Peeter Joot
peeterjoot@pm.me

Hyperbolic sine representation of mth Fibonacci number

I saw a funky looking formula for the mth Fibonacci number on twitter

$$F_m = \frac{2}{\sqrt{5} i^m} \sinh\left(m \ln\left(i\phi\right)\right), \tag{1.1}$$

where

$$\phi = \frac{1 + \sqrt{5}}{2}, \tag{1.2}$$

is the golden ratio.

This certainly doesn't look like it's a representation of the sequence

$$1, 1, 2, 3, 5, 8, 13, 21, 34, 55, \cdots \tag{1.3}$$

We can verify that it works in Mathematica, as seen in fig. 1.1.

```
In[16]:= phi := (1 + Sqrt[5]) / 2;
         f[m_] := (2 / (Sqrt[5] I ^ m)) Sinh[ m Log[I phi]]
         f[#] & /@ Range[10] // FullSimplify

Out[18]= {1, 1, 2, 3, 5, 8, 13, 21, 34, 55}
```

Figure 1.1: Verification of hyperbolic sine representation of mth Fibonacci number.

Recall that we previously found this formula for the mth Fibonacci number

$$F_m = \frac{1}{\sqrt{5}} \left(\phi^m - \bar{\phi}^m\right), \tag{1.4}$$

where $\bar{\phi}$ is the conjugate of the golden ratio

$$\bar{\phi} = \frac{1 - \sqrt{5}}{2}. \tag{1.5}$$

Let's see how these are equivalent. First observe that the golden conjugate is easily related to the inverse of the golden ratio

$$\begin{aligned}
\frac{1}{\phi} &= \frac{2}{1+\sqrt{5}} \\
&= \frac{2\left(1-\sqrt{5}\right)}{1^2-\left(\sqrt{5}\right)^2} \\
&= -\frac{1-\sqrt{5}}{2} \\
&= -\bar{\phi}.
\end{aligned} \tag{1.6}$$

Substitution gives

$$F_m = \frac{1}{\sqrt{5}}\left(\phi^m - \left(\frac{-1}{\phi}\right)^m\right). \tag{1.7}$$

Multiplying by i^m, we have

$$\begin{aligned}
i^m F_m &= \frac{1}{\sqrt{5}}\left(i^m \phi^m - \frac{1}{(-i)^m}\left(\frac{-1}{\phi}\right)^m\right) \\
&= \frac{1}{\sqrt{5}}\left((i\phi)^m - (i\phi)^{-m}\right)
\end{aligned} \tag{1.8}$$

We can write any exponent in terms of e

$$a^m = e^{\ln a^m} = e^{m \ln a}, \tag{1.9}$$

so

$$\begin{aligned}
i^m F_m &= \frac{1}{\sqrt{5}}\left(e^{m\ln(i\phi)} - e^{-m\ln(i\phi)}\right) \\
&= \frac{1}{\sqrt{5}} 2\sinh\left(m \ln\left(i\phi\right)\right),
\end{aligned} \tag{1.10}$$

as we wanted to show. It's a bit strange looking, but we see why it works.