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Shortest distance between two planes.

1.1 The problem.

Helping Karl with his linear algebra exam prep, he asked me about this problem

Exercise 1.1

Find the shortest distance between the two parallel planes, P_1, and P_2, with respective equations:

$$\begin{aligned} x - y + 2z &= -3 \\ 3x - 3y + 6z &= 1. \end{aligned}$$

1.2 A numerical way to tackle the problem.

A fairly straightforward way to tackle this problem is illustrated in the sketch of fig. 1.1. If we can find a point in the first plane, we can follow the normal to the plane to the next, and compute the length of that connecting vector.

For this problem, let

$$\mathbf{n} = (1, -1, 2), \tag{1.1}$$

and rescale the two plane equations to use the same normal. That is

$$\begin{aligned} \mathbf{x}_1 \cdot \mathbf{n} &= -3 \\ \mathbf{x}_2 \cdot \mathbf{n} &= \frac{1}{3}, \end{aligned} \tag{1.2}$$

where $\mathbf{x}_1$ are vectors in the first plane, and $\mathbf{x}_2$ are vectors in the second plane. Finding a vector in one of the planes isn't hard. Suppose, for example, that $\mathbf{x}_0 = (\alpha, \beta, \gamma)$ is a vector in the first plane, then

$$\alpha - \beta + 2\gamma = -3. \tag{1.3}$$

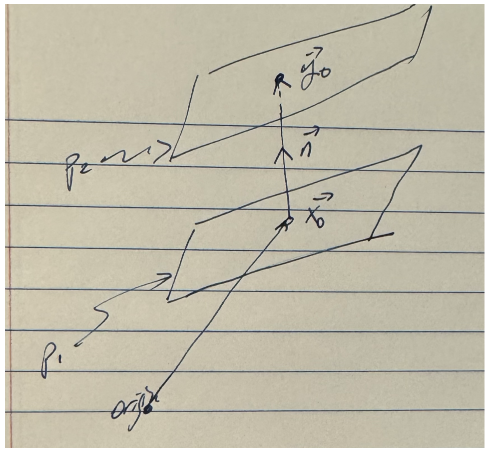

Figure 1.1: Distance between two planes.

One solution is $\alpha = -3, \beta = 0, \gamma = 0$, or $\mathbf{x}_0 = (-3, 0, 0)$. We can follow the normal from that point to the closest point in the second plane by forming

$$\mathbf{y}_0 = \mathbf{x}_0 + k\mathbf{n}, \tag{1.4}$$

where k is to be determined. If $\mathbf{y}_0$ is a point in the second plane, we must have

$$\begin{aligned} \frac{1}{3} &= \mathbf{y}_0 \cdot \mathbf{n} \\ &= (\mathbf{x}_0 + k\mathbf{n}) \cdot \mathbf{n} \\ &= (-3, 0, 0) \cdot (1, -1, 2) + k(1, -1, 2) \cdot (1, -1, 2) \\ &= -3 + 6k, \end{aligned} \tag{1.5}$$

or

$$k = \frac{10}{18} = \frac{5}{9}. \tag{1.6}$$

This means the point in plane two closest to $\mathbf{x}_0 = (-3, 0, 0)$ is

$$\begin{aligned} \mathbf{y}_0 &= (-3, 0, 0) + \frac{5}{9}(1, -1, 2) \\ &= \frac{1}{9}(-27 + 5, -5, 10) \\ &= \frac{1}{9}(-22, -5, 10), \end{aligned} \tag{1.7}$$

and the vector distance between the planes is

$$
\begin{aligned}
\mathbf{y}_0 - \mathbf{x}_0 &= \frac{1}{9}(-22, -5, 10) - (-3, 0, 0) \\
&= \frac{1}{9}(-22 + 27, -5, 10) \\
&= \frac{1}{9}(5, -5, 10).
\end{aligned} \tag{1.8}
$$

This vector's length is $\sqrt{150}/9 = (5/9)\sqrt{6}$, which is the shortest distance between the planes.

1.3 A symbolic approach.

Generally, we get more clarity if we avoid plugging in numbers until the very end, so let's try a generalization of this problem.

Exercise 1.2

Find the shortest distance between the two parallel planes, P_1, and P_2, with respective equations:

$$
\begin{aligned}
\mathbf{x}_1 \cdot \mathbf{n}_1 &= d_1 \\
\mathbf{x}_2 \cdot \mathbf{n}_2 &= d_2.
\end{aligned}
$$

We can use the same approach, but first, let's rescale the two normals. Let

$$
\mathbf{n}_2 = t\mathbf{n}_1, \tag{1.9}
$$

or

$$
\mathbf{n}_1 \cdot \mathbf{n}_2 = t\mathbf{n}_1^2, \tag{1.10}
$$

so

$$
\mathbf{n}_2 = \frac{\mathbf{n}_1 \cdot \mathbf{n}_2}{\mathbf{n}_1^2}\mathbf{n}_1, \tag{1.11}
$$

which means that our plane equations are

$$
\begin{aligned}
\mathbf{x}_1 \cdot \mathbf{n}_1 &= d_1 \\
\mathbf{x}_2 \cdot \mathbf{n}_1 &= \frac{\mathbf{n}_1^2}{\mathbf{n}_1 \cdot \mathbf{n}_2} d_2,
\end{aligned} \tag{1.12}
$$

We can further streamline our plane equation representation, setting $\hat{\mathbf{n}} = \mathbf{n}_1/\|\mathbf{n}_1\|$, which gives us

$$
\begin{aligned}
\mathbf{x}_1 \cdot \hat{\mathbf{n}} &= \frac{d_1}{\|\mathbf{n}_1\|} \\
\mathbf{x}_2 \cdot \hat{\mathbf{n}} &= \frac{d_2}{\hat{\mathbf{n}} \cdot \mathbf{n}_2}.
\end{aligned} \tag{1.13}
$$

This time, let's assume that we can find a point $\mathbf{x}_0$ in the first plane, but not actually try to find it. We can still follow the normal to the second plane from that point

$$\mathbf{y}_0 = \mathbf{x}_0 + k\hat{\mathbf{n}}, \tag{1.14}$$

but since we only care about the vector distance between the planes, we seek

$$\mathbf{y}_0 - \mathbf{x}_0 = k\hat{\mathbf{n}}. \tag{1.15}$$

Now, the constant k, once we find it, is exactly the distance between the planes that we seek. Plugging $\mathbf{y}_0$ into the P_2 equation, we find

$$\begin{aligned} \frac{d_2}{\hat{\mathbf{n}} \cdot \mathbf{n}_2} &= (\mathbf{x}_0 + k\hat{\mathbf{n}}) \cdot \hat{\mathbf{n}} \\ &= \mathbf{x}_0 \cdot \hat{\mathbf{n}} + k \\ &= \frac{d_1}{\|\mathbf{n}_1\|} + k, \end{aligned} \tag{1.16}$$

or

$$\boxed{|k| = \|\mathbf{y}_0 - \mathbf{x}_0\| = \left| \frac{d_2}{\hat{\mathbf{n}} \cdot \mathbf{n}_2} - \frac{d_1}{\hat{\mathbf{n}} \cdot \mathbf{n}_1} \right|.} \tag{1.17}$$

If $\mathbf{n}_2 = \mathbf{n}_1 = \mathbf{n}$, then we have

$$\begin{aligned} \|\mathbf{y}_0 - \mathbf{x}_0\| &= \left| \frac{d_2}{\mathbf{n}_1^2/\|\mathbf{n}_1\|} - \frac{d_1}{\|\mathbf{n}_1\|} \right| \\ &= \frac{|d_2 - d_1|}{\|\mathbf{n}\|}, \end{aligned} \tag{1.18}$$

and if $\mathbf{n}$ is a unit normal, this further reduces to just $|d_2 - d_1|$.

Let's try this for the specific problem originally given. We have $\mathbf{n}_1 = \mathbf{n}_2$, so the distance between the planes is

$$\begin{aligned} \|\mathbf{y}_0 - \mathbf{x}_0\| &= \frac{|1/3 + 3|}{\sqrt{6}} \\ &= \frac{10}{3 \times 6}\sqrt{6} \\ &= \frac{5}{9}\sqrt{6}, \end{aligned} \tag{1.19}$$

as previously calculated.