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A PV integral using contour integration.

Here's the second last real-integral sub-problem from [1], problem 31(j). Find

$$I = P \int_{-\infty}^{\infty} \frac{1}{(\omega' - \omega_0)^2 + a^2} \frac{1}{\omega' - \omega} d\omega'. \tag{1.1}$$

Our poles are sitting at ω, and

$$\alpha, \beta = \omega_0 \pm ia \tag{1.2}$$

one of which sits above the x-axis, one below, and one on the line.

This means that if we compute the usual infinite semicircular contour integral, we have a $2\pi i$ weighted residue above the line and one πi weighted residue for the x-axis pole. That is

$$\begin{aligned}
\oint \frac{1}{(z - \omega_0)^2 + a^2} \frac{1}{z - \omega} dz &= (2\pi i) \left. \frac{1}{(z - (\omega_0 - ia))(z - \omega)} \right|_{z = \omega_0 + ia} + (\pi i) \left. \frac{1}{(z - \omega_0)^2 + a^2} \right|_{z = \omega} \\
&= (2\pi i) \frac{1}{(\omega_0 + ia - (\omega_0 - ia))(\omega_0 + ia - \omega)} + (\pi i) \frac{1}{(\omega - \omega_0)^2 + a^2} \\
&= \frac{2\pi i}{2ia} \frac{1}{\omega_0 + ia - \omega} \frac{\omega_0 - ia - \omega}{\omega_0 - ia - \omega} + (\pi i) \frac{1}{(\omega - \omega_0)^2 + a^2} \\
&= \frac{\pi}{(\omega - \omega_0)^2 + a^2} \left(\frac{\omega_0 - \omega}{a} - i + i \right),
\end{aligned} \tag{1.3}$$

or

$$\boxed{I = \frac{\pi (\omega_0 - \omega)}{a \left((\omega - \omega_0)^2 + a^2 \right)}.} \tag{1.4}$$

Interestingly, Mathematica doesn't seem to be able to solve this integral, even setting PrincipleValue to True. The solution ends up with a bogus seeming $\text{Im}(\omega_0 - \omega) = \text{Re}(a)$ restriction, and as far as I can tell, the Mathematica result is also zero after simplification that it fails to do. Mathematica can solve this if we explicitly state the PV condition as a limit, as shown in fig. 1.1.

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In[4]:= Integrate[1 / ((u + ω - ω0) ^2 + a^2) / u, {u, -Infinity, Infinity}, PrincipalValue → True]
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Out[4]=
$$-\frac{\left(\sqrt{\frac{1}{a^2}}\, a\,\pi - i\,\text{Log}\left[-\frac{i}{a}\right] + i\,\text{Log}\left[\frac{i}{a}\right]\right)(\omega - \omega_0)}{a\left(a^2 + \omega^2 - 2\,\omega\,\omega_0 + \omega_0^2\right)} \quad \text{if } \text{Im}[-\omega + \omega_0] == \text{Re}[a] \;\&\&\; \text{Re}[a] \neq 0$$

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In[3]:= Limit[
         Integrate[1 / ((u + ω - ω0) ^2 + a^2) / u, {u, -Infinity, -e}]
          + Integrate[1 / ((u + ω - ω0) ^2 + a^2) / u, {u, e, Infinity}],
         e → 0, Direction → "FromAbove"]
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Out[3]=
$$-\frac{\sqrt{\frac{1}{a^2}}\,\pi\,(\omega - \omega_0)}{a^2 + \omega^2 - 2\,\omega\,\omega_0 + \omega_0^2}$$

Figure 1.1: Coercing Mathematica to evaluate this.

Bibliography

[1] F.W. Byron and R.W. Fuller. *Mathematics of Classical and Quantum Physics*. Dover Publications, 1992. 1