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Curl of Curl. Tensor and GA expansion, and GA equivalent identity.

In this blog post, we will expand $\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \mathbf{f}) = -\boldsymbol{\nabla}^2 \mathbf{f} + \boldsymbol{\nabla} (\boldsymbol{\nabla} \cdot \mathbf{f})$ two different ways, using tensor index gymnastics and using geometric algebra.

1.1 The tensor way.

To expand the curl using a tensor expansion, let's first expand the cross product in coordinates

$$\begin{aligned}\mathbf{a} \times \mathbf{b} &= (\mathbf{e}_r \times \mathbf{e}_s) a_r b_s \\ &= \mathbf{e}_t \cdot (\mathbf{e}_r \times \mathbf{e}_s) \mathbf{e}_t a_r b_s \\ &= \epsilon_{rst} a_r b_s \mathbf{e}_t.\end{aligned} \tag{1.1}$$

Here ϵ_{rst} is the completely antisymmetric (Levi-Civita) tensor, and allows us to compactly express the geometrical nature of the triple product.

We can then expand the curl of the curl by applying this twice

$$\begin{aligned}\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \mathbf{f}) &= \epsilon_{rst} \partial_r (\boldsymbol{\nabla} \times \mathbf{f})_s \mathbf{e}_t \\ &= \epsilon_{rst} \partial_r (\epsilon_{uvw} \partial_u f_v \mathbf{e}_w)_s \mathbf{e}_t \\ &= \epsilon_{rst} \partial_r \epsilon_{uvs} \partial_u f_v \mathbf{e}_t.\end{aligned} \tag{1.2}$$

It turns out that there's a nice identity to reduce the single index contraction of a pair of Levi-Civita tensors.

$$\epsilon_{abt} \epsilon_{cdt} = \delta_{ac} \delta_{bd} - \delta_{ad} \delta_{bc}. \tag{1.3}$$

To show this, consider the $t = 1$ term of this sum $\epsilon_{ab1} \epsilon_{cd1}$. This is non-zero only for $a, b, c, d \in \{2, 3\}$. If $a, b = c, d$, this is one, and if $a, b = d, c$, this is minus one. We may summarize that as

$$\epsilon_{ab1} \epsilon_{cd1} = \delta_{ac} \delta_{bd} - \delta_{ad} \delta_{bc}, \tag{1.4}$$

but this holds for $t = 2, 3$ too, so eq. (1.3) holds generally.

We may now contract the tensors to find

$$\begin{aligned}\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \mathbf{f}) &= \epsilon_{rst}\epsilon_{uvs}\mathbf{e}_t \partial_r \partial_u f_v \\ &= -\epsilon_{rts}\epsilon_{uvs}\mathbf{e}_t \partial_r \partial_u f_v \\ &= -\left(\delta_{ru}\delta_{tv} - \delta_{rv}\delta_{tu}\right)\mathbf{e}_t \partial_r \partial_u f_v \\ &= -\mathbf{e}_v \partial_u \partial_u f_v + \mathbf{e}_u \partial_v \partial_u f_v \\ &= -\boldsymbol{\nabla}^2 \mathbf{f} + \boldsymbol{\nabla}\left(\boldsymbol{\nabla} \cdot \mathbf{f}\right).\end{aligned} \tag{1.5}$$

1.2 Using geometric algebra.

Now let's pull out the GA toolbox. We start with introducing a no-op grade-1 selection, and using the identity $\mathbf{a} \times \mathbf{b} = -I\left(\mathbf{a} \wedge \mathbf{b}\right)$

$$\begin{aligned}\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \mathbf{f}) &= \left\langle \boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \mathbf{f}) \right\rangle_1 \\ &= \left\langle -I\left(\boldsymbol{\nabla} \wedge (\boldsymbol{\nabla} \times \mathbf{f})\right) \right\rangle_1\end{aligned} \tag{1.6}$$

We can now expand $\mathbf{a} \wedge \mathbf{b} = \mathbf{a}\mathbf{b} - \mathbf{a} \cdot \mathbf{b}$

$$\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \mathbf{f}) = \left\langle -I\boldsymbol{\nabla}\left(\boldsymbol{\nabla} \times \mathbf{f}\right) + I\left(\boldsymbol{\nabla} \cdot (\boldsymbol{\nabla} \times \mathbf{f})\right) \right\rangle_1 \tag{1.7}$$

but that dot product is a scalar, leaving just a pseudoscalar, which has a zero grade-1 selection. This leaves

$$\begin{aligned}\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \mathbf{f}) &= \left\langle -I\boldsymbol{\nabla}\left(-I\left(\boldsymbol{\nabla} \wedge \mathbf{f}\right)\right) \right\rangle_1 \\ &= -\left\langle \boldsymbol{\nabla}\left(\boldsymbol{\nabla} \wedge \mathbf{f}\right) \right\rangle_1.\end{aligned} \tag{1.8}$$

We use $\mathbf{a} \wedge \mathbf{b} = \mathbf{a}\mathbf{b} - \mathbf{a} \cdot \mathbf{b}$ once more

$$\begin{aligned}\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \mathbf{f}) &= -\left\langle \boldsymbol{\nabla}\left(\boldsymbol{\nabla}\mathbf{f}\right) - \boldsymbol{\nabla}\left(\boldsymbol{\nabla} \cdot \mathbf{f}\right) \right\rangle_1 \\ &= -\boldsymbol{\nabla}^2 \mathbf{f} + \boldsymbol{\nabla}\left(\boldsymbol{\nabla} \cdot \mathbf{f}\right).\end{aligned} \tag{1.9}$$

1.3 GA identity.

It's also worth noting that there's a natural GA formulation of the curl of a curl. From the Laplacian and divergence relationship that we ended up with, we need only factor out the gradient

$$\begin{aligned}\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \mathbf{f}) &= -\boldsymbol{\nabla}^2 \mathbf{f} + \boldsymbol{\nabla}\left(\boldsymbol{\nabla} \cdot \mathbf{f}\right) \\ &= -\boldsymbol{\nabla}\left(\boldsymbol{\nabla}\mathbf{f} - \boldsymbol{\nabla} \cdot \mathbf{f}\right) \\ &= -\boldsymbol{\nabla}\left(\boldsymbol{\nabla} \wedge \mathbf{f}\right).\end{aligned} \tag{1.10}$$

Because $\boldsymbol{\nabla} \wedge (\boldsymbol{\nabla} \wedge \mathbf{f}) = 0$, we may also write this as

$$\boxed{\boldsymbol{\nabla} \cdot (\boldsymbol{\nabla} \wedge \mathbf{f}) = -\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \mathbf{f}).} \tag{1.11}$$

From the GA LHS, we see by inspection that

$$\boldsymbol{\nabla} \cdot (\boldsymbol{\nabla} \wedge \mathbf{f}) = \boldsymbol{\nabla}^2 \mathbf{f} - \boldsymbol{\nabla}\left(\boldsymbol{\nabla} \cdot \mathbf{f}\right). \tag{1.12}$$