tag{A.17a}$$

$$\boxed{\frac{E_r}{E_i} = \frac{1-\alpha\beta}{1+\alpha\beta}} \tag{A.17b}$$

.

Exercise A.2 Sanity check. Verify for **B** parallel to the interface.

Answer for Exercise A.2

As a second sanity check let's rotate our field polarizations by applying a rotation $e^{\mathbf{e}_2\hat{\mathbf{m}}_p\pi/2} = \mathbf{e}_2\hat{\mathbf{m}}_p$ ($\phi_i = \phi_r = \phi_t = -\pi/2$) so that

$$-\hat{\mathbf{m}}_p \to -\hat{\mathbf{m}}_p\mathbf{e}_2\hat{\mathbf{m}}_p = \mathbf{e}_2. \tag{A.18a}$$

$$\mathbf{e}_2 \to \mathbf{e}_2\mathbf{e}_2\hat{\mathbf{m}}_p = \hat{\mathbf{m}}_p. \tag{A.18b}$$

This time we have $\mathbf{E}_p \parallel \hat{\mathbf{m}}_p$ and $\mathbf{B}_p \parallel \mathbf{e}_2$. Our boundary value equations become

$$\epsilon_1 \langle \mathbf{e}_3 \left(\hat{\mathbf{m}}_i E_i + \hat{\mathbf{m}}_r E_r \right) \rangle = \epsilon_2 \langle \mathbf{e}_3 \left(\hat{\mathbf{m}}_t E_t \right) \rangle . \tag{A.19a}$$

$$\frac{1}{v_1} \langle \mathbf{e}_3 \left(\mathbf{e}_2 E_i + \mathbf{e}_2 E_r \right) \rangle = \frac{1}{v_2} \langle \mathbf{e}_3 \left(\mathbf{e}_2 E_t \right) \rangle . \tag{A.19b}$$

$$\langle \mathbf{e}_3 \left(\hat{\mathbf{m}}_i E_i + \hat{\mathbf{m}}_r E_r \right) \rangle_2 = \langle \mathbf{e}_3 \left(\hat{\mathbf{m}}_t E_t \right) \rangle_2 . \tag{A.19c}$$

$$\frac{1}{\mu_1 v_1} \langle \mathbf{e}_3 \left(\mathbf{e}_2 E_i + \mathbf{e}_2 E_r \right) \rangle_2 = \frac{1}{\mu_2 v_2} \langle \mathbf{e}_3 \left(\mathbf{e}_2 E_t \right) \rangle_2 . \tag{A.19d}$$

This second eq. (A.19b) is a $0 = 0$ identity, and the remaining after $\hat{\mathbf{m}}_p$ substitution are

$$\epsilon_1 \left\langle \mathbf{e}_3 \left(\mathbf{e}_3 j e^{j\theta_i} E_i + (-\mathbf{e}_3) j e^{-j\theta_r} E_r \right) \right\rangle = \epsilon_2 \left\langle \mathbf{e}_3 \left(\mathbf{e}_3 j e^{j\theta_t} E_t \right) \right\rangle . \tag{A.20a}$$

$$\left\langle \mathbf{e}_3 \left(\mathbf{e}_3 j e^{j\theta_i} E_i + (-\mathbf{e}_3) j e^{-j\theta_r} E_r \right) \right\rangle_2 = \left\langle \mathbf{e}_3 \left(\mathbf{e}_3 j e^{j\theta_t} E_t \right) \right\rangle_2 . \tag{A.20b}$$

$$\frac{1}{\mu_1 v_1} \langle \mathbf{e}_3 \left(\mathbf{e}_2 E_i + \mathbf{e}_2 E_r \right) \rangle_2 = \frac{1}{\mu_2 v_2} \langle \mathbf{e}_3 \left(\mathbf{e}_2 E_t \right) \rangle_2 . \tag{A.20c}$$

Simplifying we have

$$\epsilon_1 \left(-\sin\theta_i E_i - \sin\theta_r E_r \right) = -\epsilon_2 \sin\theta_t E_t . \tag{A.21a}$$

$$\cos\theta_i E_i - \cos\theta_r E_r = \cos\theta_t E_t . \tag{A.21b}$$

$$E_i + E_r = \frac{\mu_1 v_1}{\mu_2 v_2} E_t . \tag{A.21c}$$

Noting that $\epsilon_p v_p = 1/(v_p \mu_p)$ we find

$$\frac{\epsilon_2 \sin\theta_t}{\epsilon_1 \sin\theta_i} = \frac{\epsilon_2 n_1}{\epsilon_1 n_2} = \frac{\epsilon_2 v_2}{\epsilon_1 v_1} = \frac{\mu_1 v_1}{\mu_2 v_2}. \tag{A.22}$$

showing that eq. (A.21a) and eq. (A.21c) are dependent. We are left with the system

$$E_i - E_r = \alpha E_t. \tag{A.23a}$$

$$E_i + E_r = \beta E_t. \tag{A.23b}$$

This time we find that the ratio of field strengths for **B** parallel to the interface, again using notation eq. (A.8), is

with solution

$$\boxed{\frac{E_t}{E_i} = \frac{2}{\beta + \alpha}} \tag{A.24a}$$

$$\boxed{\frac{E_r}{E_i} = \frac{\beta - \alpha}{\beta + \alpha}} \tag{A.24b}$$

.

Exercise A.3 General case. Arbitrary polarization angle.

Determine the set of simultaneous equations that would have to be solved for if the incident polarization angle was allowed to be neither TE nor TM mode.

Answer for Exercise A.3

Substituting our $\hat{\mathbf{m}}_p$ vector expressions into the boundary value constraints we have

$$\epsilon_1 \left\langle \mathbf{e}_3 \mathbf{e}_2 \left(E_i e^{\hat{\mathbf{m}}_i \mathbf{e}_2 \phi_i} + E_r e^{\hat{\mathbf{m}}_r \mathbf{e}_2 \phi_r} \right) \right\rangle = \epsilon_2 \left\langle \mathbf{e}_3 \mathbf{e}_2 E_t e^{\hat{\mathbf{m}}_t \mathbf{e}_2 \phi_t} \right\rangle. \tag{A.25a}$$

$$\frac{1}{v_1} \left\langle j e^{j\theta_i} E_i e^{\hat{\mathbf{m}}_i \mathbf{e}_2 \phi_i} - j e^{-j\theta_r} E_r e^{\hat{\mathbf{m}}_r \mathbf{e}_2 \phi_r} \right\rangle = \frac{1}{v_2} \left\langle j e^{j\theta_t} E_t e^{\hat{\mathbf{m}}_t \mathbf{e}_2 \phi_t} \right\rangle. \tag{A.25b}$$

$$\left\langle \mathbf{e}_3 \mathbf{e}_2 \left(E_i e^{\hat{\mathbf{m}}_i \mathbf{e}_2 \phi_i} + E_r e^{\hat{\mathbf{m}}_r \mathbf{e}_2 \phi_r} \right) \right\rangle_2 = \left\langle \mathbf{e}_3 \mathbf{e}_2 E_t e^{\hat{\mathbf{m}}_t \mathbf{e}_2 \phi_t} \right\rangle_2. \tag{A.25c}$$

$$\frac{1}{\mu_1 v_1}\left\langle j e^{j\theta_i} E_i e^{\hat{\mathbf{m}}_i \mathbf{e}_2 \phi_i} - j e^{-j\theta_r} E_r e^{\hat{\mathbf{m}}_r \mathbf{e}_2 \phi_r} \right\rangle_2 = \frac{1}{\mu_2 v_2}\left\langle j e^{j\theta_t} E_t e^{\hat{\mathbf{m}}_t \mathbf{e}_2 \phi_t} \right\rangle_2. \tag{A.25d}$$

We want to expand some intermediate multivector products

$$\begin{aligned}
\mathbf{e}_{32} e^{\hat{\mathbf{m}}_q \mathbf{e}_2 \phi_q} &= \mathbf{e}_{32} \cos\phi_q + \mathbf{e}_{32} \hat{\mathbf{m}}_q \mathbf{e}_2 \sin\phi_q \\
&= \mathbf{e}_{32} \cos\phi_q + \mathbf{e}_{32} \mathbf{e}_3 j e^{j\theta_q} \mathbf{e}_2 \sin\phi_q \\
&= \mathbf{e}_{32} \cos\phi_q - j e^{j\theta_q} \sin\phi_q \\
&= \mathbf{e}_{32} \cos\phi_q - \mathbf{e}_{31} \cos\theta_q \sin\phi_q + \sin\theta_q \sin\phi_q.
\end{aligned} \tag{A.26}$$

$$\begin{aligned}
\mathbf{e}_{32} e^{\hat{\mathbf{m}}_r \mathbf{e}_2 \phi_r} &= \mathbf{e}_{32} \cos\phi_r + \mathbf{e}_{32} \hat{\mathbf{m}}_r \mathbf{e}_2 \sin\phi_r \\
&= \mathbf{e}_{32} \cos\phi_r + \mathbf{e}_{32} (-\mathbf{e}_3) j e^{-j\theta_r} \mathbf{e}_2 \sin\phi_r \\
&= \mathbf{e}_{32} \cos\phi_r + j e^{-j\theta_r} \sin\phi_r \\
&= \mathbf{e}_{32} \cos\phi_r + \mathbf{e}_{31} \cos\theta_r \sin\phi_r + \sin\theta_r \sin\phi_r.
\end{aligned} \tag{A.27}$$

$$\begin{aligned}
j e^{j\theta_q} e^{\hat{\mathbf{m}}_q \mathbf{e}_2 \phi_q} &= j e^{j\theta_q} \left(\cos\phi_q + \hat{\mathbf{m}}_q \mathbf{e}_2 \sin\phi_q\right) \\
&= j e^{j\theta_q} \left(\cos\phi_q + \mathbf{e}_3 j e^{j\theta_q} \mathbf{e}_2 \sin\phi_q\right) \\
&= j e^{j\theta_q} \left(\cos\phi_q - j e^{-j\theta_q} \mathbf{e}_{32} \sin\phi_q\right) \\
&= j e^{j\theta_q} \cos\phi_q + \mathbf{e}_{32} \sin\phi_q \\
&= \mathbf{e}_{31} \cos j\theta_q \cos\phi_q + \mathbf{e}_{32} \sin\phi_q - \sin\theta_q \cos\phi_q.
\end{aligned} \tag{A.28}$$

$$\begin{aligned}
-j e^{-j\theta_r} e^{\hat{\mathbf{m}}_r \mathbf{e}_2 \phi_r} &= -j e^{-j\theta_r} \left(\cos\phi_r + \hat{\mathbf{m}}_r \mathbf{e}_2 \sin\phi_r\right) \\
&= -j e^{-j\theta_r} \left(\cos\phi_r - \mathbf{e}_3 j e^{-j\theta_r} \mathbf{e}_2 \sin\phi_r\right) \\
&= -j e^{-j\theta_r} \left(\cos\phi_r + j e^{j\theta_r} \mathbf{e}_{32} \sin\phi_r\right) \\
&= -j e^{-j\theta_r} \cos\phi_r + \mathbf{e}_{32} \sin\phi_r \\
&= -\mathbf{e}_{31} \cos\theta_r \cos\phi_r + \mathbf{e}_{32} \sin\phi_r - \sin\theta_r \cos\phi_r.
\end{aligned} \tag{A.29}$$

Our boundary value conditions are then

$$\epsilon_1 \left(E_i \sin\theta_i \sin\phi_i + E_r \sin\theta_r \sin\phi_r\right) = \epsilon_2 E_t \sin\theta_t \sin\phi_t. \tag{A.30a}$$

$$\frac{1}{v_1}\left(E_i \sin\theta_i \cos\phi_i + E_r \sin\theta_r \cos\phi_r\right) = \frac{1}{v_2} E_t \sin\theta_t \cos\phi_t. \tag{A.30b}$$

$$E_i \cos\phi_i + E_r \cos\phi_r = E_t \cos\phi_t. \tag{A.30c}$$

$$-E_i \cos\theta_i \sin\phi_i + E_r \cos\theta_r \sin\phi_r = -E_t \cos\theta_t \sin\phi_t. \tag{A.30d}$$

$$\frac{1}{\mu_1 v_1}\left(E_i \cos\theta_i \cos\phi_i - E_r \cos\theta_r \cos\phi_r\right) = \frac{1}{\mu_2 v_2} E_t \cos\theta_t \cos\phi_t. \tag{A.30e}$$

$$\frac{1}{\mu_1 v_1}\left(E_i \sin\phi_i + E_r \sin\phi_r\right) = \frac{1}{\mu_2 v_2} E_t \sin\phi_t. \tag{A.30f}$$

Note that the wedge product equations above have been separated into $\mathbf{e}_3\mathbf{e}_1$ and $\mathbf{e}_3\mathbf{e}_2$ components, yielding two equations each. Because of eq. (A.22), we see that eq. (A.30a) and eq. (A.30f) are dependent. Also, as demonstrated in eq. (A.13) we see that eq. (A.30b) and eq. (A.30c) are also dependent. We can therefore consider only the last four equations (and still have additional linear dependencies to be discovered.)

Let's write these as

$$E_i \cos\phi_i + E_r \cos\phi_r = E_t \cos\phi_t. \tag{A.31a}$$

$$-E_i \sin\phi_i + E_r \sin\phi_r = -E_t \alpha \sin\phi_t. \tag{A.31b}$$

$$E_i \cos\phi_i - E_r \cos\phi_r = \alpha\beta E_t \cos\phi_t. \tag{A.31c}$$

$$E_i \sin\phi_i + E_r \sin\phi_r = \beta E_t \sin\phi_t. \tag{A.31d}$$

Observe that if $\phi_i = \phi_r = \phi_t = 0$ (killing all the sine terms) we recover eq. (A.14), and with $\phi_i = \phi_r = \phi_t = \pi/2$ (killing all the cosines) we recover eq. (A.23).

Now, if $\phi_i = n\pi/2$ we've got a different story. Specifically it appears that should we wish to solve for the reflected and transmitted magnitudes, we also have to simultaneously solve for the polarization angles in the reflected and transmitted directions. This is now a problem of solving four simultaneous equations in two linear and two non-linear variables.

Does it make sense that we would have polarization rotation should our initial polarization angle be rotated? I think so. In discussing this problem with Prof Thywissen, he strongly suggested treating the problem as a superposition of two light waves. If we consider that, even without attempting to solve the problem, we see that we must have different reflected and transmitted magnitudes associated with the pair of incident waves since we have to calculate each of these with different Fresnel equations. This would have an effect of scaling and rotating the superimposed reflected and transmitted waves.

Exercise A.4 General case using superposition.

Using superposition determine the Fresnel equations for an arbitrary incident polarization angle. This should involve solving for both the magnitude and the polarization angle of the reflected and transmitted rays.

Answer for Exercise A.4

For a polarization of $\phi = 0$ and $\phi = \pi/2$ respectively, we found in eq. (A.17), and eq. (A.24)

$$\frac{E_{r\|}}{E_{i\|}} = \frac{1-\alpha\beta}{1+\alpha\beta}. \tag{A.32a}$$

$$\frac{E_{t\|}}{E_{i\|}} = \frac{2}{1+\alpha\beta}. \tag{A.32b}$$

$$\frac{E_{r\perp}}{E_{i\perp}} = \frac{\beta-\alpha}{\beta+\alpha}. \tag{A.32c}$$

$$\frac{E_{t\perp}}{E_{i\perp}} = \frac{2}{\beta + \alpha}. \tag{A.32d}$$

We can use these results to consider a polarization of $\phi < \pi/2$ as illustrated in fig. A.2.

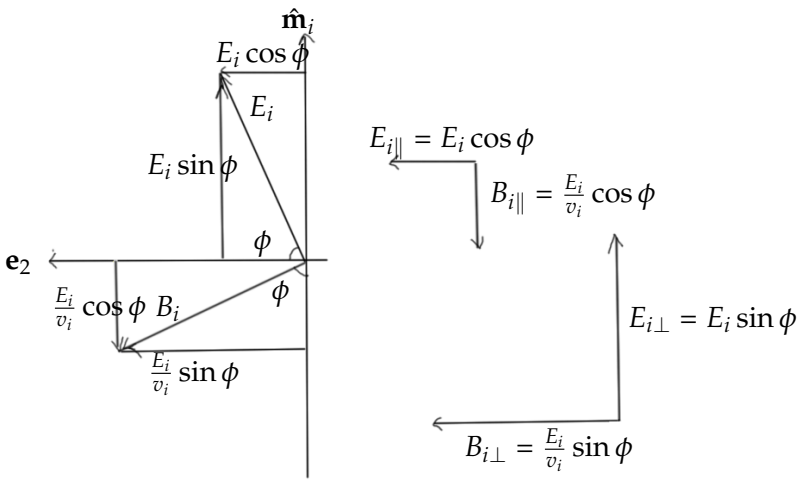

Figure A.2: Polarization of incident field to be considered

Our incident, reflected, and transmitted fields are

$$\mathbf{E}_i = E_i \mathbf{e}_2 e^{\mathbf{e}_2 \hat{\mathbf{m}}_i \phi}. \tag{A.33a}$$

$$\mathbf{E}_r = E_{i\|} \frac{1 - \alpha\beta}{1 + \alpha\beta} \mathbf{e}_2 + E_{i\perp} \frac{\beta - \alpha}{\beta + \alpha} \hat{\mathbf{m}}_r. \tag{A.33b}$$

$$\mathbf{E}_t = E_{i\|} \frac{2}{1 + \alpha\beta} \mathbf{e}_2 + E_{i\perp} \frac{2}{\beta + \alpha} \hat{\mathbf{m}}_i. \tag{A.33c}$$

However, $E_{i\|} = E_i \cos\phi$ and $E_{i\perp} = E_i \sin\phi$ leaving us with

$$\mathbf{E}_i = E_i \left(\mathbf{e}_2 \cos\phi + \mathbf{e}_1 e^{j\theta_i} \sin\phi \right). \tag{A.34a}$$

$$\mathbf{E}_r = E_i \left(\cos\phi \frac{1-\alpha\beta}{1+\alpha\beta} \mathbf{e}_2 - \sin\phi \frac{\beta-\alpha}{\beta+\alpha} \mathbf{e}_1 e^{-j\theta_r} \right). \tag{A.34b}$$

$$\mathbf{E}_t = E_i \left(\cos\phi \frac{2}{1+\alpha\beta} \mathbf{e}_2 + \sin\phi \frac{2}{\beta+\alpha} \mathbf{e}_1 e^{j\theta_t} \right). \tag{A.34c}$$

We find that the reflected and transmitted polarization angles are respectively

$$\tan\phi_r = \tan\phi \frac{\beta-\alpha}{\beta+\alpha} \frac{1+\alpha\beta}{1-\alpha\beta}. \tag{A.35a}$$

$$\tan\phi_t = \tan\phi \frac{1+\alpha\beta}{\beta+\alpha}. \tag{A.35b}$$

where the associated magnitudes are

$$\frac{E_r}{E_i} = \sqrt{\left(\cos\phi \frac{1-\alpha\beta}{1+\alpha\beta} \right)^2 + \left(\sin\phi \frac{\beta-\alpha}{\beta+\alpha} \right)^2}. \tag{A.36a}$$

$$\frac{E_t}{E_i} = \sqrt{\left(\cos\phi \frac{2}{1+\alpha\beta} \right)^2 + \left(\sin\phi \frac{2}{\beta+\alpha} \right)^2}. \tag{A.36b}$$

FIXME: in [4] he claims in §2 that "if polarized at an angle ϕ to the axis, a fraction $\sin^2\phi$ will go through". Either I have my result above wrong, or this appears to be an approximate statement?

Exercise A.5 Normal polarization angles. ([6] pr 9.14)

For normal incidence, without assuming that the reflected and transmitted waves have the same polarization as the incident wave, prove that this must be so.

Answer for Exercise A.5

Working with coordinates as illustrated in fig. A.3, the incident wave can be assumed to have the form

$$\tilde{\mathbf{E}}_\mathrm{I} = E_\mathrm{I} e^{i(kz-\omega t)} \hat{\mathbf{x}}. \tag{A.37a}$$

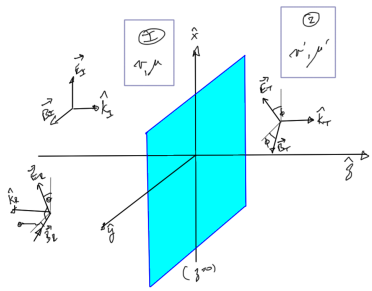

Figure A.3: Normal incidence coordinates.

$$\tilde{\mathbf{B}}_{\mathrm{I}} = \frac{1}{v}\hat{\mathbf{z}} \times \tilde{\mathbf{E}}_{\mathrm{I}} = \frac{1}{v}E_{\mathrm{I}}e^{i(kz-\omega t)}\hat{\mathbf{y}}. \tag{A.37b}$$

Assuming a polarization $\hat{\mathbf{n}} = \cos\theta\hat{\mathbf{x}} + \sin\theta\hat{\mathbf{y}}$ for the reflected wave, we have

$$\tilde{\mathbf{E}}_{\mathrm{R}} = E_{\mathrm{R}}e^{i(-kz-\omega t)}(\hat{\mathbf{x}}\cos\theta + \hat{\mathbf{y}}\sin\theta). \tag{A.38a}$$

$$\tilde{\mathbf{B}}_{\mathrm{R}} = \frac{1}{v}(-\hat{\mathbf{z}}) \times \tilde{\mathbf{E}}_{\mathrm{R}} = \frac{1}{v}E_{\mathrm{R}}e^{i(-kz-\omega t)}(\hat{\mathbf{x}}\sin\theta - \hat{\mathbf{y}}\cos\theta). \tag{A.38b}$$

And finally assuming a polarization $\hat{\mathbf{n}} = \cos\phi\hat{\mathbf{x}} + \sin\phi\hat{\mathbf{y}}$ for the transmitted wave, we have

$$\tilde{\mathbf{E}}_{\mathrm{T}} = E_{\mathrm{T}}e^{i(k'z-\omega t)}(\hat{\mathbf{x}}\cos\phi + \hat{\mathbf{y}}\sin\phi). \tag{A.39a}$$

$$\tilde{\mathbf{B}}_{\mathrm{T}} = \frac{1}{v}\hat{\mathbf{z}} \times \tilde{\mathbf{E}}_{\mathrm{T}} = \frac{1}{v'}E_{\mathrm{T}}e^{i(k'z-\omega t)}(-\hat{\mathbf{x}}\sin\phi + \hat{\mathbf{y}}\cos\phi). \tag{A.39b}$$

With no components of any of the $\tilde{\mathbf{E}}$ or $\tilde{\mathbf{B}}$ waves in the $\hat{\mathbf{z}}$ directions the boundary value conditions at $z = 0$ require the equality of the $\hat{\mathbf{x}}$ and $\hat{\mathbf{y}}$ components of

$$\left(\tilde{\mathbf{E}}_{\mathrm{I}} + \tilde{\mathbf{E}}_{\mathrm{R}}\right)_{x,y} = \left(\tilde{\mathbf{E}}_{\mathrm{T}}\right)_{x,y}. \tag{A.40a}$$

$$\left(\frac{1}{\mu}\left(\tilde{\mathbf{B}}_{\mathrm{I}} + \tilde{\mathbf{B}}_{\mathrm{R}}\right)\right)_{x,y} = \left(\frac{1}{\mu'}\tilde{\mathbf{B}}_{\mathrm{T}}\right)_{x,y}. \tag{A.40b}$$

With $\beta = \mu v/\mu' v'$, those components are

$$E_I + E_R \cos\theta = E_T \cos\phi. \tag{A.41a}$$

$$E_R \sin\theta = E_T \sin\phi. \tag{A.41b}$$

$$E_R \sin\theta = -\beta E_T \sin\phi. \tag{A.41c}$$

$$E_I - E_R \cos\theta = \beta E_T \cos\phi. \tag{A.41d}$$

Equality of eq. (A.41b), and eq. (A.41c) require

$$-\beta E_T \sin\phi = E_T \sin\phi, \tag{A.42}$$

or $(\theta, \phi) \in \{(0,0), (0,\pi), (\pi,0), (\pi,\pi)\}$. It turns out that all of these solutions correspond to the same physical waves. Let's look at each in turn

1. $(\theta, \phi) = (0, 0)$. The system eq. (A.41) is reduced to

$$\begin{aligned} E_I + E_R &= E_T \\ E_I - E_R &= \beta E_T, \end{aligned} \tag{A.43}$$

with solution

$$\begin{aligned} \frac{E_T}{E_I} &= \frac{2}{1+\beta} \\ \frac{E_R}{E_I} &= \frac{1-\beta}{1+\beta}. \end{aligned} \tag{A.44}$$

2. $(\theta, \phi) = (\pi, \pi)$. The system eq. (A.41) is reduced to

$$\begin{aligned} E_I - E_R &= -E_T \\ E_I + E_R &= -\beta E_T, \end{aligned} \tag{A.45}$$

with solution

$$\begin{aligned} \frac{E_T}{E_I} &= -\frac{2}{1+\beta} \\ \frac{E_R}{E_I} &= -\frac{1-\beta}{1+\beta}. \end{aligned} \tag{A.46}$$

Effectively the sign for the magnitude of the transmitted and reflected phasors is toggled, but the polarization vectors are also negated, with $\hat{\mathbf{n}} = -\hat{\mathbf{x}}$, and $\hat{\mathbf{n}}' = -\hat{\mathbf{x}}$. The resulting $\tilde{\mathbf{E}}_R$ and $\tilde{\mathbf{E}}_T$ are unchanged relative to those of the $(0,0)$ solution above.

3. $(\theta, \phi) = (0, \pi)$. The system eq. (A.41) is reduced to

$$\begin{aligned} E_I + E_R &= -E_T \\ E_I - E_R &= -\beta E_T, \end{aligned} \tag{A.47}$$

with solution

$$\begin{aligned} \frac{E_T}{E_I} &= -\frac{2}{1+\beta} \\ \frac{E_R}{E_I} &= \frac{1-\beta}{1+\beta}. \end{aligned} \tag{A.48}$$

Effectively the sign for the magnitude of the transmitted phasor is toggled. The polarization vectors in this case are $\hat{\mathbf{n}} = \hat{\mathbf{x}}$, and $\hat{\mathbf{n}}' = -\hat{\mathbf{x}}$, so the transmitted phasor magnitude change of sign does not change $\tilde{\mathbf{E}}_T$ relative to that of the $(0,0)$ solution above.

4. $(\theta, \phi) = (\pi, 0)$. The system eq. (A.41) is reduced to

$$\begin{aligned} E_I - E_R &= E_T \\ E_I + E_R &= \beta E_T, \end{aligned} \tag{A.49}$$

with solution

$$\begin{aligned} \frac{E_T}{E_I} &= \frac{2}{1+\beta} \\ \frac{E_R}{E_I} &= -\frac{1-\beta}{1+\beta}. \end{aligned} \tag{A.50}$$

This time, the sign for the magnitude of the reflected phasor is toggled. The polarization vectors in this case are $\hat{\mathbf{n}} = -\hat{\mathbf{x}}$, and $\hat{\mathbf{n}}' = \hat{\mathbf{x}}$. In this final variation the reflected phasor magnitude change of sign does not change $\tilde{\mathbf{E}}_R$ relative to that of the $(0,0)$ solution.

We see that there is only one solution for the polarization angle of the transmitted and reflected waves relative to the incident wave. Although we fixed the incident polarization with $\mathbf{E}$ along $\hat{\mathbf{x}}$, the polarization of the incident wave is maintained regardless of TE or TM labeling in this example, since our system is symmetric with respect to rotation.

B

VECTORAL NATURE OF LIGHT.

Exercise B.1 Poynting flux, complex 2D fields. ([5] *pr. 2.4*)

Given a complex field phasor representation of the form

$$\tilde{\mathbf{E}} = \mathbf{E}_0 e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)}. \tag{B.1}$$

$$\tilde{\mathbf{H}} = \mathbf{H}_0 e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)}. \tag{B.2}$$

Here we allow the components of $\mathbf{E}_0$ and $\mathbf{H}_0$ to be complex. As usual our fields are defined as the real parts of the phasors

$$\mathbf{E} = \mathrm{Re}(\tilde{\mathbf{E}}). \tag{B.3}$$

$$\mathbf{H} = \mathrm{Re}(\tilde{\mathbf{H}}). \tag{B.4}$$

Show that the average Poynting vector has the value

$$\langle \mathbf{S} \rangle = \langle \mathbf{E} \times \mathbf{H} \rangle = \frac{1}{2}\mathrm{Re}(\mathbf{E}_0 \times \mathbf{H}_0^*). \tag{B.5}$$

Answer for Exercise B.1

This is a problem from [5], something that I'd tried back when reading [9] but in a way that involved Geometric Algebra and the covariant representation of the energy momentum tensor. Let's try this with plain old complex vector algebra instead.

While the text works with two dimensional quantities in the x, y plane, I found this problem easier when tackled in three dimensions. Suppose we write the complex phasor components as

$$\mathbf{E}_0 = \sum_k (\mathbf{E}_{kr} + i\mathbf{E}_{ki})\mathbf{e}_k = \sum_k |\mathbf{E}_k| e^{i\phi_k}\mathbf{e}_k. \tag{B.6}$$

$$\mathbf{H}_0 = \sum_k (\mathbf{H}_{kr} + i\mathbf{H}_{ki})\mathbf{e}_k = \sum_k |\mathbf{H}_k| e^{i\psi_k} \mathbf{e}_k, \tag{B.7}$$

and also write $\phi'_k = \phi_k + \mathbf{k} \cdot \mathbf{x}$, and $\psi'_k = \psi_k + \mathbf{k} \cdot \mathbf{x}$, then our (real) fields are

$$\mathbf{E} = \sum_k |\mathbf{E}_k| \cos(\phi'_k - \omega t)\mathbf{e}_k. \tag{B.8}$$

$$\mathbf{H} = \sum_k |\mathbf{H}_k| \cos(\psi'_k - \omega t)\mathbf{e}_k, \tag{B.9}$$

and our Poynting vector before averaging (in these units) is

$$\mathbf{E} \times \mathbf{H} = \sum_{klm} |\mathbf{E}_k||\mathbf{H}_l| \cos(\phi'_k - \omega t) \cos(\psi'_l - \omega t)\epsilon_{klm}\mathbf{e}_m. \tag{B.10}$$

We are tasked with computing the average of cosines

$$\begin{aligned}
\langle \cos(a - \omega t) \cos(b - \omega t) \rangle &= \frac{1}{T} \int_0^T \cos(a - \omega t) \cos(b - \omega t) dt \\
&= \frac{1}{\omega T} \int_0^T \cos(a - \omega t) \cos(b - \omega t) \omega dt \\
&= \frac{1}{2\pi} \int_0^{2\pi} \cos(a - u) \cos(b - u) du \\
&= \frac{1}{4\pi} \int_0^{2\pi} \cos(a + b - 2u) + \cos(a - b) du \\
&= \frac{1}{2} \cos(a - b).
\end{aligned} \tag{B.11}$$

So, our average Poynting vector is

$$\langle \mathbf{E} \times \mathbf{H} \rangle = \frac{1}{2} \sum_{klm} |\mathbf{E}_k||\mathbf{H}_l| \cos(\phi_k - \psi_l)\epsilon_{klm}\mathbf{e}_m. \tag{B.12}$$

We have only to compare this to the desired expression

$$\begin{aligned}
\frac{1}{2} \operatorname{Re}(\mathbf{E}_0 \times \mathbf{H}_0^*) &= \frac{1}{2} \sum_{klm} \operatorname{Re}\left(|\mathbf{E}_k| e^{i\phi_k} |\mathbf{H}_l| e^{-i\psi_l} \right) \epsilon_{klm}\mathbf{e}_m \\
&= \frac{1}{2} \sum_{klm} |\mathbf{E}_k||\mathbf{H}_l| \cos(\phi_k - \psi_l)\epsilon_{klm}\mathbf{e}_m.
\end{aligned} \tag{B.13}$$

This proves the desired result.

Exercise B.2 Complex form of electric wave. ([5] pr. 2.5)

Show that the real electric wave

$$\mathbf{E} = E_0 \left(\hat{\mathbf{i}} \cos(kz - \omega t) + \hat{\mathbf{j}} b \cos(kz - \omega t + \phi) \right) . \tag{B.14}$$

is equivalent to the complex expression

$$\mathbf{E} = E_0 \left(\hat{\mathbf{i}} + \hat{\mathbf{j}} b e^{i\phi} \right) e^{i(kz - \omega t)}. \tag{B.15}$$

Answer for Exercise B.2

This clearly follows by inspection (only stated this problem to reference in the next.)

Exercise B.3 Some polarization plots. ([5] pr. 2.6)

For a field specified by eq. (B.15), sketch diagrams to show the type of polarization for the following parameters

1. $\phi = 0, b = 1$
2. $\phi = 0, b = 2$
3. $\phi = \pi/2, b = -1$
4. $\phi = \pi/4, b = 1$

Answer for Exercise B.3

The electric fields, with $\psi = kz - \omega t$, are

1.
$$\mathrm{Re} \left(E_0 (\hat{\mathbf{i}} + \hat{\mathbf{j}}) e^{i\psi} \right) = E_0 (\hat{\mathbf{i}} + \hat{\mathbf{j}}) \cos \psi. \tag{B.16}$$

2.
$$\mathrm{Re} \left(E_0 (\hat{\mathbf{i}} + 2\hat{\mathbf{j}}) e^{i\psi} \right) = E_0 (\hat{\mathbf{i}} + 2\hat{\mathbf{j}}) \cos \psi. \tag{B.17}$$

3.
$$\mathrm{Re} \left(E_0 (\hat{\mathbf{i}} - i\hat{\mathbf{j}}) e^{i\psi} \right) = E_0 \left(\hat{\mathbf{i}} \cos \psi + \hat{\mathbf{j}} \sin \psi \right) . \tag{B.18}$$

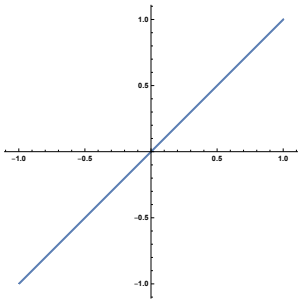

Figure B.1: Linear polarization at right angle.

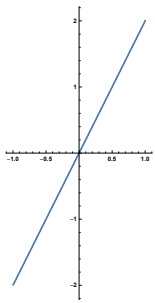

Figure B.2: Linear polarization at angle.

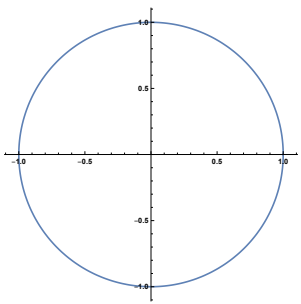

Figure B.3: Circular polarization.

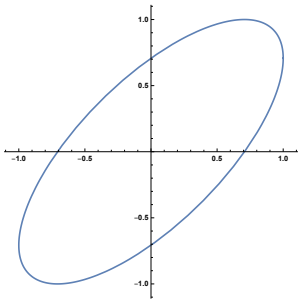

Figure B.4: Elliptical polarization.

4.

$$\mathrm{Re}\left(E_0\left(\hat{\mathbf{i}}-\frac{1}{\sqrt{2}}\hat{\mathbf{j}}(1+i)\right)e^{i\psi}\right)=E_0\left(\hat{\mathbf{i}}\cos\psi+\frac{\hat{\mathbf{j}}}{\sqrt{2}}(\cos\psi-\sin\psi)\right). \tag{B.19}$$

We have linear fig. B.1, linear fig. B.2, circular fig. B.3 and elliptical fig. B.4 polarization respectively.

Exercise B.4 Geometry of general Jones vector. ([5] *pr. 2.8*)

The general case is represented by the Jones vector

$$\begin{bmatrix} A \\ Be^{i\Delta} \end{bmatrix}. \tag{B.20}$$

Show that this represents elliptically polarized light in which the major axis of the ellipse makes an angle

$$\frac{1}{2}\tan^{-1}\left(\frac{2AB\cos\Delta}{A^2-B^2}\right), \tag{B.21}$$

with the x axis.

Answer for Exercise B.4

Prior to attempting the problem as stated, let's explore the algebra of a parametric representation of an ellipse, rotated at an angle θ as in fig. B.5. The equation of the ellipse in the rotated coordinates is

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} a\cos u \\ b\sin u \end{bmatrix}, \tag{B.22}$$

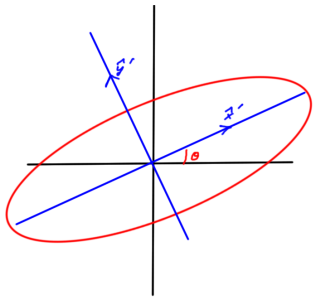

Figure B.5: Rotated ellipse.

which is easily seen to have the required form

$$\left(\frac{x'}{a}\right)^2 + \left(\frac{y'}{b}\right)^2 = 1. \tag{B.23}$$

We'd like to express x' and y' in the "fixed" frame. Consider fig. B.6 where our coordinate conventions are illustrated. With

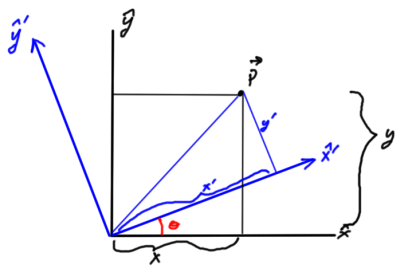

Figure B.6: 2d rotation of frame.

$$\begin{bmatrix} \hat{\mathbf{x}}' \\ \hat{\mathbf{y}}' \end{bmatrix} = \begin{bmatrix} \hat{\mathbf{x}} e^{\hat{\mathbf{x}}\hat{\mathbf{y}}\theta} \\ \hat{\mathbf{y}} e^{\hat{\mathbf{x}}\hat{\mathbf{y}}\theta} \end{bmatrix} = \begin{bmatrix} \hat{\mathbf{x}}\cos\theta + \hat{\mathbf{y}}\sin\theta \\ \hat{\mathbf{y}}\cos\theta - \hat{\mathbf{x}}\sin\theta \end{bmatrix}, \tag{B.24}$$

and $x\hat{\mathbf{x}} + y\hat{\mathbf{y}} = x'\hat{\mathbf{x}} + y'\hat{\mathbf{y}}$ we find

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}, \tag{B.25}$$

so that the equation of the ellipse can be stated as

$$\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} a\cos u \\ b\sin u \end{bmatrix}, \tag{B.26}$$

or

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} a\cos u \\ b\sin u \end{bmatrix} = \begin{bmatrix} a\cos\theta\cos u - b\sin\theta\sin u \\ a\sin\theta\cos u + b\cos\theta\sin u \end{bmatrix}. \tag{B.27}$$

Observing that

$$\cos u + \alpha\sin u = \mathrm{Re}\left((1+i\alpha)e^{-iu}\right). \tag{B.28}$$

we have, with atan2 = atan2(x, y) a Jones vector representation of our rotated ellipse

$$\begin{aligned} \begin{bmatrix} x \\ y \end{bmatrix} &= \mathrm{Re}\begin{bmatrix} (a\cos\theta - ib\sin\theta)e^{-iu} \\ (a\sin\theta + ib\cos\theta)e^{-iu} \end{bmatrix} \\ &= \mathrm{Re}\begin{bmatrix} \sqrt{a^2\cos^2\theta + b^2\sin^2\theta}\,e^{i\,\mathrm{atan2}(a\cos\theta, -b\sin\theta) - iu} \\ \sqrt{a^2\sin^2\theta + b^2\cos^2\theta}\,e^{i\,\mathrm{atan2}(a\sin\theta, b\cos\theta) - iu} \end{bmatrix}. \end{aligned} \tag{B.29}$$

Since we can absorb a constant phase factor into our $-iu$ argument, we can write this as

$$\begin{bmatrix} x \\ y \end{bmatrix} = \mathrm{Re}\left(\begin{bmatrix} \sqrt{a^2\cos^2\theta + b^2\sin^2\theta} \\ \sqrt{a^2\sin^2\theta + b^2\cos^2\theta}\,e^{i\,\mathrm{atan2}(a\sin\theta, b\cos\theta) - i\,\mathrm{atan2}(a\cos\theta, -b\sin\theta)} \end{bmatrix} e^{-iu'}\right). \tag{B.30}$$

This has the required form once we make the identifications

$$A = \sqrt{a^2\cos^2\theta + b^2\sin^2\theta}. \tag{B.31}$$

$$B = \sqrt{a^2\sin^2\theta + b^2\cos^2\theta}. \tag{B.32}$$

$$\Delta = \mathrm{atan2}(a\sin\theta, b\cos\theta) - \mathrm{atan2}(a\cos\theta, -b\sin\theta). \tag{B.33}$$

What isn't obvious is that we can do this for any A, B, and Δ. Portions of this problem I tried in ellipticalPolarizationRotationToStdForm.cdf starting from the elliptic equation derived in §8.1.3 of [8]. I'd used Mathematica since on paper I found the rotation angle that eliminated the cross terms to always be 45 degrees, but this turns out to have been because I'd first used a change of variables that scaled the equation. Here's the whole procedure without any such scaling to arrive at the desired result for this problem. Our starting point is the Jones specified field, again as above I've using $-iu = i(kz - \omega t)$

$$\mathbf{E} = \mathrm{Re}\left(\begin{bmatrix} A \\ Be^{i\Delta} \end{bmatrix} e^{-iu}\right) = \begin{bmatrix} A\cos u \\ B\cos(\Delta - u) \end{bmatrix} e^{-iu}. \tag{B.34}$$

We need our cosine angle addition formula

$$\begin{aligned} \cos(a+b) &= \mathrm{Re}\left((\cos a + i\sin a)(\cos b + i \sin b)\right) \\ &= \cos a \cos b - \sin a \sin b. \end{aligned} \tag{B.35}$$

Using this and writing $\mathbf{E} = (x, y)$ we have

$$x = A\cos u. \tag{B.36}$$

$$y = B(\cos\Delta\cos u + \sin\Delta\sin u). \tag{B.37}$$

Subtracting $x\cos\Delta/A$ from y/B we have

$$\frac{y}{B} - \frac{x}{A}\cos\Delta = \sin\Delta\sin u. \tag{B.38}$$

Squaring this and using $\sin^2 u = 1 - \cos^2 u$, and eq. (B.36) we have

$$\left(\frac{y}{B} - \frac{x}{A}\cos\Delta\right)^2 = \sin^2\Delta\left(1 - \frac{x^2}{A^2}\right), \tag{B.39}$$

which expands and simplifies to

$$\left(\frac{x}{A}\right)^2 + \left(\frac{y}{B}\right)^2 - 2\left(\frac{x}{A}\right)\left(\frac{y}{B}\right)\cos\Delta = \sin^2\Delta, \tag{B.40}$$

which is an equation of a rotated ellipse as desired. Let's figure out the angle of rotation required to kill the cross terms. Writing

$a = 1/A$, $b = 1/B$ and rotating our primed coordinate frame by θ degrees

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x' \\ y' \end{bmatrix}, \tag{B.41}$$

we have

$$\begin{aligned} \sin^2\Delta &= a^2(x'\cos\theta - y'\sin\theta)^2 + b^2(x'\sin\theta + y'\cos\theta)^2 \\ &\quad - 2ab(x'\cos\theta - y'\sin\theta)(x'\sin\theta + y'\cos\theta)\cos\Delta \\ &= (x')^2(a^2\cos^2\theta + b^2\sin^2\theta - 2ab\cos\theta\sin\theta\cos\Delta) \\ &\quad + (y')^2(a^2\sin^2\theta + b^2\cos^2\theta + 2ab\cos\theta\sin\theta\cos\Delta) \\ &\quad + 2x'y'((b^2 - a^2)\cos\theta\sin\theta + ab(\sin^2\theta - \cos^2\theta)\cos\Delta). \end{aligned} \tag{B.42}$$

To kill off the cross term we require

$$\begin{aligned} 0 &= (b^2 - a^2)\cos\theta\sin\theta + ab(\sin^2\theta - \cos^2\theta)\cos\Delta \\ &= \frac{1}{2}(b^2 - a^2)\sin(2\theta) - ab\cos(2\theta)\cos\Delta, \end{aligned} \tag{B.43}$$

or

$$\tan(2\theta) = \frac{2ab\cos\Delta}{b^2 - a^2} = \frac{2AB\cos\Delta}{A^2 - B^2}. \tag{B.44}$$

This yields eq. (B.21) as desired. We also end up with expressions for our major and minor axis lengths, which are respectively for $\sin\Delta \neq 0$

$$\sin\Delta / \sqrt{b^2 + (a^2 - b^2)\cos^2\theta - ab\sin(2\theta)\cos\Delta}. \tag{B.45}$$

$$\sin\Delta / \sqrt{b^2 + (a^2 - b^2)\sin^2\theta + ab\sin(2\theta)\cos\Delta}, \tag{B.46}$$

which completes the task of determining the geometry of the elliptic parameterization we see results from the general Jones vector description.

C

MATHEMATICA NOTEBOOKS.

These Mathematica notebooks, some just trivial ones used to generate figures, others more elaborate, and perhaps some even polished, can be found in

https://github.com/peeterjoot/mathematica/tree/master/phy485/.

The free Wolfram CDF player, is capable of read-only viewing these notebooks to some extent.

Files saved explicitly as CDF have interactive content that can be explored with the CDF player.

- Aug 5, 2012 modernOpticsProblemCh2Pr6Plots.cdf

  Plots for problem 2.6 in Fowles Modern Optics.

- Aug 9, 2012 ellipticalPolarizationRotationToStdForm.cdf

  Problem 2.9 in Fowles Modern Optics. General Jones vector

- Oct 4, 2012 modernOpticsProblemSet1.cdf

  Problem set 1 numerical and plot stuff.

- Oct 17, 2012 diffractionBesselFunctionTransformPair.cdf

  Attempt to verify the circular aperture Fourier transform result from the diffraction notes. Mathematica gives me a different result than what our Prof detailed.

- Oct 22, 2012 modernOpticsProblemSet2work.cdf

  Problem set 2 work. Verify some results. Do the plots and numerical work. This includes the integral that yields the first order Bessel function.

- Oct 24, 2012 randomVariate.cdf

  Thinking about problem set 2 problem 3b. Logical want to consider the solar originated rays as random variates in the lingo of mathematica ... functions that generate frequencies or frequency ranges as opposed to the probability that a frequency is found in a certain range.

- Oct 30, 2012 gaussianScalingVerification.cdf

  Determine the scaling and variance for a Gaussian

- Oct 30, 2012 etalon.cdf

  Plot the Etalon function. Used Evaluate and the PlotLegends package to label the level curves automatically

- Nov 01, 2012 lecture14figures.cdf

  Plots for lecture 14. One is a simple sine squared (using Ticks to mark only on 2 Pi multiples), and the other I was experimenting with Mathematica Text label placement.

- Nov 06, 2012 etalonFancyLabellingApp.cdf

  Try out Belisaris's label placement "App" for the Etalon figure.

- Nov 06, 2012 etalonFancyLabellingResult.cdf

  Results from Belisaris's label placement "App" for the Etalon figure.

- Nov 6, 2012 etalonAngularFancyLabellingApp.nb

  likely using Belisaris's labeling app

- Nov 08, 2012 lecture15figures.nb

  Plot the single slit diffraction wavefunction and N slit intensity, the latter using a Manipulate so that various parameters can be played with

- Nov 08, 2012 problemSet3.nb

  Plots and rough calculations for problem set 3

- Nov 20, 2012 lecture18figures1DQuantumSHO.nb

  Plots for lecture 18. First couple 1D Quantum SHO solutions

- Nov 21, 2012 negativeExponentialPlot.nb

  Plot of decreasing exponential

- Dec 1, 2012 lecture20figures.nb

  Plots for lecture 20.

- Dec 1, 2012 gaussianBeamHandoutNotes.nb

  Plot of the lowest order Gaussian beam envelope. Verify normalization from page 2 of the notes. Use ContourPlot3D to plot the hyperboliod of revolution for the lowest order Gaussian beam mode.

- Dec 4, 2012 fowles1028signError.nb

  Verify sign error in the characteristic poly in Fowles just before 10.28. Functions used: Collect, Solve. Also sets up a 2 by 2 matrix.

- Dec 7, 2012 midtermReflectionAbsSincPlot.nb

  Plot of Abs[Sinc[]] for Lloyd's mirror problem post midterm reflection

- Dec 10, 2012 2010finalQuestion3BesselIntegral.nb

  Bessel integral for 2010 question 3 exam practice

- Dec 11, 2012 2010finalQuestion5numericalEvaluation.nb

  Numerical evaluation for 2010 question 5a and 5b exam practice. Used the new Mathematica 9 Quantity function for easy handling of units. Provides a nice check that the right numerical combinations end up dimensionless.

- Dec 15, 2012 vanDrielz0z1z2stabilityAlgebra.nb

  Here's the algebra for the Van Driel notes that give expressions for z1 z2 z0 in terms of g1 g2, and for w(z) at these points. Too hard to do it by hand. Mathematica functions used include Notation package for subscript variables, Flatten, Solve, Eliminate, FullSimplify, and Factor.

D

COSINE TRANSFORMS.

D.1 MOTIVATION.

Cosine transforms were mentioned in the class notes. Let's work through a few basic operations ourselves to get a feel for things.

Exercise D.1 Fourier transform of an even function.

Given an even function, constructed from any arbitrary function $f(\tau)$

$$f(\tau) = \frac{1}{2}\left(g(\tau) + g(-\tau)\right), \tag{D.1}$$

determine if the Fourier transform is even or odd.
Answer for Exercise D.1

$$\begin{aligned}\int_{-\infty}^{-\infty} e^{-i\omega\tau} f(\tau) d\tau &= \frac{1}{2}\int_{-\infty}^{\infty} e^{-i\omega\tau}\left(g(\tau) + g(-\tau)\right) d\tau \\ &= \frac{1}{2}\tilde{G}(\omega) - \frac{1}{2}\int_{\infty}^{-\infty} e^{i\omega\tau} g(\tau) d\tau \\ &= \frac{1}{2}\left(\tilde{G}(\omega) + \tilde{G}(-\omega)\right).\end{aligned} \tag{D.2}$$

Yes, the Fourier transform of an even function in time is even in frequency.

Exercise D.2 Even function Fourier transform.

Express the Fourier transform of an even function in terms of cosines.
Answer for Exercise D.2

$$
\begin{aligned}
\int_{-\infty}^{\infty} e^{-i\omega\tau} f(\tau)d\tau &= \int_{-\infty}^{0} e^{-i\omega\tau} f(\tau)d\tau + \int_{0}^{\infty} e^{-i\omega\tau} f(\tau)d\tau \\
&= \int_{-\infty}^{0} e^{-i\omega\tau} f(-\tau)d\tau + \int_{0}^{\infty} e^{-i\omega\tau} f(\tau)d\tau \\
&= -\int_{\infty}^{0} e^{i\omega\tau} f(\tau)d\tau + \int_{0}^{\infty} e^{-i\omega\tau} f(\tau)d\tau \\
&= \int_{0}^{\infty} e^{i\omega\tau} f(\tau)d\tau + \int_{0}^{\infty} e^{-i\omega\tau} f(\tau)d\tau \\
&= \int_{0}^{\infty} \left(e^{i\omega\tau} + e^{-i\omega\tau}\right) f(\tau)d\tau \\
&= 2\int_{0}^{\infty} \cos(\omega\tau) f(\tau)d\tau.
\end{aligned}
\tag{D.3}
$$

Let's write

$$
\boxed{\tilde{f}_c(\omega) = \int_{0}^{\infty} \cos(\omega\tau) f(\tau)d\tau,} \tag{D.4}
$$

with

$$
\tilde{f}(\omega) = \int_{-\infty}^{\infty} e^{-i\omega\tau} f(\tau)d\tau. \tag{D.5}
$$

for the normal Fourier transform, so that

$$
\boxed{\tilde{f}(\omega) = 2\tilde{f}_c(\omega)} \tag{D.6}
$$

Exercise D.3 Inverse transform, even function.

Answer for Exercise D.3

$$
\begin{aligned}
f(\tau) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega e^{i\omega\tau}\tilde{f}(\omega) \\
&= \frac{1}{2\pi}\int_{0}^{\infty} d\omega e^{i\omega\tau}\tilde{f}(\omega) + \frac{1}{2\pi}\int_{-\infty}^{0} d\omega e^{i\omega\tau}\tilde{f}(\omega) \\
&= \frac{1}{2\pi}\int_{0}^{\infty} d\omega e^{i\omega\tau}\tilde{f}(\omega) + \frac{1}{2\pi}\int_{-\infty}^{0} d\omega e^{i\omega\tau}\tilde{f}(-\omega) \\
&= \frac{1}{2\pi}\int_{0}^{\infty} d\omega e^{i\omega\tau}\tilde{f}(\omega) - \frac{1}{2\pi}\int_{\infty}^{0} d\omega e^{-i\omega\tau}\tilde{f}(\omega) \\
&= \frac{1}{2\pi}\int_{0}^{\infty} d\omega e^{i\omega\tau}\tilde{f}(\omega) + \frac{1}{2\pi}\int_{0}^{\infty} d\omega e^{-i\omega\tau}\tilde{f}(\omega) \\
&= \frac{1}{\pi}\int_{0}^{\infty} d\omega \cos(\omega\tau)\tilde{f}(\omega).
\end{aligned}
\tag{D.7}
$$

This gives us the inverse transform relationship

$$\boxed{f(\tau) = \frac{2}{\pi}\int_0^\infty d\omega \cos(\omega\tau)\tilde{f}_c(\omega)} \tag{D.8}$$

Exercise D.4 Is the convolution of even functions even?

Answer for Exercise D.4

with

$$(f*g)(\tau) = \int_{-\infty}^{\infty} f(\tau')g(\tau'-\tau), \tag{D.9}$$

is this an even function? Let's compute at a negative time

$$\begin{aligned}(f*g)(-\tau) &= \int_{-\infty}^{\infty} d\tau' f(\tau')g(\tau'+\tau)\\ &= -\int_{\infty}^{-\infty} d\tau' f(-\tau')g(-\tau'+\tau)\\ &= \int_{-\infty}^{\infty} d\tau' f(\tau')g(-\tau'+\tau)\\ &= \int_{-\infty}^{\infty} d\tau' f(\tau')g(\tau'-\tau).\end{aligned} \tag{D.10}$$

Okay, yes, the convolution of an even function is even.

Exercise D.5 Cosine transform, even functions.

What is the cosine transformation of a convolution of even functions.

Answer for Exercise D.5

It's not obvious that we can even do this. If we start naively

$$\begin{aligned}&\int_0^\infty d\tau \cos(\omega\tau)\int_{-\infty}^{\infty} f(\tau')g(\tau'-\tau)d\tau'\\ &\quad= \int_{-\infty}^{\infty} d\tau' f(\tau')\int_0^\infty d\tau \cos(\omega\tau)g(\tau'-\tau)\\ &\quad= \int_{-\infty}^{\infty} d\tau' f(\tau')\int_{-\tau'}^{\infty} d\tau \cos(\omega(u+\tau'))g(u)\\ &\quad= \int_{-\infty}^{\infty} d\tau' f(\tau')\int_{-\tau'}^{\infty} d\tau \left(\cos(\omega u)\cos(\omega\tau') - \sin(\omega u)\sin(\omega\tau')\right) g(u).\end{aligned} \tag{D.11}$$

We see this $[0, \infty]$ interval causes us some trouble. If we had a symmetric interval, the sine term would be killed off, but we don't have that. Some thought, and explicit demonstration (above) that the convolution of even functions is also even, we can still evaluate this, but have to step back and double the interval.

$$\begin{aligned}
&\int_0^\infty d\tau \cos(\omega\tau) \int_{-\infty}^\infty f(\tau')g(\tau'-\tau)d\tau' \\
&\quad = \frac{1}{2}\int_{-\infty}^\infty d\tau \cos(\omega\tau) \int_{-\infty}^\infty f(\tau')g(\tau'-\tau)d\tau' \\
&\quad = \frac{1}{2}\int_{-\infty}^\infty d\tau e^{-i\omega\tau} \int_{-\infty}^\infty f(\tau')g(-\tau'+\tau)d\tau' \\
&\quad = \frac{1}{2}\int_{-\infty}^\infty f(\tau')d\tau' \int_{-\infty}^\infty d\tau e^{-i\omega\tau} g(-\tau'+\tau) \qquad \text{(D.12)} \\
&\quad = \frac{1}{2}\int_{-\infty}^\infty f(\tau')d\tau' \int_{-\infty}^\infty du e^{-i\omega(u+\tau')} g(u) \\
&\quad = \frac{1}{2}\tilde{g}(\omega) \int_{-\infty}^\infty f(\tau')d\tau' e^{-i\omega\tau'} \\
&\quad = \frac{1}{2}\tilde{g}(\omega)\tilde{f}(\omega).
\end{aligned}$$

We find then for the Cosine transform of a convolution

$$\boxed{\int_0^\infty d\tau \cos(\omega\tau)\left(f(\tau)*g(\tau)\right) = 2\tilde{g}_c(\omega)\tilde{f}_c(\omega).} \qquad \text{(D.13)}$$

E

CHEAT SHEET.

E.1 RULES.

Rules: Handwritten. No text or figures allowed.

E.2 GEOMETRIC OPTICS.

$$n \sin \alpha = n' \sin \alpha'. \tag{E.1}$$

$$\frac{1}{f} = \frac{1}{s} + \frac{1}{s'}. \tag{E.2}$$

$$\begin{bmatrix} y_f \\ \alpha_f \end{bmatrix}' = \begin{bmatrix} 1 & L \\ 0 & 1 \end{bmatrix} \begin{bmatrix} y_i \\ \alpha_i \end{bmatrix} \tag{E.3}$$

$$\begin{bmatrix} y_f \\ \alpha_f \end{bmatrix}' = \begin{bmatrix} 1 & 0 \\ 0 & \frac{n}{n'} \end{bmatrix} \begin{bmatrix} y_i \\ \alpha_i \end{bmatrix} \tag{E.4}$$

$$\begin{bmatrix} y_f \\ \alpha_f \end{bmatrix}' = \begin{bmatrix} 1 & 0 \\ \frac{1}{R}\left(\frac{n}{n'} - 1\right) & \frac{n}{n'} \end{bmatrix} \begin{bmatrix} y_i \\ \alpha_i \end{bmatrix} \tag{E.5}$$

$$M = \begin{bmatrix} 1 & 0 \\ 2/R & 1 \end{bmatrix} \tag{E.6}$$

$$M = \begin{bmatrix} 1 & 0 \\ -1/f & 1 \end{bmatrix} \tag{E.7}$$

$$\frac{1}{f} = \frac{n' - n}{n}\left(\frac{1}{R_1} - \frac{1}{R_2}\right), \qquad R_1 > 0, R_2 < 0. \tag{E.8}$$

$$xx' = f^2, x + f = s, x' + f = s'. \tag{E.9}$$

$$m = -\frac{s'}{s} = \frac{x'}{f}. \tag{E.10}$$

E.3 MISC TRIG.

$$\sin(\pi - \theta) = \sin\theta. \tag{E.11}$$

$$\sin(\alpha + \beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta. \tag{E.12}$$

$$\cos(\alpha + \beta) = cos(\alpha)cos(\beta) - sin(\alpha)sin(\beta). \tag{E.13}$$

$$\sin 2\theta = 2\sin\theta\cos\theta. \tag{E.14}$$

$$\cos 2\theta = 2\cos^2\theta - 1. \tag{E.15}$$

$$\cos\operatorname{atan} x = \frac{1}{\sqrt{1 + x^2}}. \tag{E.16}$$

$$\sin\operatorname{atan} x = \frac{x}{\sqrt{1 + x^2}}. \tag{E.17}$$

E.4 EIKONAL.

$$\begin{bmatrix} \mathbf{E} \\ \mathbf{B} \end{bmatrix} = \begin{bmatrix} \mathbf{E}_0(\mathbf{r}) \\ \mathbf{B}_0(\mathbf{r}) \end{bmatrix} e^{i\phi(\mathbf{r})-i\omega t}. \tag{E.18}$$

$$n = \frac{c}{v} = \sqrt{\epsilon\mu}. \tag{E.19}$$

$$\begin{bmatrix} \mathbf{E} \\ \mathbf{B} \end{bmatrix} = \begin{bmatrix} \mathbf{E}_0(\mathbf{r}) \\ \mathbf{B}_0(\mathbf{r}) \end{bmatrix} e^{i\phi(\mathbf{r})-i\omega t}. \tag{E.20}$$

$$\nabla \cdot \mathbf{E} = e^{-i\omega t} \left(\cancel{e^{i\phi(\mathbf{r})} \nabla \cdot \mathbf{E}_0(\mathbf{r})} + \mathbf{E}_0(\mathbf{r}) \cdot \left(\nabla e^{i\phi(\mathbf{r})} \right) \right). \tag{E.21}$$

and

$$\nabla \times \mathbf{E} = e^{-i\omega t} \left(\cancel{e^{i\phi(\mathbf{r})} \nabla \times \mathbf{E}_0(\mathbf{r})} - \mathbf{E}_0(\mathbf{r}) \times \left(\nabla e^{i\phi(\mathbf{r})} \right) \right). \tag{E.22}$$

$$\mathbf{E}_0 \cdot \nabla\phi = 0. \tag{E.23a}$$

$$\mathbf{B}_0 \cdot \nabla\phi = 0. \tag{E.23b}$$

$$\nabla\phi \times \mathbf{E}_0 = k_0 \mathbf{B}_0. \tag{E.23c}$$

$$\nabla\phi \times \mathbf{B}_0 = -\epsilon k_0 \mathbf{E}_0. \tag{E.23d}$$

$$|\nabla\phi|^2 = k_0^2 \epsilon(\mathbf{r}). \tag{E.24}$$

$$|\nabla\phi| = k_0 n(\mathbf{r}). \tag{E.25}$$

$$\langle \mathbf{S} \rangle = \frac{c}{8\pi k_0} |\mathbf{E}_0|^2 \nabla \phi. \tag{E.26}$$

$$\mathbf{t} = \frac{d\mathbf{r}(s)}{ds} = \frac{\nabla \phi}{|\nabla \phi|} \qquad = \frac{\nabla \phi}{n(\mathbf{r}) k_0}. \tag{E.27}$$

$$n(\mathbf{r}) \frac{d\mathbf{r}}{ds} = \frac{1}{k_0} \nabla \phi. \tag{E.28}$$

$$\frac{d}{ds} \left(n(\mathbf{r}) \frac{d\mathbf{r}}{ds} \right) = \nabla n(\mathbf{r}). \tag{E.29}$$

E.5 WAVE RELATIONS.

$$k = \frac{2\pi}{\lambda}. \tag{E.30a}$$

$$\omega = 2\pi \nu. \tag{E.30b}$$

$$k = \frac{\omega}{c}. \tag{E.30c}$$

$$k = n k_0. \tag{E.30d}$$

$$\lambda = \lambda_0 / n. \tag{E.30e}$$

E.6 ELECTRODYNAMICS.

$$\nabla \cdot \mathbf{D} = 0 \tag{E.31}$$

$$\nabla \cdot \mathbf{B} = 0 \tag{E.32}$$

$$\nabla \times \mathbf{E} = -\frac{1}{c} \frac{\partial \mathbf{B}}{\partial t} \tag{E.33}$$

$$\nabla \times \mathbf{B} = \frac{1}{c} \frac{\partial \mathbf{D}}{\partial t} \tag{E.34}$$

$$\mathbf{D} = \epsilon \mathbf{E}. \tag{E.35}$$

$$\mathbf{S} = \frac{c}{4\pi} \operatorname{Re} \mathbf{E} \times \operatorname{Re} \mathbf{B}. \tag{E.36}$$

$$\mathcal{L}_{\text{EM}} = \frac{1}{c}\rho\phi + \frac{1}{c}\mathbf{j} \cdot \mathbf{A} + \frac{1}{8\pi}\mathbf{E}^2 - \frac{1}{8\pi}\mathbf{B}^2. \tag{E.37}$$

$$\mathbf{E} = \operatorname{Re}\left(\mathbf{E}_0 e^{i\mathbf{k}\cdot\mathbf{x}-\omega t}\right) \tag{E.38}$$

$$\mathbf{B} = \operatorname{Re}\left(\mathbf{B}_0 e^{i\mathbf{k}\cdot\mathbf{x}-\omega t}\right) \tag{E.39}$$

$$\langle \mathbf{E} \times \mathbf{B} \rangle = \frac{1}{2} \operatorname{Re}(\mathbf{E}_0 \times \mathbf{B}_0^*). \tag{E.40}$$

$$I = c\epsilon_0 \left\langle |\mathbf{E}|^2 \right\rangle = \frac{c}{4\pi}\sqrt{\frac{\epsilon}{\mu}} \left\langle |\mathbf{E}|^2 \right\rangle. \tag{E.41}$$

E.7 MISC CALCULUS RESULTS.

$$(\boldsymbol{\nabla}\phi \cdot \boldsymbol{\nabla})\, \boldsymbol{\nabla}\phi = \frac{1}{2}\boldsymbol{\nabla}\left(\boldsymbol{\nabla}\phi\right)^2. \tag{E.42}$$

$$\boldsymbol{\nabla} e^{i\phi} = i(\boldsymbol{\nabla}\phi)e^{i\phi}. \tag{E.43}$$

$$\boldsymbol{\nabla} = \hat{\mathbf{r}}\frac{\partial}{\partial r} + \frac{\hat{\boldsymbol{\theta}}}{r}\frac{\partial}{\partial \theta} + \hat{\mathbf{z}}\frac{\partial}{\partial z}. \tag{E.44}$$

E.8 DIFFRACTION.

$$\mathbf{R} = \mathbf{r} - \mathbf{r}' \tag{E.45}$$
$$\mathbf{R}_s = \mathbf{r}_s - \mathbf{r}' \tag{E.46}$$
$$R = |\mathbf{R}| \tag{E.47}$$
$$R_s = |\mathbf{R}_s| \tag{E.48}$$

$$\left(\boldsymbol{\nabla}^2 + \mathbf{k}^2\right)\Psi(\mathbf{r}) = 0. \tag{E.49}$$

$$\Psi(\mathbf{r}) = \iint da' \left(\Psi(\mathbf{r}')\boldsymbol{\nabla}' G - G\boldsymbol{\nabla}'\Psi(\mathbf{r}')\right) \cdot \hat{\mathbf{n}}. \tag{E.50}$$

$$G(\mathbf{r}, \mathbf{r}') = -\frac{e^{ikR}}{4\pi R} = -\frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{4\pi|\mathbf{r}-\mathbf{r}'|}. \tag{E.51}$$

$$\boldsymbol{\nabla}\left(\frac{e^{ikr}}{r}\right) = \hat{\mathbf{r}}\left(ik - \frac{1}{r}\right)\frac{e^{ikr}}{r}. \tag{E.52}$$

$$\Psi(\mathbf{r}) = -\frac{1}{4\pi}\iint \frac{e^{ikR}}{R}\hat{\mathbf{n}} \cdot \left(\boldsymbol{\nabla}'\Psi(\mathbf{r}') + \left(ik - \frac{1}{R}\right)\frac{\mathbf{R}}{R}\Psi(\mathbf{r}')\right) da'. \tag{E.53}$$

$$\Psi(\mathbf{r}) = \frac{A}{\lambda i}\iint da' \frac{e^{ik(r+r_s)}}{rr_s} k(\theta). \tag{E.54}$$

$$k(\theta) = \frac{1}{2}\left(1 + \cos\theta\right). \tag{E.55}$$

$$kR = kr - k\hat{\mathbf{r}} \cdot \mathbf{r}' + \frac{k}{2r}\left(\mathbf{r}'^2 - \left(\hat{\mathbf{r}} \cdot \mathbf{r}'\right)^2\right) + \cdots \tag{E.56}$$

$$\Psi(\mathbf{r}) = \frac{A}{\lambda i}\frac{e^{ik(r_s+r)}}{r_s r}\iint_A da' e^{ikf(\mathbf{r}')}. \tag{E.57}$$

$$f(\mathbf{r}') = -\left(\hat{\mathbf{r}} + \hat{\mathbf{r}}_s\right) \cdot \mathbf{r}' + \frac{1}{2r}\left(\mathbf{r}'^2 - (\hat{\mathbf{r}} \cdot \mathbf{r}')^2\right) + \frac{1}{2r_s}\left((\mathbf{r}')^2 - (\hat{\mathbf{r}}_s \cdot \mathbf{r}')^2\right). \tag{E.58}$$

$$\hat{\mathbf{r}} \sim (\alpha, \beta, 1). \tag{E.59}$$

$$\Psi(\mathbf{r}) = \frac{\Psi_s}{i\lambda}\frac{e^{ikf}}{f}\iint_A e^{-ik(\alpha x' + \beta y')} da'. \tag{E.60}$$

$$kf = \frac{k}{2}\left(r_s^{-1} + r^{-1}\right)\left(x'^2 + y'^2\right) = \frac{\pi}{2}\left(u^2 + v^2\right). \tag{E.61}$$

$$\Psi(\mathbf{r}) = \frac{A}{2i}e^{ik(r_s + r)}\frac{1}{r_s + r}\int_A e^{i\frac{\pi}{2}(u^2+v^2)} du dv. \tag{E.62}$$

$$\int_{-\infty}^{\infty} dv e^{i\frac{\pi}{2}v^2} = 1 + i = \sqrt{2}e^{i\pi/4}. \tag{E.63}$$

$$S(w) = \int_0^w \sin\left(\frac{\pi}{2}u^2\right) du. \tag{E.64a}$$

$$C(w) = \int_0^w \cos\left(\frac{\pi}{2}u^2\right) du. \tag{E.64b}$$

$$1 = \left(\frac{dS}{dw}\right)^2 + \left(\frac{dC}{dw}\right)^2. \tag{E.65}$$

$$\tan\left(\frac{\pi}{2}w^2\right) = \frac{dy}{dx}. \tag{E.66}$$

E.9 COHERENCE.

$$\Psi_1 = \sqrt{I_1(\mathbf{r})}e^{i\phi_1(\mathbf{r},t)} \tag{E.67}$$

$$\Psi_2 = \sqrt{I_2(\mathbf{r})}e^{i\phi_2(\mathbf{r},t)} \tag{E.68}$$

$$I = |\Psi_1 + \Psi_2|^2 = |\Psi_1|^2 + |\Psi_2|^2 + 2\,\mathrm{Re}\,\Gamma_{12}. \tag{E.69}$$

$$\Gamma_{12} = \langle \Psi_1 \Psi_2^* \rangle\,. \tag{E.70}$$

$$\mathcal{V} \equiv \frac{I_{\max} - I_{\min}}{I_{\max} + I_{\min}} = \frac{2\sqrt{I_1 I_2}}{I_1 + I_2}|\gamma_{12}| = |\gamma_{12}|. \tag{E.71}$$

$$\gamma_{12} = \frac{\Gamma_{12}}{\sqrt{I_1}\sqrt{I_2}}. \tag{E.72}$$

E.9.1 *Temporal coherence.*

$$\langle f(t) \rangle = \lim_{T\to\infty} \frac{1}{T} \int_0^T dt' f(t'). \tag{E.73}$$

$$I = \left\langle |\Psi|^2 \right\rangle = I(\mathbf{r}_1) + I(\mathbf{r}_2) + 2\,\mathrm{Re}\,\langle \Psi(\mathbf{r}_1, t)\Psi^*(\mathbf{r}_2, t+\tau) \rangle\,. \tag{E.74}$$

$$\Gamma_{12} \equiv \langle \Psi(\mathbf{r}_1, t)\Psi^*(\mathbf{r}_2, t+\tau) \rangle\,. \tag{E.75}$$

$$\tau_{\mathrm{coh}} = \frac{1}{\Delta\omega}. \tag{E.76}$$

$$I = I_1 + I_2 + 2\sqrt{I_1 I_2}\,\mathrm{Re}\,\gamma_{12}. \tag{E.77}$$

$$\gamma(\tau) = \gamma_{12} = \mathcal{F}\{I(\omega)\}. \tag{E.78}$$

$$\mathrm{Re}(\Gamma(\tau)) = \int_0^\infty I(\omega)\cos(\omega\tau)d\omega. \tag{E.79a}$$

$$I(\omega) = \int_0^\infty \mathrm{Re}\left(\Gamma(\tau)\right)\cos(\omega\tau)d\tau. \tag{E.79b}$$

$$\tau = \frac{s_2 - s_1}{c}. \tag{E.80}$$

$$\begin{aligned} I &= |\gamma_{12}|\left(I_1 + I_2 + 2\sqrt{I_1 I_2}\cos\left(\alpha_{12}(\tau) - \delta\right)\right) \\ &\qquad + \left(1 - |\gamma_{12}|\right)\left(I_1 + I_2\right) \\ &= |\gamma_{12}| I_{\mathrm{coh}} + (1 - |\gamma_{12}|) I_{\mathrm{incoh}}. \end{aligned} \tag{E.81}$$

$$\tau_a - \tau_b = \frac{r_{1a} - r_{2a} - r_{1b} + r_{2b}}{c} = \frac{l\theta_s}{c}. \tag{E.82}$$

E.9.2 *Spatial coherence.*

$$\sum_{\mathbf{k}} I_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{l}} = \Gamma_{12} = \mathcal{F}(I_{\mathbf{k}}). \tag{E.83}$$

$$\begin{aligned} I &= \sum_{\mathbf{k}} \left|\Psi_k(\mathbf{r}_1, t) + e^{i\mathbf{k}\cdot\mathbf{l}}\Psi_k(\mathbf{r}_1, t)\right|^2 \\ &= \sum \left(2I_{\mathbf{k}} + 2\,\mathrm{Re}\left(\Psi_k^* e^{i\mathbf{k}\cdot\mathbf{l}}\Psi_k\right)\right) \\ &= 2\sum I_{\mathbf{k}} + 2\,\mathrm{Re}\left(\sum_{\mathbf{k}} I_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{l}}\right). \end{aligned} \tag{E.84}$$

$$\Gamma_{12} = \frac{1}{\lambda^2 \overline{R_1}\,\overline{R_2}} e^{ik\Delta\mathbf{r}\cdot\hat{\mathbf{r}}_{\mathrm{av}}} \int d^2 r_s g(\mathbf{r}_s) I(\mathbf{r}_s) e^{-ik\Delta\mathbf{r}\cdot\mathbf{r}_s/r_{\mathrm{av}}}. \tag{E.85a}$$

$$\Delta \mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2. \tag{E.85b}$$

$$\mathbf{r}_{\text{av}} = \mathbf{r}_2 - \Delta \mathbf{r}/2. \tag{E.85c}$$

$$k\left(R_1 - R_2\right) \approx k\left(\mathbf{r}_{\text{av}} - \mathbf{r}_s\right) \cdot \frac{\Delta \mathbf{r}}{r_{\text{av}}}. \tag{E.85d}$$

$$\gamma(\mathbf{r}_1, \mathbf{r}_2) = \frac{\Gamma_{12}}{\Gamma_{12}|_{\Delta r=0}}. \tag{E.85e}$$

$$\mathcal{V} = |\gamma_{12}| = 2\left|\frac{J_1(\pi\theta_s d/\lambda)}{\pi\theta_s d/\lambda}\right|. \tag{E.86}$$

$$l_c = \frac{\lambda}{\Delta\theta_s}. \tag{E.87}$$

E.10 MULTIPLE INTERFERENCE.

E.10.1 *Fabry-Perot.*

$$\delta = 2kL\cos\theta. \tag{E.88a}$$

$$r = e^{i\delta_r}\sqrt{R}. \tag{E.88b}$$

$$t = e^{i\delta_t}\sqrt{T}. \tag{E.88c}$$

$$\Delta = 2\delta_r + \delta = 2\pi m. \tag{E.88d}$$

$$\Psi_{\text{transmission}} = \Psi_0 t^2 \frac{1}{1 - Re^{i\Delta}}. \tag{E.88e}$$

$$I_{\text{trans}} = \frac{I_{\text{max}}}{1 + F\sin^2(\Delta/2)}. \tag{E.88f}$$

$$I_{\text{max}} = \frac{I_0 T^2}{(1-R)^2}. \tag{E.88g}$$

$$F = \frac{4R}{(1-R)^2}. \tag{E.88h}$$

$$\mathcal{F} = \pi \frac{\sqrt{R}}{1-R} = \frac{\pi}{2}\sqrt{F} \sim \frac{\pi}{T}. \tag{E.88i}$$

$$\frac{\omega_1 - \omega_2}{\overline{\omega}} = \frac{1}{\mathcal{F} m}. \tag{E.88j}$$

$$\overline{\omega} = \frac{\pi c}{L}(m + j) = \omega_0 + j\text{F S R}. \tag{E.88k}$$

$$\delta = \omega - \omega_m. \tag{E.89a}$$

$$\frac{I}{I_0} = \frac{1}{1 + \frac{\delta^2}{\Gamma^2}}. \tag{E.89b}$$

$$\Gamma = \frac{\pi c}{2L\mathcal{F}} = \frac{\text{FSR}}{2\mathcal{F}}. \tag{E.89c}$$

$$\gamma = \frac{1}{2} a \frac{\omega}{c} \sin\theta = \pi \frac{\omega}{\omega_0} \sin\theta. \tag{E.90}$$

$$\omega_0 = \frac{2\pi c}{a}. \tag{E.91}$$

E.10.2 *Diffraction grating interferometry.*

$$I = I_0 \left(\frac{\sin\beta}{\beta}\right)^2 \left(\frac{\sin N\gamma}{N\sin\gamma}\right)^2. \tag{E.92a}$$

$$\beta = \frac{1}{2}bk_y = \frac{1}{2}bk\sin\theta. \tag{E.92b}$$

$$\gamma = \frac{1}{2}k_y a = \frac{1}{2}ka\sin\theta. \tag{E.92c}$$

$$\omega_0 = \frac{2\pi c}{a}. \tag{E.92d}$$

$$\gamma = \pi\frac{\omega}{\omega_0}\sin\theta. \tag{E.92e}$$

$$\gamma = m\pi = \pi\frac{\omega}{\omega_0}\sin\theta. \tag{E.92f}$$

$$l = Nm + 1. \tag{E.92g}$$

$$\Delta\gamma = \frac{m\pi}{\omega_1}\Delta\omega = \frac{\pi}{N}. \tag{E.92h}$$

$$\frac{\Delta\omega}{\overline{\omega}} = \frac{1}{Nm}. \tag{E.92i}$$

E.11 LASERS.

$$\frac{N_e}{N_g} = \frac{P_e}{P_g} = e^{-(E_e - E_g)/kT}. \tag{E.93a}$$

$$u_\omega = \hbar\omega\,\langle n_\omega\rangle\,\mathcal{D}(\omega). \tag{E.93b}$$

$$\langle n_\omega \rangle = \frac{1}{e^{\hbar\omega/kT} - 1}. \tag{E.93c}$$

$$\mathcal{D}(\omega) = \frac{\omega^2}{\pi^2 c^3}. \tag{E.93d}$$

$$\frac{d}{dt} N_e = -AN_e + B_{\text{abs}} u N_g - B_{\text{se}} u N_e = 0. \tag{E.93e}$$

$$\frac{d}{dt} N_g = -\frac{d}{dt} N_e. \tag{E.93f}$$

$$u_\omega = \frac{\hbar\omega^3}{\pi^2 c^3} \frac{1}{e^{\hbar\omega/kT} - 1}. \tag{E.93g}$$

$$\frac{A}{B_{\text{se}}} = \frac{\hbar\omega^3}{\pi^2 c^3}. \tag{E.93h}$$

$$B_{\text{abs}}/B_{\text{se}} = 1. \tag{E.93i}$$

$$\frac{Bu}{A} = \langle n_\omega \rangle. \tag{E.93j}$$

$$\langle n \rangle^2 - (C-1) n_s \langle n \rangle - C n_s = 0. \tag{E.93k}$$

$$C \equiv \frac{NR\Gamma_{\text{st}}}{\Gamma_{\text{sp}}\Gamma_{\text{cav}}}. \tag{E.93l}$$

$$n_s \equiv \frac{\Gamma_{\text{sp}}}{\Gamma_{\text{st}}}. \tag{E.93m}$$

$$\langle n \rangle \approx \begin{cases} (C-1) n_s & C > 1 \\ \frac{C}{1-C} & C < 1 \end{cases} \tag{E.93n}$$

$$N_2 = \frac{NR}{\Gamma_{\text{sp}} + \Gamma_{\text{st}} \langle n \rangle} = \frac{C}{1 + \langle n \rangle / n_s}. \tag{E.93o}$$

E.12 GAUSSIAN BEAMS.

$$\boldsymbol{\nabla}^2\mathbf{E}(\mathbf{r}) + k^2(\mathbf{r})\mathbf{E}(\mathbf{r}) = 0. \tag{E.94a}$$

$$k^2(\mathbf{r}) = k_0^2 - k_0 k_2 r^2. \tag{E.94b}$$

$$\mathbf{E} = \mathbf{E}_0 u(r,\theta,z) e^{ik_0 z}. \tag{E.94c}$$

$$\boldsymbol{\nabla}^2 = \boldsymbol{\nabla}_{\mathrm{T}}^2 + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2} + \frac{\partial^2}{\partial z^2}. \tag{E.94d}$$

$$\boldsymbol{\nabla}_{\mathrm{T}}^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}. \tag{E.94e}$$

$$\cancel{\frac{\partial^2 u}{\partial z^2}} + 2ik_0\frac{\partial u}{\partial z} + \boldsymbol{\nabla}_{\mathrm{T}}^2 u - k_0 k_2 r^2 u = 0. \tag{E.94f}$$

$$\frac{\omega_{\mathrm{eff}}}{c} \equiv \sqrt{\frac{k_2}{k_0}}. \tag{E.94g}$$

$$m_{\mathrm{eff}} \equiv \frac{\hbar k_0}{c}. \tag{E.94h}$$

$$t_{\mathrm{eff}} \equiv \frac{z}{c}. \tag{E.94i}$$

$$-\frac{\hbar^2}{2m_{\mathrm{eff}}}\boldsymbol{\nabla}_{\mathrm{T}}^2 u + \frac{1}{2}m_{\mathrm{eff}}\left(\omega_{\mathrm{eff}}\right)^2 r^2 u = i\hbar\frac{\partial u}{\partial t_{\mathrm{eff}}}. \tag{E.94j}$$

$$u(r,z) = \frac{z_0}{iq}\exp\left(i\frac{k_0 r^2}{2q(z)}\right). \tag{E.94k}$$

$$q = z - iz_0. \tag{E.94l}$$

$$z_0 = \frac{\pi w_0^2}{\lambda} = \frac{\pi n w_0^2}{\lambda_0}. \tag{E.94m}$$

$$u = \frac{w_0}{w(z)} \exp\left(-\frac{r^2}{w^2(z)} + i\frac{k_0 r^2}{2R(z)} - i\phi(z)\right). \tag{E.94n}$$

$$\phi(z) = \operatorname{atan}\left(\frac{z}{z_0}\right). \tag{E.94o}$$

$$w^2(z) = w_0^2\left(1 + \frac{z^2}{z_0^2}\right). \tag{E.94p}$$

$$\frac{1}{R(z)} = \frac{z}{z^2 + z_0^2}. \tag{E.94q}$$

$$\frac{1}{q(z)} = \frac{1}{R(z)} + i\frac{\lambda_0}{n\pi w^2(z)}. \tag{E.94r}$$

$$\Theta_{\text{div}} = \frac{w_0}{z_0} = \frac{1}{\pi w_0}\lambda. \tag{E.94s}$$

$$-i\phi(z) + ik_0 z - i\omega t = iC. \tag{E.95a}$$

$$\frac{dz}{dt} = V_{\text{ph}} = \frac{\omega}{k_{\text{eff}}}. \tag{E.95b}$$

$$k_{\text{eff}} = k_0 - \frac{z_0}{z^2 + z_0^2}. \tag{E.95c}$$

$$u_{lm}(x,y,z) \sim \frac{w_0}{w(z)} \exp\left(-\frac{r^2}{w^2(z)} + \frac{ik_0 r^2}{R(z)} - i(m+l+1)\phi(z)\right) \times H_l\left(\frac{\sqrt{2}x}{w(z)}\right) H_m\left(\frac{\sqrt{2}y}{w(z)}\right) \tag{E.96a}$$

$$\begin{aligned} H_0(x) &= 1 \\ H_1(x) &= 2x \\ H_2(x) &= 4x^2 - 1 \end{aligned} \tag{E.96b}$$

$$k_{\text{eff}} = k_0 - (m+l+1)\frac{z_0}{z^2 + z_0^2}. \tag{E.96c}$$

$$q' = \frac{Aq + B}{Cq + D}. \tag{E.96d}$$

$$0 < g_1 g_2 < 1. \tag{E.96e}$$

$$g_{1,2} = 1 - \frac{L}{R_{1,2}}. \tag{E.96f}$$

E.13 FOURIER TRANSFORMS.

$$f(x) = g * h = \int_{-\infty}^{\infty} dx' g(x') h(x - x'). \tag{E.97}$$

$$F(k) = G(k)H(k). \tag{E.98}$$

F

PLANCK BLACKBODY SUMMATION.

F.1 MOTIVATION.

Here's a silly exercise. I'm so used to seeing imaginaries in $e^{\cdots\omega\cdots}$ expressions, when I looked at the famous blackbody summation for an exponentially decreasing probability distribution

$$\langle n_\omega \rangle = \sum_{n=0}^{\infty} nP(n) = \frac{\sum_{n=0}^{\infty} ne^{-\hbar\omega n/kT}}{\sum_{n=0}^{\infty} e^{-\hbar\omega n/kT}}, \tag{F.1}$$

I imagined (sic) an imaginary in the exponential and thought "how can that converge?". I thought things must somehow magically work out if the limits are taken carefully, so I derived the finite summation expressions using the old tricks.

F.2 GUTS.

If we want to sum a discrete power series, say

$$S_N(x) = 1 + x + x^2 + \cdots x^{N-1} = \sum_{n=0}^{N-1} x^n, \tag{F.2}$$

we have only to take the difference

$$xS_N - S_N = x^N - 1, \tag{F.3}$$

so we have, regardless of the magnitude of x

$$\boxed{S_N(x) = \frac{1 - x^N}{1 - x}.} \tag{F.4}$$

Observe that the derivative of S_N is

$$\frac{dS_N}{dx} = \sum_{n=1}^{N-1} nx^{n-1} = \frac{1}{x} \sum_{n=1}^{N-1} nx^n, \tag{F.5}$$

but we also have

$$\begin{aligned}
\frac{dS_N}{dx} &= S_N(x) \\
&= \frac{-Nx^{N-1}}{1-x} + \frac{1-x^N}{(1-x)^2} \\
&= \frac{1}{(1-x)^2}\left(-Nx^{N-1}(1-x) + 1 - x^N\right) \\
&= \frac{1}{(1-x)^2}\left(-Nx^{N-1} + Nx^N + 1 - x^N\right) \\
&= \frac{1}{(1-x)^2}\left(1 - Nx^{N-1} + (N-1)x^N\right).
\end{aligned} \tag{F.6}$$

We expect this and eq. (F.5) to differ only by a constant. For eq. (F.5), or $dS_N/dx = 1 + 2x + 3x^2 + \cdots$, we have 1 at the origin, the same as eq. (F.6). Our conclusion is

$$\boxed{\sum_{n=1}^{N-1} nx^n = \frac{x}{(1-x)^2}\left(1 - Nx^{N-1} + (N-1)x^N\right),} \tag{F.7}$$

a result that applies, no matter the magnitude of x. Now we can form the Planck summation up to some discrete summation point (say $N-1$)

$$\frac{\sum_{n=0}^{N-1} ne^{-\hbar\omega n/kT}}{\sum_{n=0}^{N-1} e^{-\hbar\omega n/kT}} = \frac{x}{1-x}\left(1 - Nx^{N-1} + (N-1)x^N\right)\frac{1}{1-x^N}. \tag{F.8}$$

I got this far and noticed there's still an issue with $N \to \infty$. Taking a second look, I see that we have a plain old real exponential, something perhaps like fig. F.1. It doesn't really matter what the value of $\hbar/kT$ is (if considering the function one of ω), it will be greater than zero, so that we have for our sum

$$\frac{\sum_{n=0}^{\infty} ne^{-\hbar\omega n/kT}}{\sum_{n=0}^{\infty} e^{-\hbar\omega n/kT}} = \frac{e^{-\hbar\omega/kT}}{1 - e^{-\hbar\omega/kT}} = \frac{1}{e^{\hbar\omega/kT} - 1}, \tag{F.9}$$

which is the Planck result.

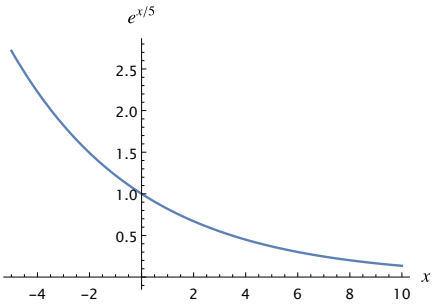

Figure F.1: Plot of $e^{-x/5}$.

G

VECTOR IDENTITIES.

G.1 CURL OF CURL.

Expanding the c component of $\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \mathbf{A})$ we have

$$\begin{aligned}
(\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \mathbf{A}))_c &= \partial_a (\boldsymbol{\nabla} \times \mathbf{A})_b \epsilon_{abc} \\
&= \partial_a \left(\partial_m A_n \epsilon_{mnb}\right) \epsilon_{abc} \\
&= \partial_a \partial_m A_n \delta_{ca}^{[mn]} \\
&= \partial_a \left(\partial_c A_a - \partial_a A_c\right) \\
&= \partial_c (\boldsymbol{\nabla} \cdot \mathbf{A}) - \boldsymbol{\nabla}^2 A_c .
\end{aligned} \tag{G.1}$$

or

$$\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \mathbf{A}) = \boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \mathbf{A}) - \boldsymbol{\nabla}^2 \mathbf{A}. \tag{G.2}$$

H

FOWLES OPTICS TYPOS.

See http://www.6911norfolk.com/dolbln/h7cf99/fowles.pdf, for a nice collection of errata notes for [5].

Two others:

- page 137: equation (5.52) A factor of k/L was dropped from the exponential.
- It also seems to me that Fowles (3.45) has an additional factor of $1/\sqrt{2\pi}$ and that the $1/\sqrt{2\pi}$ in (3.46) should have no square root (both on page 81).

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BIBLIOGRAPHY

[1] D. Bohm. *Quantum Theory*. Courier Dover Publications, 1989. (Cited on page 260.)

[2] M. Born and E. Wolf. *Principles of optics: electromagnetic theory of propagation, interference and diffraction of light*. Cambridge university press, 1980. (Cited on pages xi, 11, 27, 41, 64, and 116.)

[3] BR Desai. *Quantum mechanics with basic field theory*. Cambridge University Press, 2009. (Cited on page 218.)

[4] PAM Dirac. *The principles of quantum mechanics*. Oxford:[sn], 1974. (Cited on page 279.)

[5] G.R. Fowles. *Introduction to modern optics*. Dover Pubns, 1989. (Cited on pages xi, 65, 101, 115, 172, 206, 250, 285, 287, 289, and 325.)

[6] David Jeffrey Griffiths and Reed College. *Introduction to electrodynamics*. Prentice hall Upper Saddle River, NJ, 3rd edition, 1999. (Cited on pages 267, 269, 270, and 279.)

[7] G. Grynberg, A. Aspect, and C. Fabre. *Introduction to quantum optics: from the semi-classical approach to quantized light*. Cambridge university press, 2010. (Cited on page 234.)

[8] E. Hecht. *Optics*. 1998. (Cited on pages xi, 1, 35, 37, 65, 101, 139, 267, and 292.)

[9] JD Jackson. *Classical Electrodynamics*. John Wiley and Sons, 2nd edition, 1975. (Cited on pages 64 and 285.)

[10] Peeter Joot. *Quantum Mechanics II.*, chapter Verifying the Helmholtz Green's function. peeterjoot.com, 2011. URL `https://peeterjoot.com/archives/math2011/phy456.pdf`. [Online; accessed 22-August-2023]. (Cited on page 65.)

[11] C. Kittel and H. Kroemer. *Thermal physics*. WH Freeman, 1980. (Cited on page 206.)

[12] J. Pahikkala. *Fresnel formulas — PlanetMath*, 2012. URL https://planetmath.org/fresnelformulas. [Online; accessed 22-May-2014]. (Cited on page 75.)

[13] user1874. How do you find the velocity function of a mechanical wave? Physics Stack Exchange. URL https://physics.stackexchange.com/q/5026. URL:https://physics.stackexchange.com/q/5026 (version: 2013-03-12). (Cited on page 229.)

[14] Wikipedia. *Optical cavity — Wikipedia, The Free Encyclopedia*, 2012. URL https://en.wikipedia.org/w/index.php?title=Optical_cavity&oldid=505038610. [Online; accessed 6-December-2012]. (Cited on page 259.)

[15] H.G. Winful et al. Physical origin of the gouy phase shift. *Optics letters*, 26(8):485–487, 2001. URL http://users.unimi.it/aqm/wp-content/uploads/Feng-2001.pdf. [Online; accessed 22-May-2014]. (Cited on page 229.)

[16] A. Yariv. *Quantum Electronics*. Wiley and Sons, New York, 1989. (Cited on pages 211 and 245.)