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Evaluating some real trig integrals with unit circle contours.

Here are a couple of problems from [1].

1.1 A sine integral.

This is problem (31(a)). For $a > b > 0$, find

$$I = \int_0^{2\pi} \frac{d\theta}{a + b\sin\theta}. \tag{1.1}$$

We can proceed by making a change of variables $z = e^{i\theta}$, for which $dz = izd\theta$. Also let $\alpha = a/b$, so

$$\begin{aligned} I &= \frac{1}{b} \oint_{|z|=1} \frac{-idz}{z} \frac{1}{\alpha + (1/2i)(z - 1/z)} \\ &= \frac{2}{b} \oint_{|z|=1} \frac{dz}{2iz\alpha + z^2 - 1} \\ &= \frac{2}{b} \oint_{|z|=1} \frac{dz}{\left(z + i\alpha + i\sqrt{\alpha^2 - 1}\right)\left(z + i\alpha - i\sqrt{\alpha^2 - 1}\right)}. \end{aligned} \tag{1.2}$$

Clearly the mixed sign factor represents the pole that falls within the unit circle, so we have only one residue to include

$$\begin{aligned} I &= \frac{2}{b} 2\pi i \left. \frac{1}{z + i\alpha + i\sqrt{\alpha^2 - 1}} \right|_{z = -i\alpha + i\sqrt{\alpha^2 - 1}} \\ &= \frac{4\pi i}{b} \frac{1}{2i\sqrt{\alpha^2 - 1}} \\ &= \frac{2\pi}{\sqrt{a^2 - b^2}}. \end{aligned} \tag{1.3}$$

1.2 Sines and cosines upstairs and downstairs.

This is problem (31(b)). Given $a > b > 0$ (again), this time we want to find

$$I = \int_0^{2\pi} \frac{\sin^2\theta d\theta}{a + b\cos\theta}. \tag{1.4}$$

We'd like to make the same $z = e^{i\theta}$ substitution, but have to prepare a bit. We rewrite the sine

$$\begin{aligned} \sin^2\theta &= \frac{1}{2}\left(1 - \cos(2\theta)\right) \\ &= \frac{1}{2} - \frac{1}{4}\left(e^{2i\theta} + e^{-2i\theta}\right), \end{aligned} \tag{1.5}$$

so, again setting $\alpha = a/b$, we have

$$\begin{aligned} I &= \frac{1}{b}\oint_{|z|=1}\left(\frac{1}{2} - \frac{1}{4}\left(z^2 + 1/z^2\right)\right)\frac{-idz}{z}\frac{1}{\alpha + (1/2)\left(z + 1/z\right)} \\ &= \frac{-i}{2b}\oint_{|z|=1}\left(2 - z^2 - \frac{1}{z^2}\right)\frac{dz}{2\alpha z + z^2 + 1} \\ &= \frac{-i}{2b}\oint_{|z|=1} dz \frac{2z^2 - z^4 - 1}{z^2\left(2\alpha z + z^2 + 1\right)} \\ &= \frac{-i}{2b}\oint_{|z|=1} dz \frac{2z^2 - z^4 - 1}{z^2\left(z + \alpha + \sqrt{\alpha^2 - 1}\right)\left(z + \alpha - \sqrt{\alpha^2 - 1}\right)}. \end{aligned} \tag{1.6}$$

The enclosed poles are at $z = 0$ (a second order pole) and $z = -\alpha + \sqrt{\alpha^2 - 1}$, so the integral is

$$I = (2\pi i)\left(\frac{-i}{2b}\right)\left(\left.\left(\frac{2z^2 - z^4 - 1}{2\alpha z + z^2 + 1}\right)'\right|_{z=0} + \left.\frac{2z^2 - z^4 - 1}{z^2\left(z + \alpha + \sqrt{\alpha^2 - 1}\right)}\right|_{z=-\alpha+\sqrt{\alpha^2-1}}\right) \tag{1.7}$$

The derivative residue simplifies to

$$\begin{aligned} \left.\left(\frac{2z^2 - z^4 - 1}{2\alpha z + z^2 + 1}\right)'\right|_{z=0} &= \left.\frac{4z - 4z^3}{2\alpha z + z^2 + 1} - \frac{2z^2 - z^4 - 1}{\left(2\alpha z + z^2 + 1\right)^2}\left(2\alpha + 2z\right)\right|_{z=0} \\ &= 2\alpha, \end{aligned} \tag{1.8}$$

whereas the remaining residue is

$$\left.-\frac{\left(z^2 - 1\right)^2}{z^2\left(2\sqrt{\alpha^2 - 1}\right)}\right|_{z=-\alpha+\sqrt{\alpha^2-1}} = \left.-\left(z - \frac{1}{z}\right)^2\frac{1}{2\sqrt{\alpha^2 - 1}}\right|_{z=-\alpha+\sqrt{\alpha^2-1}}, \tag{1.9}$$

but

$$
\begin{aligned}
\frac{1}{z} &= \frac{1}{-\alpha + \sqrt{\alpha^2 - 1}} \frac{\left(\alpha + \sqrt{\alpha^2 - 1}\right)}{\left(\alpha + \sqrt{\alpha^2 - 1}\right)} \\
&= \frac{\alpha + \sqrt{\alpha^2 - 1}}{-\alpha^2 + (\alpha^2 - 1)} \\
&= -\left(\alpha + \sqrt{\alpha^2 - 1}\right),
\end{aligned}
\tag{1.10}
$$

and

$$
z - \frac{1}{z} = -\alpha + \sqrt{\alpha^2 - 1} + \alpha + \sqrt{\alpha^2 - 1} \quad = 2\sqrt{\alpha^2 - 1},
\tag{1.11}
$$

so

$$
\begin{aligned}
-\left.\frac{\left(z^2 - 1\right)^2}{z^2 \left(2\sqrt{\alpha^2 - 1}\right)}\right|_{z = -\alpha + \sqrt{\alpha^2 - 1}} &= -\frac{\left(2\sqrt{\alpha^2 - 1}\right)^2}{2\sqrt{\alpha^2 - 1}} \\
&= -2\sqrt{\alpha^2 - 1},
\end{aligned}
\tag{1.12}
$$

for a final answer of

$$
\begin{aligned}
I &= \frac{2\pi}{b} \left(\alpha - \sqrt{\alpha^2 - 1}\right) \\
&= \frac{2\pi}{b^2} \left(a - \sqrt{a^2 - b^2}\right).
\end{aligned}
\tag{1.13}
$$

1.3 Another cosine integral.

Last problem of this sort (31 (c)), was to find, again with $a > b > 0$

$$
I = \int_0^{2\pi} \frac{d\theta}{\left(a + b\cos\theta\right)^2}.
\tag{1.14}
$$

Making our $z = e^{i\theta}$ substitution, and setting $\alpha = a/b$, we have

$$
\begin{aligned}
I &= \frac{1}{b^2} \oint_{|z|=1} \frac{-idz/z}{\left(\alpha + (1/2)\left(z + 1/z\right)\right)^2} \\
&= \frac{-4i}{b^2} \oint_{|z|=1} \frac{zdz}{\left(2\alpha z + z^2 + 1\right)^2} \\
&= \frac{-4i}{b^2} \oint_{|z|=1} \frac{zdz}{\left(z + \alpha + \sqrt{\alpha^2 - 1}\right)^2 \left(z + \alpha - \sqrt{\alpha^2 - 1}\right)^2}.
\end{aligned}
\tag{1.15}
$$

Again, only this mixed sign pole will be within the unit circle, so

$$I = \left(\frac{-4i}{b^2}\right)(2\pi i)\left(\left.\left(\frac{z}{\left(z+\alpha+\sqrt{\alpha^2-1}\right)^2}\right)'\right|_{z=-\alpha+\sqrt{\alpha^2-1}}\right) \tag{1.16}$$

That derivative is

$$\begin{aligned}
\left(\frac{z}{\left(z+\alpha+\sqrt{\alpha^2-1}\right)^2}\right)' &= \frac{1}{\left(z+\alpha+\sqrt{\alpha^2-1}\right)^2} - \frac{2z}{\left(z+\alpha+\sqrt{\alpha^2-1}\right)^3} \\
&= \frac{z+\alpha+\sqrt{\alpha^2-1}-2z}{\left(z+\alpha+\sqrt{\alpha^2-1}\right)^3} \\
&= \frac{-z+\alpha+\sqrt{\alpha^2-1}}{\left(z+\alpha+\sqrt{\alpha^2-1}\right)^3}.
\end{aligned} \tag{1.17}$$

Evaluating it at our pole $z = -\alpha + \sqrt{\alpha^2-1}$, we have

$$\begin{aligned}
\frac{-z+\alpha+\sqrt{\alpha^2-1}}{\left(z+\alpha+\sqrt{\alpha^2-1}\right)^3} &= \frac{\alpha-\sqrt{\alpha^2-1}+\alpha+\sqrt{\alpha^2-1}}{\left(-\alpha+\sqrt{\alpha^2-1}+\alpha+\sqrt{\alpha^2-1}\right)^3} \\
&= \frac{2\alpha}{\left(2\sqrt{\alpha^2-1}\right)^3} \\
&= \frac{1}{4}\frac{\alpha}{\left(\alpha^2-1\right)^{3/2}},
\end{aligned} \tag{1.18}$$

so

$$\begin{aligned}
I &= \frac{8\pi}{b^2}\frac{1}{4}\frac{\alpha}{\left(\alpha^2-1\right)^{3/2}} \\
&= \frac{2\pi a}{b^3\left(\alpha^2-1\right)^{3/2}},
\end{aligned} \tag{1.19}$$

but $b^3 = \left(b^2\right)^{3/2}$, for

$$I = \frac{2\pi a}{\left(a^2-b^2\right)^{3/2}}. \tag{1.20}$$

Bibliography

[1] F.W. Byron and R.W. Fuller. *Mathematics of Classical and Quantum Physics*. Dover Publications, 1992. 1