Peeter Joot
peeterjoot@pm.me

Evaluating a sum using a contour integral.

One of my favorite Dover books, [1], is a powerhouse of a reference, and has a huge set of the mathematical tricks and techniques that any engineer or physicist would ever want.

Reading it a bit today, I encountered the following interesting looking theorem for evaluating sums using contour integrals.

Theorem 1.1

Given a meromorphic function $f(z)$ that shares no poles with $\cot(\pi z)$, where C encloses the zeros of $\sin(\pi z$, located at $z = a, a+1, \cdots b$, then

$$\sum_{m=a}^{b} f(m) = \frac{1}{2\pi i} \oint_C \pi \cot(\pi z) f(z) dz - \sum_{\text{poles of } f(z) \text{ in } C} \text{Res}\left(\pi \cot(\pi z) f(z)\right).$$

The enclosing contour may look like fig. 1.1.

Proof. We basically want to evaluate

$$\oint_C \pi \cot(\pi z) f(z) dz, \tag{1.1}$$

using residues. To see why this works, observe that $\cot(\pi z)$ is periodic, as plotted in fig. 1.2.

In particular, if $z = m + \epsilon$, we have

$$\begin{aligned} \cot(\pi z) &= \frac{\cos(\pi(m+\epsilon))}{\sin(\pi(m+\epsilon))} \\ &= \frac{(-1)^m \cos(\pi\epsilon)}{(-1)^m \sin(\pi\epsilon)} \\ &= \cot(\pi\epsilon). \end{aligned} \tag{1.2}$$

The residue of $\pi \cot(\pi z)$, at $z = 0$, or at any other integer point, is

$$\frac{\pi}{\pi z - (\pi z)^3/6 + \cdots} = 1. \tag{1.3}$$

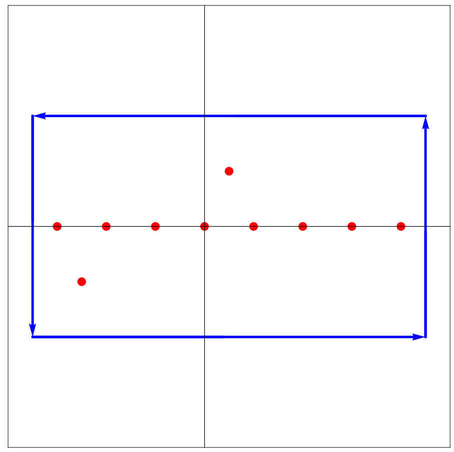

Figure 1.1: Sample contour

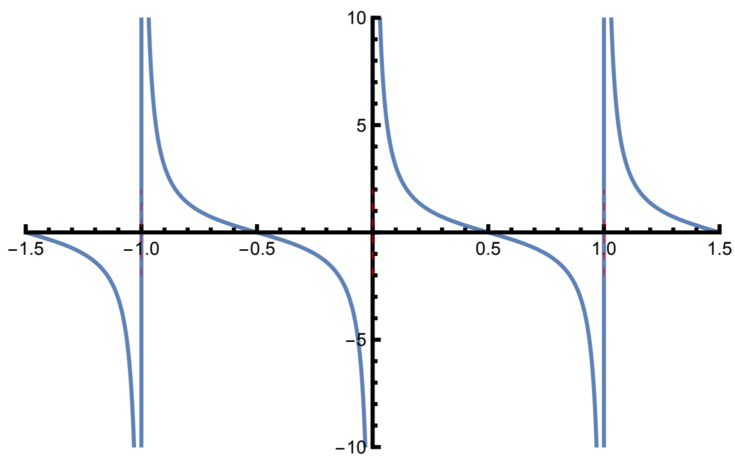

Figure 1.2: Cotangent.

This means that we have

$$\oint_C \pi \cot(\pi z) f(z) dz = 2\pi i \sum_{m=a}^{b} f(m) + 2\pi i \sum_{\text{poles of } f(z) \text{ in } C} \operatorname{Res}\left(\pi \cot(\pi z) f(z)\right). \tag{1.4}$$

We just have to rearrange and scale to complete the proof. □

In the book the sample application was to use this to show that

$$\coth x - \frac{1}{x} = \sum_{m=1}^{\infty} \frac{2x}{x^2 + m^2\pi^2}. \tag{1.5}$$

That's then integrated to show that

$$\frac{\sinh x}{x} = \prod_{m=1}^{\infty} \left(1 + \frac{x^2}{m^2\pi^2}\right), \tag{1.6}$$

or with $x = i\theta$,

$$\sin\theta = \theta \prod_{m=1}^{\infty} \left(1 - \frac{\theta^2}{m^2\pi^2}\right), \tag{1.7}$$

and finally equating θ^3 terms in this infinite product, we find

$$\sum_{m=1}^{\infty} \frac{1}{m^2} = \frac{\pi^2}{6}, \tag{1.8}$$

which is $\zeta(2)$, a specific value of the Riemann zeta function.

All this is done in a couple spectacularly dense pages of calculation, and illustrates the kind of gems in this book. At about 700 pages, it's got a lot of gems.

Bibliography

[1] F.W. Byron and R.W. Fuller. *Mathematics of Classical and Quantum Physics*. Dover Publications, 1992. 1