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A contour integral with a third order pole.

Here's problem 31(e) from [1]. Find

$$I = \int_0^\infty \frac{x^2 dx}{\left(a^2 + x^2\right)^3}. \tag{1.1}$$

Again, we use the contour C illustrated in fig. 1.1

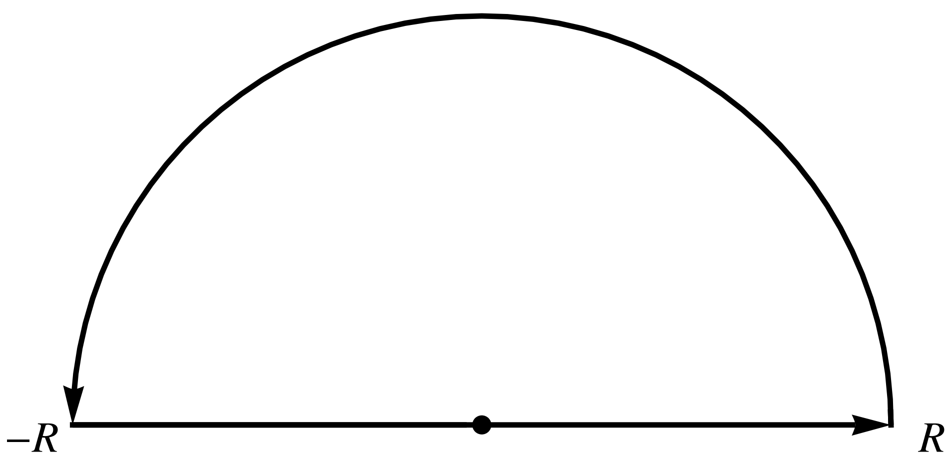

Figure 1.1: Standard above the x-axis, semicircular contour.

Along the infinite semicircle, with $z = Re^{i\theta}$,

$$\left| \int \frac{z^2 dz}{\left(a^2 + z^2\right)^3} \right| = O(R^3/R^6), \tag{1.2}$$

which tends to zero. We are left to just evaluate some residues

$$\begin{aligned} I &= \frac{1}{2} \oint \frac{z^2 dz}{\left(a^2 + z^2\right)^3} \\ &= \frac{1}{2} \oint \frac{z^2 dz}{\left(z - ia\right)^3 \left(z + ia\right)^3} \\ &= \frac{1}{2} \left(2\pi i\right) \frac{1}{2!} \left(\frac{z^2}{\left(z + ia\right)^3}\right)'' \Bigg|_{z=ia} \end{aligned} \tag{1.3}$$

Evaluating the derivatives, we have

$$\begin{aligned} \left(\frac{z^2}{\left(z + ia\right)^3}\right)' &= \frac{2z\left(z + ia\right) - 3z^2}{\left(z + ia\right)^4} \\ &= \frac{-z^2 + 2iaz}{\left(z + ia\right)^4}, \end{aligned} \tag{1.4}$$

and

$$\begin{aligned} \left(\frac{z^2}{\left(z + ia\right)^3}\right)'' &= \left(\frac{-z^2 + 2iaz}{\left(z + ia\right)^4}\right)' \\ &= \frac{\left(-2z + 2ia\right)\left(z + ia\right) - 4\left(-z^2 + 2iaz\right)}{\left(z + ia\right)^5}, \end{aligned} \tag{1.5}$$

so

$$\begin{aligned} \left(\frac{z^2}{\left(z + ia\right)^3}\right)'' \Bigg|_{z=ia} &= \frac{\left(-2ia + 2ia\right)\left(2ia\right) - 4\left(a^2 - 2a^2\right)}{\left(2ia\right)^5} \\ &= \frac{4a^2}{\left(2ia\right)^5} \\ &= \frac{1}{8a^3 i}. \end{aligned} \tag{1.6}$$

Putting all the pieces together, we have

$$\boxed{I = \frac{\pi}{16a^3}.} \tag{1.7}$$

Bibliography

[1] F.W. Byron and R.W. Fuller. *Mathematics of Classical and Quantum Physics*. Dover Publications, 1992. 1