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A fun ellipse related integral.

1.1 Motivation.

This was a problem I found on twitter ([2])

Find

$$I = \int_0^\pi \frac{dx}{a^2 \cos^2 x + b^2 \sin^2 x}. \tag{1.1}$$

I posted my solution there (as a screenshot), but had a sign wrong. Here's a correction.

1.2 Solution.

Let's first assume we aren't interested in the $a^2 = b^2$, nor either of the $a = 0, b = 0$ cases (if either of a, b is zero, then the integral is divergent.)

We can make a couple simple transformations to start with

$$\begin{aligned} \cos^2 x &= \frac{\cos(2x) + 1}{2} \\ \sin^2 x &= \frac{1 - \cos(2x)}{2}, \end{aligned} \tag{1.2}$$

and then $u = 2x$, for $dx = du/2$

$$\begin{aligned} I &= \int_0^{2\pi} 2 \frac{du/2}{a^2 (1 + \cos u) + b^2 (1 - \cos u)} \\ &= \int_0^{2\pi} \frac{du}{(a^2 - b^2) \cos u + a^2 + b^2}. \end{aligned} \tag{1.3}$$

There is probably a simple way to evaluate this integral, but let's try it the fun way, using contour integration. Following examples from [1], let $z = e^{iu}$, where $dz = izdu$, and $\alpha = \left(a^2 + b^2\right) / \left(a^2 - b^2\right)$, for

$$\begin{aligned} I &= \oint_{|z|=1} \frac{dz/(iz)}{\left(a^2 - b^2\right)\left(z + \frac{1}{z}\right)/2 + a^2 + b^2} \\ &= \frac{2}{i\left(a^2 - b^2\right)} \oint_{|z|=1} \frac{dz}{z\left(z + \frac{1}{z} + 2\alpha\right)} \\ &= \frac{2}{i\left(a^2 - b^2\right)} \oint_{|z|=1} \frac{dz}{z^2 + 2\alpha z + 1} \\ &= \frac{2}{i\left(a^2 - b^2\right)} \oint_{|z|=1} \frac{dz}{\left(z + \alpha - \sqrt{\alpha^2 - 1}\right)\left(z + \alpha + \sqrt{\alpha^2 - 1}\right)}. \end{aligned} \tag{1.4}$$

There is a single enclosed pole on the real axis. For $a^2 > b^2$ where $\alpha > 0$ that pole is at $z = -\alpha + \sqrt{\alpha^2 - 1}$, so the integral is

$$\begin{aligned} I &= 2\pi i \frac{2}{i\left(a^2 - b^2\right)} \left. \frac{1}{z + \alpha + \sqrt{\alpha^2 - 1}} \right|_{z = -\alpha + \sqrt{\alpha^2 - 1}} \\ &= \frac{4\pi}{a^2 - b^2} \frac{1}{2\sqrt{\alpha^2 - 1}} \\ &= \frac{4\pi}{2\sqrt{\left(a^2 + b^2\right)^2 - \left(a^2 - b^2\right)^2}} \\ &= \frac{2\pi}{\sqrt{4a^2b^2}} \\ &= \frac{\pi}{|ab|}, \end{aligned} \tag{1.5}$$

and for $a^2 < b^2$ where $\alpha < 0$, the enclosed pole is at $z = -\alpha - \sqrt{\alpha^2 - 1}$, where

$$\begin{aligned} I &= 2\pi i \frac{2}{i\left(a^2 - b^2\right)} \left. \frac{1}{z + \alpha - \sqrt{\alpha^2 - 1}} \right|_{z = -\alpha - \sqrt{\alpha^2 - 1}} \\ &= \frac{4\pi}{a^2 - b^2} \frac{1}{-2\sqrt{\alpha^2 - 1}} \\ &= \frac{4\pi}{b^2 - a^2} \frac{1}{2\sqrt{\alpha^2 - 1}} \\ &= \frac{4\pi}{2\sqrt{\left(a^2 + b^2\right)^2 - \left(b^2 - a^2\right)^2}} \\ &= \frac{2\pi}{\sqrt{4a^2b^2}} \\ &= \frac{\pi}{|ab|}. \end{aligned} \tag{1.6}$$

Observe that this also holds for the $a = b$ case.

Bibliography

[1] F.W. Byron and R.W. Fuller. *Mathematics of Classical and Quantum Physics*. Dover Publications, 1992. 1.2

[2] CalcInsights. *A decent integral problem to try out*, 2025. URL https://x.com/CalcInsights_/status/1880110549146431780. [Online; accessed 18-Jan-2025]. 1.1