Peeter Joot
peeterjoot@pm.me

A funny looking log identity

On twitter, I saw a funny looking identity

$$\log_{ab} x = \frac{\log_a x \log_b x}{\log_a x + \log_b x}. \tag{1.1}$$

To verify this, let

$$\begin{aligned} u &= \log_a x \\ v &= \log_b x. \end{aligned} \tag{1.2}$$

This means that

$$\log_{ab} x = \log_{ab} a^u = \log_{ab} b^v. \tag{1.3}$$

We may rewrite either of these in terms of ab, for example

$$\begin{aligned} \log_{ab} x &= \log_{ab} b^v \\ &= v \log_{ab} b \\ &= v \log_{ab} \frac{ab}{a} \\ &= v \left(1 - \log_{ab} a\right), \end{aligned} \tag{1.4}$$

so

$$u \log_{ab} a = v \left(1 - \log_{ab} a\right), \tag{1.5}$$

or

$$(u + v) \log_{ab} a = v, \tag{1.6}$$

or

$$u \log_{ab} a = \frac{uv}{u + v}, \tag{1.7}$$

and since $x = a^u$, our proof is complete.