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Equation of a hyperplane, and shortest distance between two hyperplanes.

1.1 Scalar equation for a hyperplane.

In our last post, we found, in a round about way, that

Theorem 1.1

The equation of a $\mathbb{R}^N$ hyperplane, with distance d from the origin, and normal $\hat{\mathbf{n}}$ is

$$\mathbf{x} \cdot \hat{\mathbf{n}} = d.$$

Proof. Let $\beta = \left\{ \hat{\mathbf{f}}_1, \cdots \hat{\mathbf{f}}_{N-1} \right\}$ be an orthonormal basis for the hyperplane normal to $\hat{\mathbf{n}}$, and $\mathbf{d} = d\hat{\mathbf{n}}$ be the vector in that hyperplane, closest to the origin, as illustrated in fig. 1.1.

The hyperplane d distant from the origin with normal $\hat{\mathbf{n}}$ has the parametric representation

$$\mathbf{x}(a_1, \cdots, a_{N-1}) = d\hat{\mathbf{n}} + \sum_{i=1}^{N-1} a_i \hat{\mathbf{f}}_i. \tag{1.1}$$

Equivalently, suppressing the parameterization, with $\mathbf{x} = \mathbf{x}(a_1, \cdots, a_{N-1})$, representing any vector in that hyperplane, by dotting with $\hat{\mathbf{n}}$, we have

$$\mathbf{x} \cdot \hat{\mathbf{n}} = d\hat{\mathbf{n}} \cdot \hat{\mathbf{n}}, \tag{1.2}$$

where all the $\hat{\mathbf{f}}_i \cdot \hat{\mathbf{n}}$ dot products are zero by construction. Since $\hat{\mathbf{n}} \cdot \hat{\mathbf{n}} = 0$, the proof is complete. □

Incidentally, observe we can also write the hyperplane equation in dual form, as

$$\mathbf{x} \wedge (\hat{\mathbf{n}} I) = dI, \tag{1.3}$$

where I is an $\mathbb{R}^N$ pseudoscalar (such as $I = \hat{\mathbf{n}}\hat{\mathbf{f}}_1 \cdots \hat{\mathbf{f}}_{N-1}$).

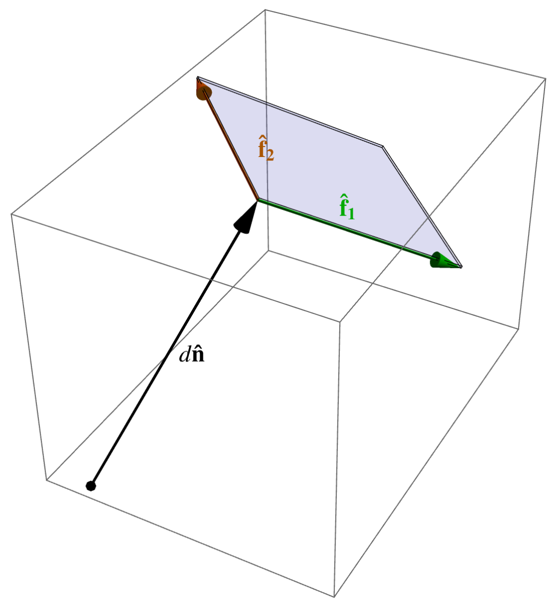

Figure 1.1: $\mathbb{R}^3$ plane with normal $\hat{\mathbf{n}}$.

1.2 Our previous parallel plane separation problem.

The standard $\mathbb{R}^3$ scalar form for an equation of a plane is

$$ax + by + cz = d, \tag{1.4}$$

where d looses it's geometrical meaning. If we form $\mathbf{n} = (a, b, c)$, then we can rewrite this as

$$\mathbf{x} \cdot \mathbf{n} = d, \tag{1.5}$$

for this representation of an equation of a plane, we see that $d/\|\mathbf{n}\|$ is the shortest distance from the origin to the plane. This means that if we have a pair of parallel plane equations

$$\begin{aligned} \mathbf{x} \cdot \mathbf{n} &= d_1 \\ \mathbf{x} \cdot \mathbf{n} &= d_2, \end{aligned} \tag{1.6}$$

then the distance between those planes, by inspection, is

$$\left| \frac{d_2}{\|\mathbf{n}\|} - \frac{d_1}{\|\mathbf{n}\|} \right|, \tag{1.7}$$

which reduces to just $|d_2 - d_1|$ if $\mathbf{n}$ is a unit normal for the plane. In our previous post, the problem to solve was to find the shortest distance between the parallel planes given by

$$\begin{aligned} x - y + 2z &= -3 \\ 3x - 3y + 6z &= 1. \end{aligned} \tag{1.8}$$

The more natural geometrical form for these plane equations is

$$\begin{aligned} \mathbf{x} \cdot \hat{\mathbf{n}} &= -\frac{3}{\sqrt{6}} \\ \mathbf{x} \cdot \hat{\mathbf{n}} &= \frac{1}{3\sqrt{6}}, \end{aligned} \tag{1.9}$$

where $\hat{\mathbf{n}} = (1, -1, 2)/\sqrt{6}$, as illustrated in fig. 1.2.

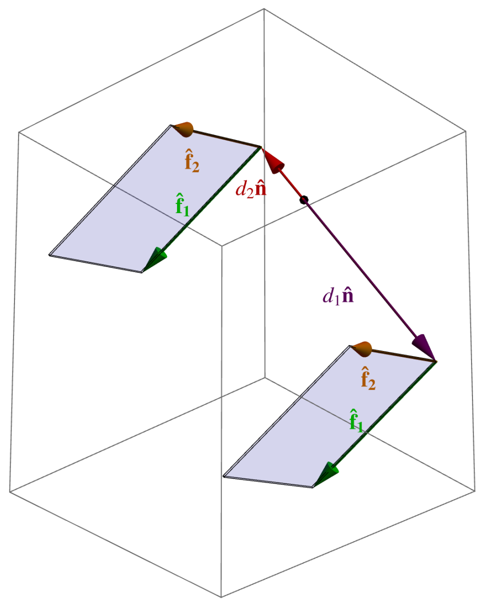

Figure 1.2: The two planes.

Given that representation, we can find the distance between the planes just by taking the absolute difference of the respective distances to the origin

$$\begin{aligned} \left| -\frac{3}{\sqrt{6}} - \frac{1}{3\sqrt{6}} \right| &= \frac{\sqrt{6}}{6} \left(3 + \frac{1}{3} \right) \\ &= \frac{10}{18}\sqrt{6} \\ &= \frac{5}{9}\sqrt{6}. \end{aligned} \tag{1.10}$$