Peeter Joot
peeterjoot@pm.me

Another real integral using contour integration.

Here's (31(d)) from [1]. Find

$$I = \int_0^\infty \frac{dx}{1+x^4} = \frac{1}{2}\int_{-\infty}^\infty \frac{dx}{1+x^4}. \tag{1.1}$$

This one is easy conceptually, but a bit messy algebraically. We integrate over the contour C illustrated in fig. 1.1.

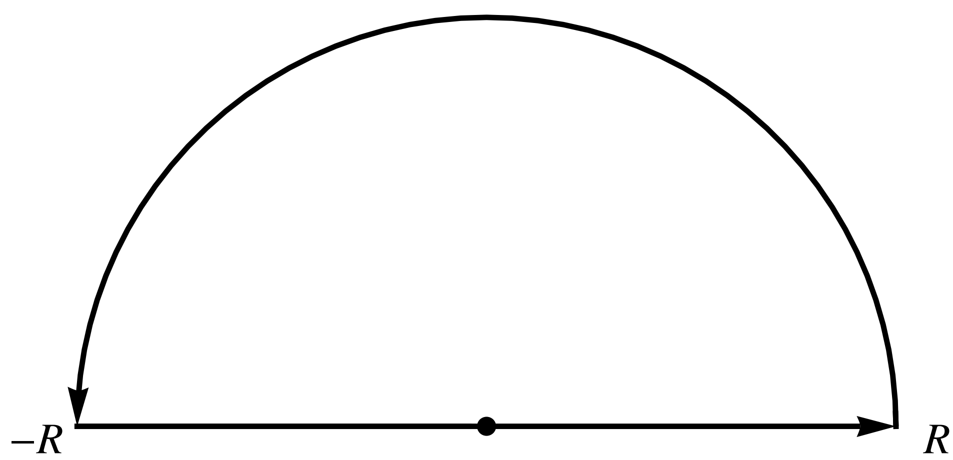

Figure 1.1: Standard above the x-axis, semicircular contour.

We want to evaluate

$$2I = \oint_C \frac{dz}{1+z^4}, \tag{1.2}$$

because the semicircular part of the integral is $O(R^{-3})$, which tends to zero in the $R \rightarrow \infty$ limit.

The poles are at the points

$$\begin{aligned} z^4 &= -1 \\ &= e^{i\pi + 2\pi i k}, \end{aligned} \tag{1.3}$$

or

$$z = e^{i\pi/4 + \pi i k/2}, \tag{1.4}$$

These are the points $z = (\pm 1 \pm i)/\sqrt{2}$, two of which are enclosed by our contour. Specifically

$$\begin{aligned}
2I &= \oint_C \frac{dz}{\left(z - \frac{1+i}{\sqrt{2}}\right)\left(z - \frac{-1+i}{\sqrt{2}}\right)\left(z - \frac{1-i}{\sqrt{2}}\right)\left(z - \frac{-1-i}{\sqrt{2}}\right)} \\
&= \oint_C \frac{dz}{\left(z - \frac{1+i}{\sqrt{2}}\right)\left(z - \frac{-1+i}{\sqrt{2}}\right)\left(\left(z + \frac{i}{\sqrt{2}}\right)^2 - \frac{1}{2}\right)} \\
&= \left.\frac{2\pi i}{\left(z - \frac{-1+i}{\sqrt{2}}\right)\left(\left(z + \frac{i}{\sqrt{2}}\right)^2 - \frac{1}{2}\right)}\right|_{z=\frac{1+i}{\sqrt{2}}} + \left.\frac{2\pi i}{\left(z - \frac{1+i}{\sqrt{2}}\right)\left(\left(z + \frac{i}{\sqrt{2}}\right)^2 - \frac{1}{2}\right)}\right|_{z=\frac{-1+i}{\sqrt{2}}} \\
&= \left.\frac{(2\pi i)(2\sqrt{2})}{(z' + 1 - i)\left((z' + i)^2 - 1\right)}\right|_{z'=1+i} + \left.\frac{(2\pi i)(2\sqrt{2})}{(z' - 1 - i)\left((z' + i)^2 - 1\right)}\right|_{z'=-1+i} \\
&= \frac{2\pi i\sqrt{2}}{(2i+1)^2 - 1} - \frac{2\pi i\sqrt{2}}{(2i-1)^2 - 1} \\
&= \frac{\pi i\sqrt{2}}{2(-1+i)} + \frac{\pi i\sqrt{2}}{2(1+i)} \\
&= (-1-i)\frac{\pi i}{2\sqrt{2}} + (1-i)\frac{\pi i}{2\sqrt{2}} \\
&= \frac{\pi}{\sqrt{2}}
\end{aligned} \tag{1.5}$$

or

$$\boxed{I = \frac{\pi}{2\sqrt{2}}.} \tag{1.6}$$

Bibliography

[1] F.W. Byron and R.W. Fuller. *Mathematics of Classical and Quantum Physics*. Dover Publications, 1992. 1