Peeter Joot
peeterjoot@pm.me

# Sum of squares and cubes, using difference calculus.

## 1.1 Motivation.

I showed Karl Gauss's trick for summing a $1, 2, \cdots, n$ sequence. Add it up twice, reversing the sum and adding by columns

$$\begin{array}{c|c|c|c|c} 1 & 2 & \cdots & n-1 & n \\ n & n-1 & \cdots & 2 & 1 \end{array} \tag{1.1}$$

We get $n + 1$, $n$ times, so

$$\sum_{k=1}^{n} k = \frac{n}{2}(n+1). \tag{1.2}$$

Karl pointed out to me that he'd looked up the formula for the sum of squares, and found

$$\sum_{k=1}^{n} k^2 = \frac{n}{6}(2n+1)(n+1). \tag{1.3}$$

I couldn't think of some equivalent of the Guassian trick to sum that, but had the vague memory that we could figure it out using difference calculus. I have a little Dover book [1] on the subject that I read some of when I was in school. Without resorting to that book, I tried to dredge up the memory of how this result could be derived.

## 1.2 Difference operator.

The key is defining a difference operator, akin to a derivative

**Definition 1.1: Difference operator (reverse.)**

Given a sequence $y_n$, let

$$\Delta y_n = y_n - y_{n-1}.$$

It's also possible to define forward difference operators $\Delta y_n = y_{n+1} - y_n$, or both, and it turns out that the text uses forward differences. I'll use reverse difference operator here, since that's what I tried. The ideas should hold either way.

We can apply the difference operator to some simple sequences, such as $y_n = \text{constant}, y_n = n, y_n = n^2, \cdots$. For those, we find

$$\begin{aligned}
\Delta 1 &= 0 \\
\Delta n &= n - (n-1) \\
&= 1 \\
\Delta n^2 &= n^2 - (n-1)^2 \\
&= 2n - 1 \\
\Delta n^3 &= n^3 - (n-1)^3 \\
&= 3n^2 - 3n + 1.
\end{aligned} \tag{1.4}$$

Rearranging, we find

$$\begin{aligned}
1 &= \Delta n \\
n &= \frac{1}{2}\left(\Delta n^2 + 1\right) \\
&= \frac{1}{2}\left(\Delta n^2 + \Delta n\right) \\
&= \frac{1}{2}\Delta\left(n^2 + n\right) \\
n^2 &= \frac{1}{3}\left(\Delta n^3 + 3n - 1\right) \\
&= \frac{1}{3}\left(\Delta n^3 + \frac{3}{2}\Delta\left(n^2 + n\right) - \Delta n\right) \\
&= \frac{1}{6}\Delta\left(2n^3 + 3\left(n^2 + n\right) - 2n\right) \\
&= \frac{1}{6}\Delta\left(2n^3 + 3n^2 + n\right).
\end{aligned} \tag{1.5}$$

## 1.3 Sum of squares.

We can now proceed to find the difference of our sum of squares sequence. Let

$$y_n = \sum_{k=1}^{n}, \tag{1.6}$$

for which we have

$$\Delta y_n = n^2 = \Delta \frac{2n^3 + 3n^2 + n}{6}. \tag{1.7}$$

Akin to integrating, we've determined $y_n$ up to a constant

$$y_n = \frac{2n^3 + 3n^2 + n}{6} + C, \tag{1.8}$$

but since $y_1 = 1$, and

$$y_1 = \frac{2 \times 1^3 + 3 \times 1^2 + 1}{6} + C = 1 + C, \tag{1.9}$$

so $C = 0$. We need only factor to find the desired result

$$\sum_{k=1}^{n} k^2 = \frac{n}{6} \left(2n + 1\right) \left(n + 1\right). \tag{1.10}$$

## 1.4 Sum of cubes.

Let's also apply this to compute a formula for the sum of cubes. We need one more difference computation

$$\begin{aligned} \Delta n^4 &= n^4 - (n - 1)^4 \\ &= 4n^3 - 6n^2 + 4n - 1, \end{aligned} \tag{1.11}$$

or

$$\begin{aligned} n^3 &= \frac{1}{4} \left( \Delta n^4 + 6n^2 - 4n + 1 \right) \\ &= \frac{1}{4} \left( \Delta n^4 + \Delta \left(2n^3 + 3n^2 + n\right) - 2\Delta \left(n^2 + n\right) + \Delta n \right) \\ &= \frac{1}{4} \Delta \left( n^4 + 2n^3 + n^2 \right), \end{aligned} \tag{1.12}$$

so

$$\sum_{k=1}^{n} n^3 = \frac{1}{4} \left( n^4 + 2n^3 + n^2 \right) + C, \tag{1.13}$$

but we see that $C = 0$ is required to satisfy the $n = 1$ case. That is

$$\sum_{k=1}^{n} n^3 = \frac{1}{4} n^2 \left(n + 1\right)^2. \tag{1.14}$$

Difference calculus is a pretty fun tool!

# Bibliography

[1] Hyman Levy and Freda Lessman. *Finite difference equations*. Courier Corporation, 1992. 1.1